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POSITIVE RATIONAL SEMIGROUPS AND COMMUTATIVE POWER JOINED CANCELLATIVE SEMIGROUPS WITHOUT IDEMPOTENT

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I. Introduction A commutative semigroup S is called archimedean if for every ordered pair of elements $a, b \in S$ there is an element $c \in S$ and a positive integer m such that $a^m = bc$. Following PETRICH [7] an $\mathfrak{N}$-semigroup is a commutative archimedean cancellative semigroup without idempotent. An $\mathfrak{N}$-semigroup S is determined by an abelian group G and a non-negative valued function I (called ϑ-function). The following fundamental theorem on $\mathfrak{N}$-semigroups was originally obtained in 1957.

Theorem 1. *Let G be an abelian group and N be the set of all non-negative integers. Suppose a function $I : G \times G \to N$ satisfies the following conditions:*

(1.1) $I(\alpha, \beta) = I(\beta, \alpha)$ *for all* $\alpha, \beta \in G$.
(1.2) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$ *for all* $\alpha, \beta, \gamma \in G$.
(1.3) $I(\varepsilon, \varepsilon) = 1$ ε *being the identity element of* G.
(1.4) *For every* $\alpha \in G$ *there is a positive integer* m *such that* $I(\alpha^m, \alpha) > 0$.

We define an operation on the set $S_0 = N \times G = \{(m, \alpha) : m \in N, \alpha \in G\}$ *by*

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta).$$

Then S_0 *is an* $\mathfrak{N}$*-semigroup. If* S *is an* $\mathfrak{N}$*-semigroup,* S *is isomorphic to some* S_0 *obtained in this manner.*

The proof of Theorem 1 can be seen in [9], [10]. Also see p. 136, [3].

S_0 is denoted by $S_0 = (G; I)$. If S is an $\mathfrak{N}$-semigroup, then G_a is associated with an element $a \in S$ in such a way that $S = (G_a; I_a)$, $G_a \cong S/\varrho_a$ where $x\varrho_a y$ iff $a^m x = a^n y$ for some positive integers m, n. G_a is called the structure group of S with respect to a. The ϑ-function I_a is also associated with a.

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A semigroup S is called power joined if for every $a, b \in S$ there are positive integers m and n such that $a^m = b^n$. The study of commutative power joined semigroups is basically important in the following sense: Every commutative semigroup is the disjoint union of power joined subsemigroups and a commutative power joined semigroup can not be decomposed into the disjoint union of more than one subsemigroup. However we will not enter to the deep study here of this problem. The following is due to CHRISLOCK [1, 2].

Lemma 2. *An $\mathfrak{N}$-semigroup S is power joined (finitely generated) if and only if G_a is periodic (finite) for every a, equivalently for some a of S.*

Throughout this paper a "positive integer semigroup" means a subsemigroup of the semigroup of all positive integers under addition, and a "positive rational semigroup" is a subsemigroup of the semigroup of all positive rational numbers under addition. By a subdirect product of semigroups A and B we mean a subsemigroup S of the direct product of A and B where the projections of S into A and B are equal to A and B respectively.

It is obvious that finitely generated $\mathfrak{N}$-semigroups are power joined. HIGGINS [4] proved that an $\mathfrak{N}$-semigroup is finitely generated if and only if it is isomorphic onto a subdirect product of a finite abelian group and a positive integer semigroup. In this paper we generalize Higgins' result as follows: A power joined $\mathfrak{N}$-semigroup is isomorphic onto a subdirect product of a periodic abelian group and a positive rational semigroup. Positive rational semigroups are not only typical examples but also play an important rôle in the theory of power joined $\mathfrak{N}$-semigroups. We will also discuss the characterization of positive rational additive semigroups.

2. Homomorphisms into R_+. R_+ denotes the semigroup of all positive rational numbers under addition. In this section we will prove that every power joined $\mathfrak{N}$-semigroup has a unique homomorphism into R_+ in some sense. Theorem 3 does not assume cancellation.

Theorem 3. *If S is a commutative power joined semigroup without idempotent, then S is homomorphic onto a positive rational semigroup. Let φ and φ_0 be homomorphisms of S into R_+, then $\varphi(x) = r \cdot \varphi_0(x)$, $x \in S$, where r is a positive rational number and $r \cdot \varphi_0(x)$ is the usual multiplication of r and $\varphi_0(x)$. Then the semigroups $\varphi(S)$ and $\varphi_0(S)$ are isomorphic.*

Proof. First we prove the existence of homomorphisms. Let $a \in S$ be fixed. Since S is power joined, for each $x \in S$ there are positive integers m, n such that

$$a^m = x^n .$$

Define a map φ_0 of S into R_+ by

$$\varphi_0(x) = \frac{m}{n}.$$

φ_0 is well defined: $a^m = x^n$, $a^{m'} = x^{n'}$ implies $x^{mn'} = x^{m'n}$ hence $mn' = m'n$ because S has no idempotent. Let $\varphi_0(x) = m/n$ and $\varphi_0(y) = l/k$. Then $a^{mk+ln} = (xy)^{nk}$ follows from $a^m = x^n$ and $a^l = y^k$. Therefore we have $\varphi_0(xy) = \varphi_0(x) + \varphi_0(y)$. Thus φ_0 is a homomorphism of S into R_+. Let φ be an arbitrary homomorphism of S into R_+. For each $x \in S$, $a^m = x^n$ implies

$$n\,\varphi_0(x) = m\,\varphi_0(a) \quad \text{and} \quad n\,\varphi(x) = m\,\varphi(a),$$

and then $\varphi(x) = r \cdot \varphi_0(x)$ where $r = \varphi(a)/\varphi_0(a) = \varphi(a)$ since $\varphi_0(a) = 1$. It is obvious that the semigroups $\varphi(S)$ and $\varphi_0(S)$ are isomorphic.

Corollary 4. *Let S and S' be positive rational semigroups. If φ is a homomorphism of S onto S' then φ is an isomorphism and $\varphi(x) = r \cdot x$ where r is a positive rational number.*

A semigroup S is called power cancellative if S satisfies the following condition

$$a^n = b^n \Rightarrow a = b \quad n = 2, 3, \ldots$$

For example positive rational semigroups are power cancellative. The following was stated in [6] but we have it here as a consequence of Theorem 3.

Theorem 5. *Let S be a non-trivial commutative power joined semigroup. S is power cancellative if and only if S is isomorphic onto a positive rational semigroup.*

Proof. The "if" part is obvious. To prove the "only if" part, suppose S is power cancellative and S has an idempotent e. Since S is power joined, for every $x \in S$, there is a positive integer m such that

$$x^m = e = e^m.$$

This implies $x = e$ and hence S would be trivial $S = \{e\}$, a contradiction. Therefore S has no idempotent. We can apply Theorem 3 to this case and there exists a homomorphism φ_0 of S to a positive rational semigroup. Assume $\varphi_0(x) = \varphi_0(y)$, that is, $a^m = x^n$, $a^k = y^l$ and $ml = nk$. By power cancellation, $x^{nk} = y^{ml}$ implies $x = y$. Therefore S is isomorphic into R_+ under φ_0.

As seen in Theorem 3, $\varphi(x) = \varphi(y)$ if and only if $\varphi_0(x) = \varphi_0(y)$. Accordingly the congruences on S induced by the homomorphisms of S into R_+ are uniquely determined. In other words there is a unique nontrivial power cancellative factor semigroup of a commutative power joined semigroup without idempotent.

Let $\mathrm{Hom}(S, R_+)$ denote the semigroup of all homomorphisms of a commutative power joined semigroup S without idempotent into the semigroup R_+ of all positive rational numbers with addition. The operation in $\mathrm{Hom}(S, R_+)$ is defined by

$$(\varphi + \Psi)(x) = \varphi(x) + \Psi(x).$$

It is easy to see

$$\mathrm{Hom}(S, R_+) \cong R_+ .$$

3. Positive Rational Semigroups. In this section we characterize positive rational semigroups in some way and discuss how to construct these semigroups. We already have a characterization of positive rational semigroups in Theorem 5: A non-trivial semigroup is commutative power joined and power cancellative if and only if it is isomorphic to a positive rational semigroup.

First we observe positive integer semigroups. It is easy to see the following proposition:

Proposition 6. *A non-trivial semigroup is a finitely generated, commutative power joined, power cancellative semigroup if and only if it is isomorphic to a positive integer semigroup.*

A positive integer semigroup S has the base i.e. a unique minimal generating set $\{a_1, \ldots, a_n\}$, $a_1 < a_2 < \ldots < a_n$, where $n \leqq a_1$. Denote S by $S = [a_1, \ldots, a_n]$. Then $[a_1, \ldots, a_n] \cong [b_1, \ldots, b_m]$ if and only if $n = m$ and $b_1/a_1 = \ldots = b_n/a_n$. Thus S is determined in terms of the base. This fact is considered as a special case of Corollary 4.

Consider a family $\{D_i : i = 1, 2, \ldots\}$ of semigroups with isomorphisms φ_{ji} of D_i into D_j, $i \leqq j$, such that for $i \leqq j \leqq k$, $\varphi_{ki}(x) = \varphi_{kj}\varphi_{ji}(x)$ and $\varphi_{ii}(x) = x$. The semigroup D of the set union $\bigcup_{i=1}^{\infty} D_i$ which, for every i, j and x such that $i \leqq j$ and $x \in D_i$, identifies x with $\varphi_{ji}(x)$ is called the direct limit of $\{D_i : i = 1, 2, \ldots\}$ with respect to the isomorphism family $F = \{\varphi_{ji} : i = 1, 2, \ldots; j = 1, 2, \ldots; i \leqq j\}$ and is denoted by $D = \lim\limits_{\longrightarrow} (D_i; F)$. When it is not necessary to specify φ_{ji}, D is called a direct limit of $\{D_i : i = 1, 2, \ldots\}$.

Theorem 7. *A semigroup S is isomorphic to a positive rational semigroup if and only if S is isomorphic to a direct limit of non-trivial finitely generated commutative power joined, power cancellative semigroups.*

Proof. Suppose S is isomorphic onto a positive rational semigroup. For convenience suppose that S itself is a positive rational semigroup. Let $S_i = S \cap [1/i!]$ where $[1/i!]$ is the additive subsemigroup generated by $1/i!$. S_i is isomorphic onto a positive integer semigroup. We define φ_{ji} by the inclusion of S_i into S_j in the natural sense, that is,

$$\varphi_{ji}\left(\frac{x}{i!}\right) = \frac{(i+1)(i+2)\ldots(j-1)j \cdot x}{(i+1)!}, \quad \frac{x}{i!} \in S_i, \quad i \leqq j .$$

Then $S \cong \varinjlim (S_i; \varphi_{ji})$. Conversely, if S_i $(i = 1, 2, \ldots)$ is a non-trivial finitely generated commutative power joined, power cancellative semigroup, then it is easy to show that a direct limit of S_i is power joined and power cancellative.

According to Corollary 4 the isomorphism φ_{ji} of S_i into S_j is determined by a positive rational number r_{ji} such that $\varphi_{ji}(x) = r_{ji} \cdot x$. A positive rational semigroup S is determined by an ascending chain $\{S_n : n = 1, 2, \ldots\}$ of positive integer semigroups and a sequence $\{s_i : i = 1, 2, \ldots\}$ of positive rational numbers. $\{S_n\}$ can be given arbitrarily because of Proposition 8 below, and $\{s_i\}$ is chosen such that $\varphi_{i+1,i}(x) = s_i x$ is an isomorphism of S_i into S_{i+1}.

Proposition 8. *Any two positive integer semigroups can be embedded into each other.*

This is proved by using the following Lemma 9.

A positive integer semigroup U is called a segment if U consists of all the positive integers greater than or equal to m for some positive integer m.

Lemma 9. *The following assertions are equivalent.*

(3.1) *A positive integer semigroup $S = [a_1, \ldots, a_n]$ contains a segment.*

(3.2) *The greatest common divisor of all elements of S is equal to* 1.

(3.3) *The greatest common divisor of $a_1, \ldots, a_n$ is equal to* 1.

Notice that every positive integer semigroup is isomorphic onto a positive integer semigroup satisfying one of the conditions (3.1), (3,2) and (3.3).

4. Power joined $\mathfrak{N}$-semigroups. Let S be a power joined $\mathfrak{N}$-semigroup, $S = (G; I)$, where G is a periodic abelian group.

Theorem 10. *If a function $\bar{\varphi} : G \to R_+$ is defined by*

$$\bar{\varphi}(\alpha) = \Big(\sum_{i=1}^{s} I(\alpha, \alpha^i)\Big)/s \tag{4.1}$$

where s is the order of the element α of G, then $\bar{\varphi}$ satisfies the following conditions:

(4.2) $\bar{\varphi}(\varepsilon) = 1$ *if ε is the identity element of G.*

(4.3) $\bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta)$ *is a non-negative integer, and*

(4.4) $I(\alpha, \beta) = \bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta)$.

Conversely, let $\bar{\varphi}$ be a function, $G \to R_+$, which satisfies (4.2) *and* (4.3). *If for $\bar{\varphi}$ we define I, $G \times G \to N$, by* (4.4), *then I satisfies* (1.1) *through* (1.4) *and* (4.1).

Proof. The function $\bar{\varphi}$ defined by (4.1) is certainly a positive rational valued function on G. $\bar{\varphi}(\alpha)$ is invariant even if s is replaced by its multiple because if n is a multiple of s, then

$$\left(\sum_{i=1}^{n} I(\alpha, \alpha^i)\right)/n = \left(\frac{n}{s} \cdot \sum_{i=1}^{s} I(\alpha, \alpha^i)\right) \Big/ n = \left(\sum_{i=1}^{s} I(\alpha, \alpha^i)\right)/s .$$

Hence we have

$$\bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta) = \left(\sum_{i=1}^{s} I(\alpha, \alpha^i)\right)/s + \left(\sum_{i=1}^{t} I(\beta, \beta^i)\right)/t - \left(\sum_{i=1}^{st} I(\alpha\beta, (\alpha\beta)^i)\right)/st$$

since $\alpha^s = \beta^t = \varepsilon$ implies $(\alpha\beta)^{st} = \varepsilon$. By the condition (1.2) of the ϑ-function I,

$$\begin{aligned}
&I(\alpha, \beta) + I(\alpha\beta, \alpha^i\beta^i) = I(\alpha, \alpha^i\beta^{i+1}) + I(\beta, \alpha^i\beta^i) , \\
&I(\alpha, \alpha^i\beta^{i+1}) + I(\alpha^i, \beta^{i+1}) = I(\alpha, \alpha^i) + I(\alpha^{i+1}, \beta^{i+1}) , \\
&I(\alpha^i, \beta^i) + I(\alpha^i\beta^i, \beta) = I(\alpha^i, \beta^{i+1}) + I(\beta^i, \beta) .
\end{aligned}$$

Adding the above three equalities, we obtain

$$I(\alpha\beta, \alpha^i\beta^i) = I(\alpha, \alpha^i) + I(\beta, \beta^i) + I(\alpha^{i+1}, \beta^{i+1}) - I(\alpha, \beta) - I(\alpha^i, \beta^i)$$

and

$$\begin{aligned}
\sum_{i=1}^{st} I(\alpha\beta, (\alpha\beta)^i) &= \sum_{i=1}^{st} \left(I(\alpha, \alpha^i) + I(\beta, \beta^i) + I(\alpha^{i+1}, \beta^{i+1}) - I(\alpha, \beta) - I(\alpha^i, \beta^i)\right) = \\
&= -st\, I(\alpha, \beta) + t \sum_{i=1}^{s} I(\alpha, \alpha^i) + s \sum_{i=1}^{t} I(\beta, \beta^i) .
\end{aligned}$$

Consequently we have $\bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta) = I(\alpha, \beta)$. $I(\alpha, \beta)$ is a non-negative integer and $\bar{\varphi}(\varepsilon) = I(\varepsilon, \varepsilon) = 1$.

We will prove the converse. It is obvious that I is defined on $G \times G$ and is non-negative integer valued.

$$\begin{aligned}
I(\alpha, \beta) &= \bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta) = I(\beta, \alpha) , \\
I(\alpha, \beta) + I(\alpha\beta, \gamma) &= \bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta) + \bar{\varphi}(\alpha\beta) + \bar{\varphi}(\gamma) - \bar{\varphi}(\alpha\beta\gamma) = \\
&= \bar{\varphi}(\alpha) + \bar{\varphi}(\beta\gamma) - \bar{\varphi}(\alpha\beta\gamma) + \bar{\varphi}(\beta) + \bar{\varphi}(\gamma) - \bar{\varphi}(\beta\gamma) = \\
&= I(\alpha, \beta\gamma) + I(\beta, \gamma) , \\
I(\varepsilon, \alpha) &= \bar{\varphi}(\varepsilon) + \bar{\varphi}(\alpha) - \bar{\varphi}(\varepsilon\alpha) = \bar{\varphi}(\varepsilon) = 1 .
\end{aligned}$$

Since G is periodic, for any $\alpha \in G$ there is a positive integer n sueh that $\alpha^n = \varepsilon$, hence $I(\alpha, \alpha^n) = 1 > 0$. Thus we have proved that I satisfies (1.1) through (1.4). To prove (4.1),

$$\sum_{i=1}^{s} I(\alpha, \alpha^i) = \sum_{i=1}^{s} \left(\bar{\varphi}(\alpha) + \bar{\varphi}(\alpha^i) - \bar{\varphi}(\alpha^{i+1})\right) = (s + 1)\, \bar{\varphi}(\alpha) - \bar{\varphi}(\alpha^{s+1}) = s\, \bar{\varphi}(\alpha)$$

$$\text{since } \alpha^s = \varepsilon .$$

The proof of the theorem has been completed.

By Theorem 10, a power joined $\mathfrak{N}$-semigroup is determined by G and a function $\bar{\varphi} : G \to R_+$ which satisfies (4.2) and (4.3). We will call such a function a $\bar{\varphi}$-function. According to (4.3) $\bar{\varphi}$ induces a homomorphism of G into the additive group of all rational numbers mod 1.

Finitely generated $\mathfrak{N}$-semigroups [4] are obtained as a special case.

Corollary 11. *If an $\mathfrak{N}$-semigroup $S = (G; I)$ is finitely generated, then G is finite and*

$$|G| \cdot \bar{\varphi}(\alpha) = \sum_{\xi \in G} I(\alpha, \xi), \quad |G| \text{ is the order of } G .$$

Proof. By Lemma 2 G is finite. The following is the decomposition of G induced by the cyclic subgroup $[\alpha]$ generated by $\alpha \in G$.

$$G = [\alpha]\, \tau_1 \cup \ldots \cup [\alpha]\, \tau_r$$

where $\tau_1 = \varepsilon$, $r = |G|/s$, $s = |[\alpha]|$. Hence we have

$$\sum_{\xi \in G} I(\alpha, \xi) = \sum_{j=1}^{r} \sum_{i=1}^{s} I(\alpha, \alpha^i \tau_j) = \sum_{j=1}^{r} \left(\sum_{i=1}^{s} \left(I(\alpha, \alpha^i) + I(\alpha^{i+1}, \tau_j) - I(\alpha^i, \tau_j) \right) \right) =$$

$$= \sum_{j=1}^{r} \left(\sum_{i=1}^{s} I(\alpha, \alpha^i) \right) = |G| \cdot \left(\sum_{i=1}^{s} I(\alpha, \alpha^i) \right) / s = |G| \cdot \bar{\varphi}(\alpha) .$$

The following theorem will play an important role in the proof of Theorem 13.

Theorem 12. *Let $S = (G; I)$ be a power joined $\mathfrak{N}$-semigroup. Every homomorphism φ_r of S into R_+ is given by*

$$\varphi_r((m, \alpha)) = r \cdot (m + \bar{\varphi}(\alpha)) \tag{4.5}$$

where r is arbitrary positive rational number and $\bar{\varphi}(\alpha)$ is defined by (4.1).

Proof. By Theorem 3 we need to prove only that φ_1 is a homomorphism of S into R_+.

$$\varphi_1((m, \alpha)(n, \beta)) = \varphi_1((m + n + I(\alpha, \beta), \alpha\beta)) = m + n + I(\alpha, \beta) + \bar{\varphi}(\alpha\beta) =$$

$$= m + n + \bar{\varphi}(\alpha) + \bar{\varphi}(\beta) = \varphi_1((m, \alpha)) + \varphi_1((n, \beta)) \quad \text{by (4.4)} .$$

Theorem 13. *A semigroup S is a power joined $\mathfrak{N}$-semigroup if and only if S is isomorphic onto a subdirect product of a periodic abelian group and a positive rational semigroup.*

Proof. Let $S = (G; I)$ be a power joined $\mathfrak{N}$-semigroup. By [1, 2] G is periodic. Let ζ be the homomorphism of S onto G in the natural way: $\zeta((m, \alpha)) = \alpha$. Let

$\varphi = \varphi_1$ defined by (4.5) in the case of $r = 1$. To prove that S is isomorphic onto a subdirect product of G and $\varphi(S)$, all we have to do is to prove

$$\zeta((m, \alpha)) = \zeta((n, \beta)) \quad \text{and} \quad \varphi(m, \alpha)) = \varphi((n, \beta)) \quad \text{imply} \quad (m, \alpha) = (n, \beta)\,.$$

From the first we have $\alpha = \beta$, and then from the second,

$$m + \bar{\varphi}(\alpha) = n + \bar{\varphi}(\beta)$$

which implies $m = n$, hence $(m, \alpha) = (n, \beta)$. This proves that if ϱ_ζ and ϱ_φ are the congruences on S induced by ζ and φ respectively, then $\varrho_\zeta \cap \varrho_\varphi = \iota$, ι being the equality relation on S. Therefore S is isomorphic onto a subdirect product of $S/\varrho_\zeta\,(\cong G)$ and $S/\varrho_\varphi\,(\cong \varphi(S))$.

Next we will prove the converse. Let S be a subdirect product of a periodic abelian group G and a positive rational semigroup M. The elements of S are denoted by $((r, \alpha))$, $r \in M$, $\alpha \in G$. Let $\xi \in G$, and let

$$M_\xi = \{s \in M : ((s, \xi)) \in S\}\,.$$

Then

$$S = \{((r, \xi)) : r \in M_\xi,\ \xi \in G\}\,, \quad M = \bigcup_{\xi \in G} M_\xi\,,$$

and

$$((r, \xi)) \,.\, ((s, \eta)) = ((r + s, \xi\eta))\,.$$

We easily see that S is commutative, cancellative and has no idempotent. We will prove S is power joined, and hence archimedean. Let $((r, \xi)), ((s, \eta)) \in S$. There exist positive integers m, n, p, q such that

$$mr = ns\,, \quad \xi^p = \eta^q = \varepsilon\,, \quad \varepsilon \text{ is the identity element of } G$$

where we notice G is periodic and mr is the usual multiplication. Then we have

$$((r, \xi))^{mpq} = ((mpqr, \xi^{mpq})) = ((npqs, \varepsilon)) = ((s, \eta))^{npq}\,.$$

This completes the proof.

Let S be a power joined $\mathfrak{N}$-semigroup, $S = (G; I)$. $\bar{\varphi}$ is defined by (4.1) and φ is defined in the proof of Theorem 13. We define a mapping $f : S \to R_+ \times G$ by

$$f((m, \alpha)) = ((m + \bar{\varphi}(\alpha), \alpha))\,.$$

Let $M = \varphi(S)$. Then f is the isomorphism of S onto a subdirect product of M and G. We notice that ϱ_φ is the unique power cancellative congruence on S.

By Theorem 10 a power joined $\mathfrak{N}$-semigroup S is determined by a periodic abelian group G and a positive rational valued function $\bar{\varphi}$ satisfying (4.2), (4.3). So S is denoted by $S = (G; \bar{\varphi})$. By using Theorem 12 or 13 we can characterize positive rational semigroups in terms of $\bar{\varphi}$.

Proposition 14. *A power joined $\mathfrak{N}$-semigroup $(G; \bar{\varphi})$ is isomorphic onto a positive rational semigroup if and only if*

$$\bar{\varphi}(\alpha) \equiv \bar{\varphi}(\beta) \pmod 1 \quad \textit{implies} \quad \alpha = \beta . \tag{4.6}$$

Proof. S is not trivial since it is an $\mathfrak{N}$-semigroup. According to Theorem 5 S is isomorphic onto a positive rational semigroup if and only if S is power cancellative. By the remark after Theorem 13, S is power cancellative if and only if $\varrho_{\varphi} = \iota$, that is, φ is one to one where $\varphi = \varphi_1$ as mentioned in the proof of Theorem 13. By Theorem 12 our desired condition is that

$$m + \bar{\varphi}(\alpha) = n + \bar{\varphi}(\beta) \quad \text{implies} \quad (m, \alpha) = (n, \beta) . \tag{4.6'}$$

It is easy to see that (4.6′) is equivalent to (4.6).

As mentioned before, (4.3) tells us that $\bar{\varphi}$ induces a homomorphism $\bar{\varphi}_{(1)}$ of G into $R/(1)$, the additive semigroup of all rational numbers mod 1. Therefore the condition (4.6) can be restated by

$$\bar{\varphi}_{(1)} \text{ is an isomorphism of } G \text{ into } R/(1) . \tag{4.7}$$

5. Problems. There are several problems which are left unsolved in this paper.

(5.1) When a periodic abelian group G is given, find a general method to determine concretely all $\bar{\varphi} : G \to R_+$ satisfying (4.2) and (4.3).

(5.2) When a periodic abelian group G and a positive rational semigroup M are given, determine all subdirect products of G and M.

(5.3) A power joined $\mathfrak{N}$-semigroup is called strong if its homomorphic image into R_+ is isomorphic to a positive integer semigroup. Every power joined $\mathfrak{N}$-semigroup is isomorphic to a direct limit of strong power joined $\mathfrak{N}$-semigroups. Characterize strong power joined $\mathfrak{N}$-semigroups. The examples of such semigroups are (1) finitely generated $\mathfrak{N}$-semigroups, (2) $\mathfrak{N}$-semigroups having the property that the orders of elements of a structure groups is bounded, (3) the direct product of a periodic abelian group and a positive integer semigroup.

(5.4) The isomorphism problem of two power joined $\mathfrak{N}$-semigroups seems to be complicated.

(1) Let S_i and S'_i be positive integer semigroups. Let $S = \varinjlim (S_i; \varphi_{ji})$ and $S' = \varinjlim (S'_i; \varphi'_{ji})$. Describe the isomorphism condition for S and S' in terms of $S_i, S'_i, \varphi_{ji}, \varphi'_{ji}$.

(2) Find the isomorphism condition of S and S' in the case where S_i and S'_i are strong power joined $\mathfrak{N}$-semigroups.

(3) Find the isomorphism condition in terms of the subdirect factors in the sense of Theorem 13.

(4) Minimal power relation is an important concept in power joined semigroups. Let a and b be elements of a power joined $\mathfrak{N}$-semigroup. The relation $a^{m_0} = b^{n_0}$ is called the minimal power relation between a and b if $a^{m_0} = b^{n_0}$ and $a^m = b^n$ imply $m_0 \leqq m$, $n_0 \leqq n$.

The minimal power relation always exists between a and b of S. If $a^{m_0} = b^{n_0}$ is the minimal power relation and if $a^m = b^n$ then $m/m_0 = n/n_0$ is an integer. If S is power cancellative, m_0 and n_0 are relatively prime.

Let S be a non-trivial commutative power joined, power cancellative semigroup. By Theorems 5, 7, $S \cong \varinjlim (S_n, \varphi_{m,n})$. We can pick at most countable set $A \subseteq S$

$$A = \{a_1, a_2, \ldots, a_i, \ldots\}$$

such that $\{a_1, \ldots, a_i\}$ is a generating set of S_i, which need not be minimal,and S is generated by A. Let

$$\text{(MP)} \qquad a_1^{m_{11}} = a_2^{m_2},\ a_1^{m_{21}} = a_3^{m_3},\ \ldots,\ a_1^{m_{i1}} = a_i^{m_i},\ \ldots$$

be the minimal power relations between a_1 and a_i $(i \neq 1)$. As mentioned above m_{i1} and m_i are relatively prime. Let S be the commutative power cancellative semigroup generated by A subject to the relations (MP), that is, there is no relation other than those derived from (MP) and by the power cancellation. Every commutative power joined power cancellative semigroup S is obtained in this manner, in other words, S is determined by a set A and a map Φ of A into $N \times N$.

$$\Phi(a_i) = (m_{i1}, m_i)\,, \quad \Phi(a_1) = (1, 1)$$

where m_{i1} and m_i are relatively prime.

Characterize commutative power joined power cancellative semigroups and power joined $\mathfrak{N}$-semigroups in terms of the minimal power relations.

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