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Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 4, 535–566

Persistent URL: http://dml.cz/dmlcz/101055

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

A LINEAR AND WEAKLY NONLINEAR EQUATION OF A BEAM: THE BOUNDARY-VALUE PROBLEM FOR FREE EXTREMITIES AND ITS PERIODIC SOLUTIONS

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(Received March, 3, 1969)

The aim of this paper is to investigate the existence of a classical solution to the mixed problem $(\mathscr{M})$ given by the equations

$$u_{tt}(t, x) + u_{xxxx}(t, x) = g(t, x) + \varepsilon f(t, x, u(t, x), u_x(t, x), u_{xx}(t, x), u_t(t, x), \varepsilon) \tag{0.1}$$

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0 \tag{0.2}$$

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x) \tag{0.3}$$

on the one hand, and on the other hand, the existence of a ω-periodic solution to the problem $(\mathscr{P}_\omega)$ given by (0.1), (0.2) with f ω-periodic in t. Of course, we shall start with the study of the corresponding linear limit cases for $\varepsilon = 0$.

To find a periodic solution we make use of the Poincaré method whose general characterization we have outlined in [1], and for the use of which we have all prepared by the solution of the problem $(\mathscr{M})$. Let us remark, however, that if only the existence of periodic solutions is requested, a more direct and even a more advantageous method consists in looking for a periodic solution in the form of an Fourier expansion in t. (The advantage is that no conditions on the behaviour of g and f for $x = 0$ and $x = \pi$ are imposed. This approach is followed in papers prepared by M. SOVA and M. KOPÁČKOVÁ.)

For the existence of a solution to the problem $(\mathscr{P}_\omega)$ with $\omega = 2\pi n$ or $\omega = 2\pi p/q$ (n, p, q naturals) we have derived necessary and sufficient conditions in the linear case and necessary or sufficient conditions in the weakly nonlinear case. For $\omega = 2\pi\alpha$, α irrational, it is possible to find only certain sufficient conditions in both cases.

In the last time, a special case of the problem $(\mathscr{P}_\omega)$, namely that with $\omega = 2\pi$, $g(t, x) \equiv 0$ and $f = f(t, x, u(t, x))$, where roughly speaking f behaves monotonically with respect to u, was studied. (See [2], [3], [4]). Here, under some additional assumptions, the existence of at least a generalized solution can be proved.

The presented paper consists of the following paragraphs:

§1. SOME FUNCTIONAL SPACES

Let us introduce some functional spaces which we shall use in the sequel.

Let us denote $\mathcal{H}_0^m$ the subspace of functions $h(x)$ from the Sobolev space $\mathcal{W}_2^m(0, \pi)$ for which

$$h^{(2l)}(0) = h^{(2l)}(\pi) = 0$$

for $2l < m$, l nonnegative integer, with the norm

$$\|h\|_{\mathcal{H}_0^m}^2 = \int_0^\pi |h^{(m)}(x)|^2 \, dx \, .$$

(By the well known embedding theorem, the functions $h^{(j)}$ for $j < m$ may be considered as continuous on $\langle 0, \pi \rangle$ and thus their values at a point $x \in \langle 0, \pi \rangle$ are defined.)

Further, let us denote $\bar{\mathcal{H}}_0^m$ the space of functions $h(x) \in \mathcal{L}_2(0, \pi)$ which have a Fourier series of the form

$$\sum_{j=1}^{\infty} h_j \sin jx \quad \text{with} \quad \sum_{j=1}^{\infty} j^{2m} h_j^2 < +\infty \, .$$

We define the norm in $\bar{\mathcal{H}}_0^m$ as

$$\|h\|_{\bar{\mathcal{H}}_0^m} = \left(\frac{\pi}{2} \sum_{j=1}^{\infty} j^{2m} h_j^2 \right)^{1/2} . \tag{1.1}$$

Lemma 1.1. *The spaces $\mathcal{H}_0^m$ and $\bar{\mathcal{H}}_0^m$ are identical.*

Proof. Choosing $m = 2k + 1$, writing $h^{(2k+1)}$ as a cosine-series and taking into account that

$$\int_0^\pi h^{(2k+1)}(x) \cos jx \, dx = (-1)^k j^{2k+1} \int_0^\pi h(x) \sin jx \, dx \, ,$$

we easily find that $\mathscr{H}_0^{2k+1} \subset \bar{\mathscr{H}}_0^{2k+1}$. On the other hand, the equality

$$\lim_{n\to\infty} \int_0^\pi \sum_{j=1}^n h_j \sin j\xi \,.\, \varphi^{(2k+1)}(\xi)\,\mathrm{d}\xi = (-1)^{2k+1} \lim_{n\to\infty} \int_0^\pi (-1)^k \sum_{j=1}^n j^{2k+1} h_j \cos j\xi \,.\, \varphi(\xi)\,\mathrm{d}\xi$$

holding for all $\varphi \in \mathscr{C}_0^\infty(0, \pi)$, yields $\bar{\mathscr{H}}_0^{2k+1} \subset \mathscr{W}_2^{2k+1}(0, \pi)$. The inclusion $\bar{\mathscr{H}}_0^{2k+1} \subset \mathscr{H}_0^{2k+1}$ follows from the fact that the continuous representative of $h^{(2l)}$ ($l = 0, 1, \ldots, k$) is given by the uniformly convergent series

$$\sum_{j=1}^\infty (-1)^l h_j j^{2l} \sin jx = \tilde{h}^{(2l)}(x)$$

for which $\tilde{h}^{(2l)}(0) = \tilde{h}^{(2l)}(\pi) = 0$. For $m = 2k$ the proof is quite similar.

Remark 1.1. In the sequel, in virtue of this lemma, we make no difference between $\mathscr{H}$ and $\bar{\mathscr{H}}$ and denote both simply $\mathscr{H}_0^m$.

Remark 1.2. Further we shall make no difference between functions belonging to $\mathscr{H}_0^m$ defined on $\langle 0, \pi\rangle$ and their 2π-periodic, odd extentions onto $(-\infty, +\infty)$.

Let $\mathfrak{h}^m$ be the space of such sequences $\{h_j\}_{j=1}^\infty$ that $\sum_{j=1}^\infty j^{2m} h_j^2 < +\infty$, with the norm given in (1.1).

Lemma 1.2. *The spaces $\mathscr{H}_0^m$ and $\mathfrak{h}^m$ are isomorfic and isometric.*

The proof follows immediately from Lemma 1.1.

Let $\mathscr{C}^{(k)}(\langle 0, T\rangle; \mathscr{H}_0^m)$ denote the space of functions $v : \langle 0, T\rangle \to \mathscr{H}_0^m$, with k continuous derivatives and with the norm

$$\|v\|_{C^{(k)}(\langle 0,T\rangle;\mathscr{H}_0{}^m)} = \sum_{l=0}^k \max_{t\in\langle 0,T\rangle} \|v^{(l)}(t)\|_{\mathscr{H}_0{}^m}\,.$$

Let $\mathscr{U}$ be the space

$$\mathscr{U} = \mathscr{C}^{(0)}(J; \mathscr{H}_0^5) \cap \mathscr{C}^{(1)}(J; \mathscr{H}_0^3) \cap \mathscr{C}^{(2)}(J; \mathscr{H}_0^1)\,, \quad J = \langle 0, T\rangle$$

with the norm

$$\|u\|_{\mathscr{U}} = \|u\|_{\mathscr{C}^{(0)}(J;\mathscr{H}_0{}^5)} + \|u\|_{\mathscr{C}^{(1)}(J;\mathscr{H}_0{}^3)} + \|u\|_{\mathscr{C}^{(2)}(J;\mathscr{H}_0{}^1)}\,.$$

Further, let us denote $\bar{\mathscr{U}}$ the space of functions $u : \langle 0, T\rangle \to \mathscr{L}_2(0, \pi)$ which have the Fourier series

$$\sum_1^\infty u_k(t) \sin kx\,,$$

where $u_k(t)$, $u_k'(t)$ and $u_k''(t)$ are continuous on $\langle 0, T\rangle$ and the series

$$(1.2) \qquad \sum_1^\infty k^{10}\, u_k^2(t)\,, \quad \sum_1^\infty k^6\, u_k'^2(t)\,, \quad \sum_1^\infty k^2\, u_k''^2(t)$$

converge uniformly in $\langle 0, T \rangle$, with the norm

$$(1.3) \qquad \|u\|_{\overline{\mathscr{U}}} = \max_{t\in\langle 0,T\rangle}\left(\frac{\pi}{2}\sum_1^\infty k^{10}\, u_k^2(t)\right)^{1/2} + \left(\max_{t\in\langle 0,T\rangle}\left(\frac{\pi}{2}\sum_1^\infty k^6 u_k^2(t)\right)^{1/2} + \right.$$

$$+ \max_{t\in\langle 0,T\rangle}\left(\frac{\pi}{2}\sum_1^\infty k^6\, u_k'^2(t)\right)^{1/2}\Bigg) + \left(\max_{t\in\langle 0,T\rangle}\left(\frac{\pi}{2}\sum_1^\infty k^2\, u_k^2(t)\right)^{1/2} + \right.$$

$$\left. + \max_{t\in\langle 0,T\rangle}\left(\frac{\pi}{2}\sum_1^\infty k^2\, u_k'^2(t)\right)^{1/2} + \max_{t\in\langle 0,T\rangle}\left(\frac{\pi}{2}\sum_1^\infty k^2\, u_k''^2(t)\right)^{1/2}\right).$$

Lemma 1.3. *The spaces $\mathscr{U}$ and $\overline{\mathscr{U}}$ are identical.*

Proof. Let $\bar{u} \in \overline{\mathscr{U}}$, with the Fourier series $\sum_{k=1}^\infty \bar{u}_k(t) \sin kx$. According to Lemma 1.1. this series represents a function $u(t) \in \mathscr{H}_0^5$ for each $t \in \langle 0, T \rangle$. In virtue of the uniform convergence of series in (1.1) it may be proved immediately that this function $u \in \mathscr{U}$. Hence, $\overline{\mathscr{U}} \subset \mathscr{U}$. On the other hand, let $u \in \mathscr{U}$. Then, again by Lemma 1.1., for each $t \in \langle 0, T \rangle$, $u(t) \in \mathscr{H}_0^5$ may be represented by a series $\sum_1^\infty u_k(t) \sin kx$. Then, using the Schwartz lemma and the definition of the space $\mathscr{U}$, it may be verified, that for every $k = 1, 2, \ldots$ the functions $u_k(t)$ are elements of the space $\mathscr{C}^2\langle 0, T \rangle$ and that $u'(t)$, $u''(t)$, respectively, is represented by the series $\sum_1^\infty u_k'(t) \sin kx$, $\sum_1^\infty u_k''(t) \sin kx$, respectively. The Dini theorem on the uniform convergence of a series with continuous non-negative terms yields the uniform convergence of the series (1.2) and thus, $\mathscr{U} \subset \overline{\mathscr{U}}$. The identity of both norms is evident. Similarly as in the case of spaces $\mathscr{H}_0^m$, $\overline{\mathscr{H}}_0^m$, we shall not make any difference between the spaces $\mathscr{U}$ and $\overline{\mathscr{U}}$.

By the standard method the following lemma may be proved:

Lemma 1.4. *The space $\mathscr{U}$ is a Banach space.*

Analogously as above, let us define the space $\mathfrak{u}$ of sequences $\{u_k(t)\}_{k=1}^\infty$, where $u_k(t) \in \mathscr{C}^2\langle 0, T \rangle$, fulfilling (1.2), with the norm (1.3). Then the following lemma holds.

Lemma 1.5. *The spaces $\mathscr{U}$ and $\mathfrak{u}$ are isomorfic and isometric.*

Remark 1.3. The function $u : \langle 0, T \rangle \to \mathscr{L}_2$ may be considered as a function $u = u(t, x)$ of two variables $t \in \langle 0, T \rangle$ and $x \in \langle 0, \pi \rangle$ while, for a fixed $t \in \langle 0, T \rangle$, $u(t, .)$ is a representative of the element $u(t) \in \mathscr{L}_2$. From the context it will be usually clear in what sense a given function is considered.

Remark 1.4. Any function $u \in \mathscr{U}$ considered as a function of the variables t and x is equivalent to a function $u = u(t, x)$ continuous together with its derivatives

$u_{xxxx}(t, x)$ and $u_{tt}(t, x)$ on $\langle 0, T\rangle \times \langle 0, \pi\rangle$. The proof is based on the following fact: let $v \in \mathscr{C}(\langle 0, T\rangle; \mathscr{H}_0^1)$, then v has a representative continuous on $\langle 0, T\rangle \times \langle 0, \pi\rangle$. Indeed, by the Sobolev embedding theorem, for each $t \in \langle 0, T\rangle$, the element $u(t) \in \mathscr{H}_0^1$ has a representative $u(t, x)$ continuous in x on $\langle 0, \pi\rangle$ and $|u(t, x)| \leqq C\|u(t)\|_{\mathscr{H}_0^1}$ $(C > 0)$. Then

$$|u(t, x) - u(\tau, \xi)| \leqq |u(t, x) - u(t, \xi)| + |u(t, \xi) - u(\tau, \xi)| \leqq$$
$$\leqq |u(t, x) - u(t, \xi)| + c\|u(t) - u(\tau)\|_{\mathscr{H}_0^1},$$

from where our assertion follows.

§2. THE LINEAR MIXED PROBLEM $(\mathscr{M}^0)$

In the rectangle $0 \leqq t \leqq T$, $0 \leqq x \leqq \pi$ let the mixed problem $(\mathscr{M}^0)$ be given by the equations

$$u_{tt}(t, x) + u_{xxxx}(t, x) = g(t, x), \tag{2.1}$$

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0 \tag{2.2}$$

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x) \tag{2.3}$$

and let us investigate under which conditions it has a solution $u \in U$.

Since the boundary-value problem

$$X'''' - \lambda X = 0, \quad X(0) = X(\pi) = X''(0) = X''(\pi) = 0$$

has eigenfunctions $\sin nx$ corresponding to eigenvalues n^4, we easily find using the Fourier method that a formal solution to the problem $(\mathscr{M}^0)$ has the form

$$u(t, x) = \sum_{n=1}^{\infty} \left[\varphi_n \cos n^2 t + \frac{\psi_n}{n^2} \sin n^2 t + \right.$$
$$\left. + \frac{1}{n^2} \int_0^t \sin n^2(t - \tau)\, g_n(\tau)\, \mathrm{d}\tau \right] \sin nx, \tag{2.4}$$

where φ_n, ψ_n, $g_n(t)$ are Fourier coefficients of functions $\varphi(x)$, $\psi(x)$, $g(t, x)$, respectively.

Theorem 2.1. *Let the problem $(\mathscr{M}^0)$ be given. Let the following assumptions be fulfilled:*

(i) $g \in \mathscr{C}(\langle 0, T\rangle; \mathscr{H}_0^3)$

(ii) $\varphi \in \mathscr{H}_0^5$, $\psi \in \mathscr{H}_0^3$.

Then the problem $(\mathscr{M}^0)$ has a unique solution $u \in \mathscr{U}$.

Proof. Let us prove that under our assumptions the function u, given by (2.4), is an element of the space $\mathscr{U}$. The norm of u fulfils

$$\|u\|_{\mathscr{U}} \leqq 3 \max_{t \in \langle 0,T \rangle} \left(\frac{\pi}{2} \sum_{k=1}^{\infty} k^{10} \left[\varphi_k \cos k^2 t + \frac{\psi_k}{k^2} \sin k^2 t + \right.\right.$$

$$\left.\left. + \frac{1}{k^2} \int_0^t g_k(\tau) \sin k^2(t - \tau) \, d\tau \right]^2 \right)^{1/2} +$$

$$+ 2 \max_{t \in \langle 0,T \rangle} \left(\frac{\pi}{2} \sum_{k=1}^{\infty} k^6 \left[-\varphi_k k^2 \sin k^2 t + \psi_k \cos k^2 t + \right.\right.$$

$$\left.\left. + \int_0^t g_k(\tau) \cos k^2(t - \tau) \, d\tau \right]^2 \right)^{1/2} +$$

$$+ \max_{t \in \langle 0,T \rangle} \left(\frac{\pi}{2} \sum_{k=1}^{\infty} k^2 \left[-\varphi_k k^4 \cos k^2 t - \psi_k k^2 \sin k^2 t - \right.\right.$$

$$\left.\left. - k^2 \int_0^t g_k(\tau) \sin k^2(t - \tau) \, d\tau + g_k(t) \right]^2 \right)^{1/2}$$

whereform using the Schwartz inequality and the Levi theorem $\|u\|_{\mathscr{U}} \leqq C(\|\varphi\|_{\mathscr{H}_0^5} + \|\psi\|_{\mathscr{H}_0^3} + \|g\|_{\mathscr{C}(\langle 0,T \rangle; \mathscr{H}_0^3)})$.

Further, let $v \in \mathscr{U}$ be a solution to the problem $(\mathscr{M}^0)$ with $g(t, x) \equiv 0$, $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$. Then,

$$0 = \int_0^\pi \int_0^t v_t [v_{tt} + v_{xxxx}] \, d\tau \, dx = \frac{1}{2} \int_0^\pi \int_0^t \frac{d}{dt} [v_t^2 + v_{xx}^2] \, d\tau \, dx =$$

$$= \frac{1}{2} \int_0^\pi [v_t^2(t, x) + v_{xx}^2(t, x)] \, dx \,, \quad 0 \leqq t \leqq T \,,$$

which, according to (2.2), yields $v(t, x) \equiv 0$ on $\langle 0, T \rangle \times \langle 0, \pi \rangle$, and this proves the uniqueness of the found solution.

Looking for a periodic solution, it is sometimes more advantageous to have the sought solution in a more closed form then that of an infinite series. So, let us try to express the solution in the form of an integral. From the literature it is known that a solution to the Cauchy problem (2.1), (2.3) $(-\infty < x < +\infty)$ with $g(t, x) \equiv 0$ can be written in the form

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} (\cos \lambda^2 + \sin \lambda^2) \, \varphi(x - 2(\sqrt{t}) \, \lambda) \, d\lambda -$$

$$- \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} (\cos \lambda^2 - \sin \lambda^2) \, \Psi(x - 2(\sqrt{t}) \, \lambda) \, d\lambda \,,$$

(Ψ being the second primitive function to ψ) provided that $|\varphi|$ and $|\psi|$ decrease sufficiently rapidly to 0 for $|x| \to +\infty$. The boundary conditions (2.2) suggest to extend the functions φ, ψ and g as odd and 2π-periodic onto $(-\infty, +\infty)$. As the integrals of the higher derivatives of the expressions with thus extended functions under the integral sign do not exist in the usual sense we have to make use of some generalized notion of the integral. We shall take that of Abel's integral defined by

$$\int_{(a,b)}^{\to} f(x)\,\mathrm{d}x = \lim_{\delta\to 0+} \int_a^b \exp(-\delta x^2) f(x)\,\mathrm{d}x\,,$$

$$-\infty \leqq a < b \leqq +\infty\,.$$

Theorem 2.1′. *Let the problem $(\mathscr{M}^0)$ be given. Let the assumptions of Theorem* 2.1 *be fulfilled. Then the unique solution $u \in \mathscr{U}$ to the problem $(\mathscr{M}^0)$ can be written in the form*

$$(2.5)\quad u(t, x) = \frac{1}{\sqrt{(2\pi)}}\left\{\int_{(-\infty,+\infty)}^{\to} (\cos\lambda^2 + \sin\lambda^2)\,\varphi(x - 2(\sqrt{t})\,\lambda)\,\mathrm{d}\lambda - \right.$$

$$- \int_{(-\infty,+\infty)}^{\to} (\cos\lambda^2 - \sin\lambda^2)\,\Psi(x - 2(\sqrt{t})\,\lambda)\,\mathrm{d}\lambda -$$

$$\left. - \int_0^t \int_{(-\infty,+\infty)}^{\to} (\cos\lambda^2 - \sin\lambda^2)\,G(\vartheta, x - 2(\sqrt{(t-\vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta\right\}$$

where

$$(2.6)\qquad \Psi(x) = \int_0^x \int_0^\xi \psi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi - \frac{x}{2\pi}\int_0^{2\pi}\int_0^\xi \psi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,,$$

$$G(t, x) = \int_0^x \int_0^\xi g(t, \eta)\,\mathrm{d}\eta\,\mathrm{d}\xi - \frac{x}{2\pi}\int_0^{2\pi}\int_0^\xi g(t, \eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,.$$

Remark 2.1. The fourth derivative with respect to x and the second derivative with respect to t of the integrals occuring in (2.5) equal to the integrals of the derivatives in question of the expressions under the integral sign.

Remark 2.2. The second primitive functions Ψ, G are chosen to be 2π-periodic and odd in x.

Proof of Theorem 2.1′. First, let us note, that if the functions φ, ψ and g are from $\mathscr{H}_0^5$, $\mathscr{H}_0^3$ and $\mathscr{C}(\langle 0, T\rangle; \mathscr{H}_0^3)$, respectively, then their extensions onto $-\infty < x < +\infty$ are such that φ'''', ψ'' and $(\partial^2/\partial x^2)\,g(t, x)$ $(t \in \langle 0, T\rangle)$ are absolutely continuous on every closed bounded interval, 2π-periodic and odd and they are given by their Fourier series. Utilizing

$$(2.7)\quad \int_{-\infty}^{+\infty} \exp(-\delta\lambda^2)(\cos\lambda^2 \pm \sin\lambda^2)\cos 2k(\sqrt{t})\,\lambda\,\mathrm{d}\lambda =$$

$$= \left(\sqrt{\frac{\pi}{2(\delta^2+1)}}\right)\exp\left(-\frac{k^2 t\delta}{\delta^2+1}\right)\left\{(\sqrt{(\delta+\sqrt{(\delta^2+1)})} \pm \sqrt{(-\delta+\sqrt{(\delta^2+1)})})\,.\right.$$

$$\left.\cdot\cos\frac{k^2 t}{\delta^2+1} + (\sqrt{(-\delta+\sqrt{(\delta^2+1)})} \mp \sqrt{(\delta+\sqrt{(\delta^2+1)})})\sin\frac{k^2 t}{\delta^2+1}\right\}$$

and inserting for φ, ψ and g their Fourier series into (2.5), we get by rather lengthy but standard calculations that the expressions (2.5) and (2.4) are identical. Analogously, for the derivatives mentioned in the Remark 2.1.

§3. PERIODIC SOLUTIONS IN THE LINEAR CASE

3.1. General situation. Let ω be a real positive number. In all this paragraph we suppose the function g to satisfy the assumption (i) from Theorem 2.1 with $T = \omega$ and

$$(3.1.1)\qquad g(t, x) = g(t + \omega, x)$$

for $t \in (-\infty, +\infty)$, $x \in \langle 0, \pi\rangle$.

We ask if there exists a solution of (2.1), (2.2) satisfying the periodicity condition

$$(3.1.2)\qquad u(t, x) = u(t + \omega, x)\,.$$

The last condition is evidently equivalent to

$$(3.1.3)\qquad \begin{aligned} u(0, x) - u(\omega, x) &= 0 \\ u_t(0, x) - u_t(\omega, x) &= 0 \end{aligned}$$

for $x \in \langle 0, \pi\rangle$.

This problem defined by (2.1), (2.2), (3.1.3) will be denoted $(\mathscr{P}^0_\omega)$.

Using the Poincaré method, we shall investigate if the initial functions φ, ψ may be chosen in such a way that the corresponding solution of the problem $(\mathscr{M}^0)$ is a solution of the problem $(\mathscr{P}^0_\omega)$, too.

It turns out useful to distinguish three different cases:

(A) $\omega = 2\pi n$, n natural,

(B) $\omega = 2\pi p/q$, p, q natural, relatively prime, $q \neq 1$,

(C) $\omega = 2\pi\alpha$, α irrational.

3.2. The case (A). From (2.4) follows immediately

Theorem 3.2.1. *Let the problem* $(\mathscr{P}^0_{2\pi n})$ *be given. Let the function* $g \in \mathscr{C}(\langle 0, 2\pi n\rangle; \mathscr{H}^3_0)$ *and let it be* $2\pi n$*-periodic in* t.

Then a solution exists if and only if

$$(3.2.1_1) \qquad \int_0^{2\pi n} \cos k^2\tau \,.\, g_k(\tau)\,\mathrm{d}\tau = 0$$

$$(3.2.1_2) \qquad \int_0^{2\pi n} \sin k^2\tau \,.\, g_k(\tau)\,\mathrm{d}\tau = 0\,,$$

for $k = 1, 2, \ldots$

If these conditions are fulfilled, every solution of the problem $(\mathscr{P}^0_{2\pi n})$ *is given by* (2.4) *or* (2.5) *where* φ *and* ψ *are arbitrary functions satisfying the assumption* (ii) *from Theorem* 2.1.

Considering the solution of $(\mathscr{M}^0)$ in the form (2.5) and taking into account that the first two terms in (2.5) are identical with the two terms in (2.4) (which means they are $2\pi n$-periodic in t) we get instantaneously that a solution of $(\mathscr{P}^0_{2\pi n})$ exists if and only if

$$(3.2.2_1) \quad \Phi_1(x) \equiv \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos\lambda^2 - \sin\lambda^2)\,.$$
$$.\, G(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$(3.2.2_2) \quad \Phi_2(x) \equiv \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos\lambda^2 - \sin\lambda^2)\,(2\pi n - \vartheta)^{-1/2}\,.$$
$$.\, G_x(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0\,.$$

(The interchange of order of integration with respect to λ and the derivation with respect to t is allowed according to Remark 2.1.) The condition $(3.2.2_1)$ is equivalent to $\Phi_1(0) = \Phi_1'(0) = 0$, $\Phi_1''(x) = 0$ for $x \in \langle 0, \pi\rangle$, or

$$(3.2.3) \quad \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos\lambda^2 - \sin\lambda^2)\, G(\vartheta, 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$(3.2.4) \quad \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos\lambda^2 - \sin\lambda^2)\, G_x(\vartheta, 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$(3.2.5) \quad \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos\lambda^2 - \sin\lambda^2)\, g(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0\,.$$

The condition (3.2.3) may be omitted since this integral equals 0 according to $G(t, x) = -G(t, -x)$. Condition (3.2.4) may be omitted since it is a consequence of (3.2.3), (3.2.5) and of the $2\pi n$-periodicity of the function $\Phi_1(x)$. Integrating by parts with respect to λ in (3.2.2$_2$) we get

$$(3.2.6) \quad \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 + \sin \lambda^2)\, g(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\, \lambda)\, d\lambda\, d\vartheta = 0\,.$$

Adding and subtracting conditions (3.2.5) and (3.2.6), we obtain

Theorem 3.2.1'. *The assertion of Theorem* 3.2.1 *remains valid if we replace the conditions* (3.2.1) *by the following ones*

$$(3.2.7) \quad \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, g(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\, \lambda)\, d\lambda\, d\vartheta = 0$$

$$\int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, g(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\, \lambda)\, d\lambda\, d\vartheta = 0\,.$$

3.3. The case (B). First let us prove the following

Lemma 3.3.1. *Let* $\omega = 2\pi\, p/q$, p, q *natural, relatively prime,* $q \neq 1$; $q = q_{(1)} \cdot q_{(2)}^2$, $q_{(1)}$ *square-free.*

Then

(i) $\sin(\pi k^2\, p/q) = 0$, k *natural if and only if* $k = n \,.\, q_{(1)} \cdot q_{(2)}$, n *natural,*

(ii) *there exists a constant* $c > 0$ *so that* $|\sin(\pi k^2\, p/q)|^{-1} \leqq c$ *for all natural* $k \neq n \,.\, q_{(1)} \cdot q_{(2)}$, $n = 1, 2, \ldots$.

Proof. First we prove the assertion (i). Let us suppose $\sin(\pi k^2\, p/q) = 0$. This implies the existence of an natural l such that $k^2 p = lq$. As lq is divisible by p and p, q are relatively prime, there must exist an natural l such that $l = lp$, then $k^2 = lq$. Expressing $k = k_1^{i_1} \ldots k_r^{i_r}$, $q = q_1^{2m_1+n_1} \ldots q_s^{2m_s+n_s}$, where $k_1, \ldots, k_r$, $q_1, \ldots, q_s$ are primes, $i_1, \ldots, i_r$, $m_1, \ldots, m_s$, $n_1, \ldots, n_s$ nonnegative integers, $n_j = 0$ or 1 and putting $q_{(1)} = q_1^{n_1} \ldots q_s^{n_s}$, $q_{(2)} = q_1^{m_1} \ldots q_s^{m_s}$, we see that $q_{(2)}$ divides k and setting $k = \bar{k} \,.\, q_{(2)}$, $\bar{k}$ natural, we easily find that $q_{(1)}$ divides $\bar{k}$, which proves the necessity of the condition formulated in (i). Sufficiency is obvious.

Now let us prove (ii). Let l be such a natural that $|\pi k^2\, p/q - l\pi| \leqq \frac{1}{2}\pi$. Then

$$|\sin(\pi k^2\, p/q)|^{-1} = (|\pi k^2\, p/q - l\pi| : |\sin(\pi k^2\, p/q - l\pi)|) \,.\, (q\pi^{-1}) \,.\, |k^2 p - lq|^{-1}\,.$$

The product of the first two factors is bounded by $\frac{1}{2}q$ and $|k^2 p - lq| \geqq 1$ by (i), which completes the proof.

We shall denote $\mathscr{S}(q)$ the set of all naturals which may be written as $n \,.\, q_{(1)} \cdot q_{(2)}$, n natural. $\overline{\mathscr{S}}(q)$ denotes the complement of $\mathscr{S}(q)$ in the set of all natural numbers.

Inserting the solution of $(\mathscr{M}^0)$ given by (2.4) into (3.1.3) and equating the cofactors of $\sin kx$, we find the following systems of equations for φ_k, ψ_k.

$$(3.3.1)\qquad \varphi_k(\cos k^2\omega - 1) + \psi_k \,.\, k^{-2}\sin k^2\omega + k^{-2}\int_0^\omega \sin k^2(\omega - \tau)\,.\,g_k(\tau)\,\mathrm{d}\tau = 0$$

$$-\varphi_k \sin k^2\omega + \psi_k k^{-2}(\cos k^2\omega - 1) + k^{-2}\int_0^\omega \cos k^2(\omega - \tau)\,.\,g_k(\tau)\,\mathrm{d}\tau = 0\,.$$

The determinant D_k of the k-th system of equations is

$$D_k = 4k^{-2}\sin^2\left(\tfrac{1}{2}k^2\omega\right) = 4k^{-2}\,.\,\sin^2\left(\pi k^2\,p/q\right).$$

For $k \in \overline{\mathscr{S}}(q)$ we have

$$(3.3.2)\qquad \varphi_k = \frac{1}{2k^2}\left[\frac{\cos(k^2\omega/2)}{\sin(k^2\omega/2)}\int_0^\omega \cos k^2\tau\,.\,g_k(\tau)\,\mathrm{d}\tau + \int_0^\omega \sin k^2\tau\,.\,g_k(\tau)\,\mathrm{d}\tau\right],$$

$$\psi_k = \frac{1}{2}\left[\frac{\cos(k^2\omega/2)}{\sin(k^2\omega/2)}\int_0^\omega \sin k^2\tau\,.\,g_k(\tau)\,\mathrm{d}\tau - \int_0^\omega \cos k^2\tau\,.\,g_k(\tau)\,\mathrm{d}\tau\right].$$

We easily find that in virtue of the assertion (ii) from Lemma 3.3.1. the series

$$(3.3.3)\qquad u^*(t, x) = \sum_{k\in\overline{\mathscr{S}}(q)}\left(\varphi_k\,.\,\cos k^2t + \frac{\psi_k}{k^2}\sin k^2t\right)\sin kx\,,$$

where φ_k, ψ_k are defined by (3.3.2), is an element of $\mathscr{U}$.

Further let k be an element of $\mathscr{S}(q)$. Then $D_k = 0$ and all coefficients of the corresponding system (3.3.1) equal 0 too. Hence, the necessary and sufficient conditions that (3.3.1) have a solution are

$$(3.3.4)\qquad \int_0^\omega \cos k^2\tau\,.\,g_k(\tau)\,\mathrm{d}\tau = 0$$

$$\int_0^\omega \sin k^2\tau\,.\,g_k(\tau)\,\mathrm{d}\tau = 0\,.$$

Theorem 3.3.1. *Let the problem $(\mathscr{P}^0_{2\pi\,p/q})$ be given. Let the function $g \in \mathscr{C}(\langle 0, T\rangle; \mathscr{H}^3_0)$ and let it be $2\pi\,p/q$-periodic in t. Then a solution to $(\mathscr{P}^0_{2\pi\,p/q})$ exists if and only if the conditions (3.3.4) for all $k \in \mathscr{S}(q)$ are fulfilled. If they are fulfilled, then every solution $u \in \mathscr{U}$ to $(\mathscr{P}^0_{2\pi\,p/q})$ has the form*

$$u(t, x) = \sum_{k\in\mathscr{S}(q)}\left(\varphi_k\,.\,\cos k^2t + \frac{\psi_k}{k^2}\,.\,\sin k^2t\right)\sin kx + u^*(t, x)\,,$$

where u^* is defined by (3.3.3) and φ_k, ψ_k $(k \in \mathscr{S}(q))$ are such that $\Sigma k^{10}\varphi_k^2 < +\infty$, $\Sigma k^6\psi_k^2 < +\infty$.

For particular values p and q it holds

Theorem 3.3.2. *Let the problem* $(\mathscr{P}^0_\omega)$ *with* $\omega = 2\pi(2r - 1)/(2s)$, r, s *natural be given. Let the function* $g \in \mathscr{C}(\langle 0, \omega \rangle; \mathscr{H}^3_0)$ *and let it be ω-periodic in t and satisfy the condition*

$$g(t, x) = g(t, \pi - x), \quad x \in \langle 0, \pi \rangle .$$

Then a solution to this problem always exists and if we require $u(t, x) = u(t, \pi - x)$, *it is determined uniquely.*

Proof. Clearly, for $q = 2s$, $q_{(1)} \cdot q_{(2)}$ is even. Hence $\mathscr{S}(q)$ contains only even naturals. Further it is easily verified that a Fourier series belonging to a 2π-periodic function $g(x)$ satisfying $g(x) = g(\pi - x)$ contains only odd harmonics and thus

$$g(t, x) = \sum_{k=1}^{\infty} g_{2k-1}(t) \, . \sin(2k - 1) x .$$

Hence all conditions (3.3.4) are satisfied *a fortiori*. It is easily found that the function $u^*(t, x)$ from (3.3.3), where it suffices to summ up over the odd indexes only, fulfils the relation $u^*(t, x) = u^*(t, \pi - x)$. Finally requiring $u(t, x) = u(t, \pi - x)$, the arbitrary part $v(t, x)$ of the solution must fulfil the same relation; but on the other hand this function being odd in x and $2\pi/q_{(1)} \cdot q_{(2)}$-periodic, while $q_{(1)} \cdot q_{(2)} = 2s^*$, it holds

$$v(t, x) = v(t, \pi - x) = v(t, -\pi - x) = -v(t, x + \pi) ,$$

$$v(t, x) = v(t, x + s^* \, . (2\pi/2s^*)) = v(t, x + \pi) ,$$

which yields

$$(3.3.5) \qquad v(t, x) = \sum_{k \in \mathscr{S}(q)} \left(\varphi_k \, . \cos k^2 t + \frac{\psi_k}{k^2} \, . \sin k^2 t \right) . \sin kx \equiv 0 .$$

Returning to the general case let us rewrite conditions (3.3.4) in a different form. Inserting $u(t + j\omega, x)$ for $j = 0, 1, 2, \ldots, q - 1$ into (3.1.2) and adding we obtain

$$u(q\omega, x) - u(0, x) = 0 , \quad u_t(q\omega, x) - u_t(0, x) = 0 .$$

This shows that a solution to $(\mathscr{P}^0_{2\pi \, p/q})$ only exists if the conditions

$$(3.3.6_1) \qquad \int_0^{\omega q} \int_{(-\infty, +\infty)}^{\rightarrow} \cos \lambda^2 \, . \, g(\vartheta, x - 2(\sqrt{(q\omega - \vartheta)}) \lambda) \, d\lambda \, d\vartheta = 0$$

$$(3.3.6_2) \qquad \int_0^{\omega q} \int_{(-\infty, +\infty)}^{\rightarrow} \sin \lambda^2 \, . \, g(\vartheta, x - 2(\sqrt{(q\omega - \vartheta)}) \lambda) \, d\lambda \, d\vartheta = 0$$

are fulfilled. (We have performed here the same arrangements as in case (A) to come from G to g.) The conditions may be slightly modified to

$$(3.3.7_1) \qquad \sum_{j=1}^{q} \int_0^{\omega} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, g(\vartheta, x - 2(\sqrt{(j\omega - \vartheta)})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\vartheta = 0\,,$$

$$(3.3.7_2) \qquad \sum_{j=1}^{q} \int_0^{\omega} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, g(\vartheta, x - 2(\sqrt{(j\omega - \vartheta)})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\vartheta = 0\,.$$

Inserting into (3.3.6) and (3.3.7) for its Fourier series we easily get that these conditions are equivalent to (3.3.4) and thus they are not only necessary but also sufficient.

To be able to state easier the reading of the next theorem, let us introduce some notations. Let us denote $\mathscr{H}^m_{0,q}(m = 1, 2, \ldots)$ the subspace of $\mathscr{H}^m_0$ which contains functions having the period $2\pi/q_{(1)} \cdot q_{(2)}$, i.e. those functions $h(x) \in \mathscr{H}^m_0$ whose Fourier expansion has the form

$$h(x) = \sum_{k \in \mathscr{S}(q)} h_k \,.\, \sin kx\,.$$

Let us denote $(\mathscr{H}^m_{0,q})^{\perp}$ the orthogonal complement of $\mathscr{H}^m_{0,q}$ in H^m_0, i.e. those functions $h \in \mathscr{H}^m_0$, whose Fourier expansion has the form

$$h(x) = \sum_{k \in \overline{\mathscr{S}}(q)} h_k \,.\, \sin kx\,.$$

Let (3.3.7) be fulfilled and let $\hat{\varphi} \in \mathscr{H}^5_{0,q}$, $\hat{\psi} \in \mathscr{H}^3_{0,q}$. Defining φ, ψ as

$$(3.3.8) \qquad \varphi(x) = \hat{\varphi}(x) + \frac{1}{\sqrt{(2\pi)}} \int_0^{\omega} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 - \sin \lambda^2) \sum_{j=1}^{q-1} \frac{q-j}{q} \,.$$
$$.\, G(\vartheta, x - 2(\sqrt{(j\omega - \vartheta)})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\vartheta$$

$$\psi(x) = \hat{\psi}(x) + \frac{1}{\sqrt{(2\pi)}} \int_0^{\omega} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 + \sin \lambda^2) \sum_{j=1}^{q-1} \frac{q-j}{q} \,.$$
$$.\, g(\vartheta, x - 2(\sqrt{(j\omega - \vartheta)})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\vartheta\,,$$

it may be verified by a straightforward calculation that $u(t, x)$ defined by (2.5), where φ, ψ are given by (3.3.8) is a solution $2\pi\, p/q$-periodic in t. Hence, the following theorem holds:

Theorem 3.3.1′. *Theorem* 3.3.1. *remains valid if the conditions* (3.3.4) *are replaced by* (3.3.6) *or* (3.3.7). *If they are satisfied, the solution of the problem* $(\mathscr{P}^0_{2\pi\, p/q})$ *is defined by* (2.5) *where* φ *and* ψ *are given by* (3.3.8), *while* $\hat{\varphi} \in \mathscr{H}^5_{0,q}$ *and* $\hat{\psi} \in \mathscr{H}^3_{0,q}$.

3.4. The case (C). Clearly the problem $(\mathscr{P}^0_{2\pi\alpha})$ has a solution only if the systems (3.3.1) for $\omega = 2\pi\alpha$ for $k = 1, 2, \ldots$ have convenient solutions. In this case the systems

(3.3.1), (since $D_k = 4k^{-2} . \sin(k^2\pi\alpha) \neq 0$) have always a solution, namely

$$(3.4.1)\quad \varphi_k = \frac{1}{2k^2}\left[\frac{\cos(k^2\omega/2)}{\sin(k^2\omega/2)}\int_0^\omega \cos k^2\tau . g_k(\tau)\,d\tau + \int_0^\omega \sin k^2\tau . g_k(\tau)\,d\tau\right]$$

$$\psi_k = \frac{1}{2}\left[\frac{\cos(k^2\omega/2)}{\sin(k^2\omega/2)}\int_0^\omega \sin k^2\tau . g_k(\tau)\,d\tau - \int_0^\omega \cos k^2\tau . g_k(\tau)\,d\tau\right].$$

But these solutions are convenient only if the expression (2.4) with φ_k, ψ_k determined by (3.4.1) belongs to $\mathscr{U}$. This can be proved only under some special assumptions on the number-theoretical character of α and on the smoothness of function g.

Theorem 3.4.1. *Let the problem* $(\mathscr{P}^0_{2\pi\alpha})$ *be given. Let numbers* $c > 0$, $\varrho \geqq 3$ *(natural) exist such that*

$$(3.4.2)\qquad \left|\alpha - \frac{l}{k^2}\right| \geqq \frac{c}{k^\varrho}$$

for l, k *natural. Further let* $g \in \mathscr{C}(\langle 0, 2\pi\alpha\rangle; \mathscr{H}_0^{\varrho+1})$ *and let it be* $2\pi\alpha$*-periodic in* t.

Then there exists a unique solution $u \in \mathscr{U}$ *to the problem* $(\mathscr{P}^0_{2\pi\alpha})$, *defined by* (2.4) *with* φ_k, ψ_k, $k = 1, 2, 3, \ldots$, *determined by* (3.4.1).

Proof. The third term in (2.4) belongs surely to $\mathscr{U}$. Hence it suffices to show the same for the first two terms. The continuity of $\varphi_k . \cos k^2t$, $\psi_k . \sin k^2t$ and of their first and second derivatives is clear. The uniform convergence of the corresponding series will be guaranteed if we show that

$$\sum_{k=1}^\infty k^{10} . \varphi_k^2 < +\infty\,, \quad \sum_{k=1}^\infty k^6 . \psi_k^2 < +\infty\,.$$

We have

$$\left|\sin\frac{k^2\omega}{2}\right|^{-1} = \left(\left|\frac{k^2\omega}{2} - l\pi\right| : \left|\sin\left(\frac{k^2\omega}{2} - l\pi\right)\right|\right).$$

$$. \left|\frac{k^2\omega}{2} - l\pi\right|^{-1} \leqq \frac{\pi}{2}\frac{1}{\pi k^2}\left|\alpha - \frac{l}{k^2}\right|^{-1} \leqq \frac{1}{2c}k^{\varrho-2}\,, \quad \text{where } \left|\frac{k^2\omega}{2} - l\pi\right| \leqq \frac{\pi}{2}$$

so that

$$|\varphi_k| \leqq c_1(k^{\varrho-4} + k^{-2})\int_0^\omega |g_k(\tau)|\,d\tau$$

and therefore

$$\sum_{k=1}^\infty k^{10} . \varphi_k^2 \leqq c_2 . \sum_{k=1}^\infty (k^{2(\varrho+1)} + k^6)\int_0^\omega |g_k(\tau)|^2\,d\tau \leqq$$

$$\leqq c_3 . \int_0^\omega \left(\sum_{k=1}^\infty k^{2(\varrho+1)} . |g_k(\tau)|^2\right) d\tau < +\infty\,,$$

and similarly for $\Sigma k^6\psi_k^2$, which completes the proof.

Remark 3.4.1. Especially in the nonlinear case it is interesting to know how many α fulfil the inequality (3.4.2) for different values ϱ. Let us recall the following facts. Given a natural number $k > 1$ the set of numbers $\alpha \in \langle 0, 1 \rangle$ fulfilling the inequality

$$\left|\alpha - \frac{l}{k^2}\right| < \frac{1}{k^3 \lg^2 k} \tag{3.4.3}$$

for the appropriate natural l, has the measure $2/k \,.\, \lg^2 k$ at most. Since the series $\Sigma(1/k \,.\, \lg^2 k)$ is convergent the set of all α fulfilling (3.4.3) for infinitely many natural l, k has the measure 0.

(On the other hand there exist irrational numbers α such that e.g. the inequality $|\alpha - l \,.\, k^{-2}| < 2^{-k}$ has infinitely many solutions.) From this follows that there exists a set of irrational numbers from $(0, 1)$ having the measure 1, such that to each its element α there exists a constant c such that (3.4.2) is satisfied with $\varrho = 4$ (cf. Theorem 6.4.1.).

§4. ANOTHER FORM OF CONDITIONS FOR THE EXISTENCE OF PERIODIC SOLUTIONS

4.1. General situation. Let the problem $(\mathscr{P}_\omega^0)$ be given by the equations (assuming $g \in \mathscr{C}(\langle 0, \omega \rangle; \mathscr{H}_0^3))$

$$L(u) \equiv u_{tt} + u_{xxxx} = g(t, x) \tag{4.1.1}$$

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0 \tag{4.1.2}$$

$$u(\omega, x) - u(0, x) = 0 \tag{4.1.3$_1$}$$

$$u_t(\omega, x) - u_t(0, x) = 0\,. \tag{4.1.3$_2$}$$

Using standard methods we find that the problem $(\mathscr{P}_\omega^{0*})$ formally adjoint to the homogeneous problem $(\mathscr{P}_\omega^0)$ is given by

$$v_{tt} + v_{xxxx} = 0 \tag{4.1.4}$$

$$v(t, 0) = v(t, \pi) = v_{xx}(t, 0) = v_{xx}(t, \pi) = 0 \tag{4.1.5}$$

$$v(\omega, x) - v(0, x) = 0 \tag{4.1.6$_1$}$$

$$v_t(\omega, x) - v_t(0, x) = 0\,. \tag{4.1.6$_2$}$$

(so that the given problem is selfadjoint). Hence for u and v belonging to $\mathscr{U}$ and satis fying (4.1.2), (4.1.3) it holds

$$\int_0^\omega \int_0^\pi (uLv - vLu)\,\mathrm{d}x\,\mathrm{d}t = 0\,, \tag{4.1.7}$$

consequently if $u \in \mathscr{U}$ is a solution of $(\mathscr{P}^0_\omega)$ and $v \in \mathscr{U}$ is a solution of $(\mathscr{P}^{0*}_\omega)$, then

$$\int_0^\omega \int_0^\pi v(t, x) \,.\, g(t, x)\, \mathrm{d}x\, \mathrm{d}t = 0\,. \tag{4.1.8}$$

From here the following theorem follows immediately.

Theorem 4.1.1. *The problem* $(\mathscr{P}^0_\omega)$ *has a solution* $u \in \mathscr{U}$ *only if the function* g *satisfies the relation* (4.1.8) *for every solution* $v \in \mathscr{U}$ *of* $(\mathscr{P}^{0*}_\omega)$.

Hereafter we investigate separately the different cases of ω as in paragraph 3.

4.2. The case (A). Let us show the sufficiency of the condition (4.1.8). By Theorem 3.2.1 every solution v of $(\mathscr{P}^{0*}_{2\pi n})$ may be written as

$$v(t, x) = \frac{1}{\sqrt{(2\pi)}}\left(\int^{\rightarrow}_{(-\infty,+\infty)} (\cos \lambda^2 + \sin \lambda^2)\, \varphi(x - 2(\sqrt{t})\, \lambda)\, \mathrm{d}\lambda - \right.$$
$$\left. - \int^{\rightarrow}_{(-\infty,+\infty)} (\cos \lambda^2 - \sin \lambda^2)\, \Psi(x - 2(\sqrt{t})\, \lambda)\, \mathrm{d}\lambda\right) \tag{4.2.1}$$

where φ, Ψ belong to $\mathscr{H}^5_0$. Inserting (4.2.1) with $\Psi = 0$ into (4.1.8) and performing some arrangements (whose justification can be shown by standard methods) we obtain (putting $S(\lambda) = \cos \lambda^2 + \sin \lambda^2$)

$$0 = \int_0^\omega \int_0^\pi g(\vartheta, \xi) \left\{\int^{\rightarrow}_{(-\infty,+\infty)} S(\lambda) \,.\, \varphi(\xi - 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\right\} \mathrm{d}\xi\, \mathrm{d}\vartheta =$$
$$= \sum_{n=-\infty}^{+\infty} \left(\int_{2n\pi}^{(2n+1)\pi} \varphi(y) \,.\, \int_0^\omega \int_{-y/2\sqrt{\vartheta}}^{-(y-\pi)/2\sqrt{\vartheta}} S(\lambda) \,.\, g(\vartheta, y + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\vartheta\, \mathrm{d}y + \right.$$
$$\left. + \int_{(2n-1)\pi}^{2n\pi} \varphi(y) \int_0^\omega \int_{-y/2\sqrt{\vartheta}}^{-(y-\pi)/2\sqrt{\vartheta}} S(\lambda) \,.\, g(\vartheta, y + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\vartheta\, \mathrm{d}y\right) =$$
$$= \int_0^\omega \left(\sum_{n=-\infty}^{+\infty} \int_0^\pi \varphi(x) \,.\, \int_{-(x+2n\pi)/2\sqrt{\vartheta}}^{-(x+(2n-1)\pi)/2\sqrt{\vartheta}} S(\lambda) \,.\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}x\right) \mathrm{d}\vartheta +$$
$$+ \int_0^\omega \left(\sum_{n=-\infty}^{+\infty} \int_0^\pi \varphi(x) \,.\, \int_{-(x-(2n-1)\pi)/2\sqrt{\vartheta}}^{-(x-2n\pi)/2\sqrt{\vartheta}} S(\lambda) \,.\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}x\right) \mathrm{d}\vartheta =$$
$$= \int_0^\omega \left(\int_0^\pi \varphi(x) \,.\, \int^{\rightarrow}_{(-\infty,(-x+\pi)/2\sqrt{\vartheta})} S(\lambda) \,.\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}x + \right.$$
$$\left. + \int_0^\pi \varphi(x) \,.\, \int^{\rightarrow}_{((-x+\pi)/2\sqrt{\vartheta},+\infty)} S(\lambda) \,.\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}x\right) \mathrm{d}\vartheta =$$
$$= \int_0^\omega \int_0^\pi \varphi(x) \,.\, \int^{\rightarrow}_{(-\infty,+\infty)} S(\lambda) \,.\, g(\lambda, x + 2(\sqrt{\vartheta})\, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}x\, \mathrm{d}\vartheta$$

which yields

$$0 = \int_0^\pi \varphi(x) \cdot \left(\int_0^\omega \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 + \sin \lambda^2)\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, d\lambda\, d\vartheta \right) dx$$

and in similar manner (putting $\varphi = 0$) we get

$$0 = \int_0^\pi \Psi(x) \cdot \left(\int_0^\omega \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 - \sin \lambda^2)\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, d\lambda\, d\vartheta \right) dx\,.$$

As the last two relations are fulfilled for all $\varphi, \Psi \in \mathscr{H}_0^5$ and the space $\mathscr{H}_0^5$ is dense in $\mathscr{L}_2$, we get that the cofactors of φ and Ψ must be equal to 0 almost everywhere, but it means according to their continuity for all x, i.e.

$$\int_0^\omega \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 + \sin \lambda^2)\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, d\lambda\, d\vartheta = 0$$

$$\int_0^\omega \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 - \sin \lambda^2)\, g(\vartheta, x + 2(\sqrt{\vartheta})\, \lambda)\, d\lambda\, d\vartheta = 0$$

which by Theorem 3.2.1′ represents sufficient conditions for the existence of a solution to $(\mathscr{P}^0_{2\pi n})$. This together with Theorem 4.1.1 yields

Theorem 4.2.1. *Let the problem $(\mathscr{P}^0_{2\pi n})$ be given. Let $g \in \mathscr{C}(\langle 0, 2\pi n\rangle; \mathscr{H}_0^3)$ and be $2\pi n$-periodic in t.*

Then the problem $(\mathscr{P}^0_{2\pi n})$ has a solution if and only if g satisfies the condition (4.1.8) *for every solution v to the problem $(\mathscr{P}^{0*}_{2\pi n})$.*

Remark. To show various approaches we made use of the form (2.5) for the solution to $(\mathscr{P}^{0*}_{2\pi n})$, but we would have come more quickly to the same result having applied the formula (2.4) (as we shall do in the next section).

4.3. The case (B). Let us show the sufficiency of the condition (4.1.8). Hence let us suppose g fulfils (4.1.8) with $v = \cos k^2 t \,.\, \sin kx$ or $\sin k^2 t \,.\, \sin kx$, for $k = 1, 2, \ldots$, i.e.

$$0 = \int_0^\omega \int_0^\pi \cos k^2\tau \,.\, \sin kx \,.\, g(\tau, x)\, dx\, d\tau = \frac{\pi}{2} \int_0^\omega g_k(\tau) \,.\, \cos k^2\tau\, d\tau$$

$$0 = \int_0^\omega \int_0^\pi \sin k^2\tau \,.\, \sin kx \,.\, g(\tau, x)\, dx\, d\tau = \frac{\pi}{2} \int_0^\omega g_k(\tau) \,.\, \sin k^2\tau\, d\tau\,.$$

Thus by Theorem 3.3.1 and Theorem 4.1.1 we have

Theorem 4.3.1. *Let the problem $(\mathscr{P}^0_{2\pi\, p/q})$ be given. Let $g \in \mathscr{C}(\langle 0, 2\pi\, p/q\rangle; \mathscr{H}_0^3)$ and let it be $2\pi\, p/q$-periodic in t. Then the problem $(\mathscr{P}^0_{2\pi\, p/q})$ has a solution if and only if g satisfies the condition* (4.1.8) *for every solution v to the problem $(\mathscr{P}^{0*}_{2\pi\, p/q})$.*

4.4. The case (C). Assume α satisfies (3.4.2) for some $\varrho \geqq 3$. Then by Theorem 3.4.1 the unique solution to $(\mathscr{P}^{0*}_{2\pi\alpha})$ is 0 and hence, in this case, the condition (4.1.8) does not impose any further limitations on g. But it may be shown (cf. a paper prepared by B. NOVÁK) that some conditions on the smoothness of g (in connection with the number-theoretical character of α) are not only sufficient but also neccesary. This means that in case $(\mathscr{P}^{0}_{2\pi\alpha})$ the formal adjoint problem is not a useful tool for its study.

§5. THE WEAKLY NONLINEAR MIXED PROBLEM $(\mathscr{M})$

5.1. Preliminaries and some auxiliary lemmas. We introduce here two lemmas from [5] (where their proofs are sketched on pp. 355, 356) which we need in the sequel.

Lemma 5.1.1. *Let the equation*

$$P(u, r)(\varepsilon) \equiv -u + L(r) + \varepsilon R(u)(\varepsilon) = 0 \tag{5.1.1}$$

be given, where $P(u, r)(\varepsilon)$ maps the direct product $\mathscr{U} \times \mathscr{R}$ into $\mathscr{U}$ ($\mathscr{U}, \mathscr{R}$ being B-spaces) for every value of the numerical parameter ε from $\mathscr{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$.

Let $L \in [\mathscr{R} \to \mathscr{U}]$. Let $R(u)(\varepsilon)$ be continuous in u and ε and have a G-derivative $R'_u(u)(\varepsilon)$ continuous in u and ε for any $u \in \mathscr{U}$ and $\varepsilon \in \mathscr{E}$.

Then to every $\tilde{r} \in \mathscr{R}$ there exist numbers δ and ε^*, $\delta > 0$, $0 < \varepsilon^* \leqq \varepsilon_0$ such that the equation (5.1.1) has a unique solution $U(r)(\varepsilon) \in \mathscr{U}$ for each $r \in B(\tilde{r}; \delta)$ and $\varepsilon \in \langle 0, \varepsilon^* \rangle$. This solution has a G-derivative $U'_r(r)(\varepsilon)$ continuous in r and ε.

Here and everywhere below $B(c; \delta)$ denotes the ball with the centre c and the radius δ and $[\mathscr{B}_1 \to \mathscr{B}_2]$ denotes the space of all continuous linear mappings from $\mathscr{B}_1$ into $\mathscr{B}_2$.

Lemma 5.1.2. *Let the equation*

$$G(r)(\varepsilon) = 0 \tag{5.1.2}$$

be given, where $G(r)(\varepsilon)$ maps a B-space $\mathscr{R}_1$ into B-space $\mathscr{R}_2$ for all $\varepsilon \in \mathscr{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$. Let the following assumptions be fulfilled:

(i) *The equation*

$$G(r_0)(0) = 0 \tag{5.1.3}$$

has a solution $r_0 = r_0^* \in \mathscr{R}_1$.

(ii) *The operator $G(r)(\varepsilon)$ is continuous in r and ε and has a G-derivative $G'_r(r)(\varepsilon)$ continuous in r and ε for $r \in B(r_0^*; \delta)$ ($\delta > 0$ being a suitably chosen number) and $\varepsilon \in \mathscr{E}$.*

(iii) *There exists*

$$H = [G'_r(r_0^*)(0)]^{-1} \in [\mathscr{R}_2 \to \mathscr{R}_1] .$$

Then there exists $\varepsilon^* > 0$ *such that the equation* (5.1.2) *has for* $0 \leqq \varepsilon \leqq \varepsilon^*$ *a unique solution* $r = r^*(\varepsilon) \in \mathscr{R}_1$, *continuous in* ε *such that* $r^*(0) = r_0^*$.

For reader's convenience we shall state here together diverse assumptions on f utilized in the last two paragraphs. For the sake of simplicity we write sometimes u_0, u_1, u_2, u_3 instead of u, u_x, u_{xx}, u_t, respectively.

($\mathscr{A}_1$) In the set $\mathscr{Q}_1 = \langle 0, T\rangle \times \langle 0, \pi\rangle \times (-\infty, +\infty)^4 \times \langle 0, \varepsilon_0\rangle$ the function $f(t, x, u_0, u_1, u_2, u_3, \varepsilon)$ has continuous partial derivatives of the form

$$\frac{\partial^{i_1+i_2+i_3+i_4+i_5}}{\partial x^{i_1}\, \partial u_0^{i_2}\, \partial u_1^{i_3}\, \partial u_2^{i_4}\, \partial u_3^{i_5}} \tag{5.1.4}$$

for $0 \leqq i_1 \leqq 3$, $0 \leqq i_k \leqq 4$ $(k = 2, 3, 4, 5)$, $\sum_{k=1}^{5} i_k \leqq 4$ and

$$f, \quad \frac{\partial f}{\partial u_1}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial u_1^2}, \quad \frac{\partial^2 f}{\partial x\, \partial u_j}, \quad \frac{\partial^2 f}{\partial u_j\, \partial u_k} \tag{5.1.5}$$

equal 0 for $j, k = 0, 2, 3$, $t \in \langle 0, T\rangle$, $x = 0, \pi$, $u_0 = u_2 = u_3 = 0$, $u_1 \in (-\infty, +\infty)$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$.

($\mathscr{A}_2$) In the set $\mathscr{Q}_2 = \langle 0, T\rangle \times \langle 0, \pi\rangle \times (-\infty, +\infty)^2 \times \langle 0, \varepsilon_0\rangle$ the function $f(t, x, u_0, u_1, \varepsilon)$ has continuous partial derivatives of the form

$$\frac{\partial^{i_1+i_2+i_3}}{\partial x^{i_1}\, \partial u_0^{i_2}\, \partial u_1^{i_3}}$$

for $0 \leqq i_1 \leqq 4$, $0 \leqq i_k \leqq 5$, $(k = 2, 3)$, $\sum_{k=1}^{3} i_k \leqq 5$ and

$$f, \quad \frac{\partial f}{\partial u_1}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial u_1^2}, \quad \frac{\partial^2 f}{\partial x\, \partial u_0}, \quad \frac{\partial^2 f}{\partial u_0^2}$$

equal 0 for $t \in \langle 0, T\rangle$, $x = 0, \pi$, $u_0 = 0$, $u_1 \in (-\infty, +\infty)$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$.

($\mathscr{A}_3$) In the set $\mathscr{Q}_3 = \langle 0, T\rangle \times \langle 0, \pi\rangle \times (-\infty, +\infty) \times \langle 0, \varepsilon_0\rangle$ the function $f(t, x, u_0, \varepsilon)$ has continuous partial derivatives of the form

$$\frac{\partial^{j+i_0}}{\partial x^j\, \partial u_0^{i_0}}$$

for $0 \leqq j \leqq 5$, $0 \leqq i_0 \leqq 6$, $j + i_0 \leqq 6$ and

$$f, \quad \frac{\partial^{j+i_0} f}{\partial x^j\, \partial u_0^{i_0}}$$

equal 0 for $t \in \langle 0, T\rangle$, $x \in 0, \pi$, $u_0 = 0$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$ and $j + i_0 = 2, 4$, $j \geqq 0$, $i_0 \geqq 0$

($\mathscr{B}$) The function φ belongs to $\mathscr{H}_0^5$ and the function ψ belongs to $\mathscr{H}_0^3$.

In the sequel we shal often understand under f the function extended by the relations

$$(5.1.6) \qquad f(t, x, u_0, u_1, u_2, u_3, \varepsilon) = -f(t, -x, -u_0, u_1, -u_2, -u_3, \varepsilon) = \\ = f(t, x + 2\pi, u_0, u_1, u_2, u_3, \varepsilon)$$

from $\mathscr{Q}_1$ onto $\mathscr{Q}_1'$ which contains all points from $\langle 0, T\rangle \times (-\infty, +\infty)^5 \times \langle 0, \varepsilon_0\rangle$ except for points with $x = j\pi$ $(j = 0, \pm 1, \pm 2, \ldots) \wedge |u_0| + |u_2| + |u_3| > 0$. The extended function is continuous in $\mathscr{Q}_1'$ together with its derivatives mentioned in (5.1.4) and all expressions from (5.1.5) equal 0 for $t \in \langle 0, T\rangle$, $x = j\pi$ $(j = 0, \pm 1, \pm 2, \ldots)$, $u_0 = u_2 = u_3 = 0$, $u_1 \in (-\infty, +\infty)$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$. Similarly for function satisfying the assumption ($\mathscr{A}_2$) or ($\mathscr{A}_3$). Below, the following notations will be used

$$(5.1.7) \quad f(u)(\varepsilon)(t, x) = f(t, x, u(t, x), u_x(t, x), u_{xx}(t, x), u_t(t, x), \varepsilon)$$

$$(5.1.8) \quad F(u)(\varepsilon)(t, x) = \int_0^x \int_0^\xi f(u)(\varepsilon)(t, \eta)\, d\eta\, d\xi - \frac{x}{2\pi} \int_0^{2\pi} \int_0^\xi f(u)(\varepsilon)(t, \eta)\, d\eta\, d\xi\,.$$

The following lemma may be easily verified:

Lemma 5.1.3. *The operator $R(u)(\varepsilon)$ defined by*

$$(5.1.9) \qquad R(u)(\varepsilon)(t, x) = \\ = \frac{1}{\sqrt{(2\pi)}} \int_0^t \int_{(-\infty, +\infty)}^{\rightarrow} (\cos \lambda^2 - \sin \lambda^2)\, F(u)(\varepsilon)(\vartheta, x - 2(\sqrt{(t - \vartheta)})\, \lambda)\, d\lambda\, d\vartheta$$

from $\mathscr{U}$ into $\mathscr{U}$ is continuous in u and ε and has the G-derivative $R_u'(u)(\varepsilon)$ continuous in u and ε as well.

5.2. The problem ($\mathscr{M}$). Let the problem ($\mathscr{M}$) be given by

$$(5.2.1) \qquad u_{tt} + u_{xxxx} = \varepsilon\,.\, f(t, x, u, u_x, u_{xx}, u_t, \varepsilon)\,,$$

$$(5.2.2) \qquad u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0\,,$$

$$(5.2.3) \qquad u(0, x) = \varphi(x)\,, \quad u_t(0, x) = \psi(x)$$

for $x \in \langle 0, \pi\rangle$, $t \in \langle 0, T\rangle$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$, with f satisfying ($\mathscr{A}_1$).

We find easily by paragraph 2 that every solution $u \in \mathscr{U}$ to ($\mathscr{M}$) must satisfy

$$(5.2.4) \qquad -u(t, x) + L(\varphi, \psi)(t, x) + \varepsilon\, R(u)(\varepsilon)(t, x) = 0\,,$$

where

$$L(\varphi, \psi)(t, x) = \frac{1}{\sqrt{(2\pi)}} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 + \sin \lambda^2)\, \varphi(x - 2(\sqrt{t})\,\lambda)\, d\lambda -$$

$$- \frac{1}{\sqrt{(2\pi)}} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 - \sin \lambda^2)\, \Psi(x - 2(\sqrt{t})\,\lambda)\, d\lambda .$$

Ψ is defined by (2.6) and $R(u)(\varepsilon)$ is defined by (5.1.9). On the other hand, every solution $u \in \mathscr{U}$ of (5.2.4) is a solution to $(\mathscr{M})$. Using lemmas (5.1.1) and (5.1.3) we obtain immediately

Theorem 5.2.1. *Let the problem $(\mathscr{M})$ be given. Let f satisfy the assumption $(\mathscr{A}_1)$. Then to every couple $\tilde{r} = (\tilde{\varphi}, \tilde{\psi}) \in \mathscr{H}_0^5 \times \mathscr{H}_0^3$ there exist numbers δ and ε^*, $\delta > 0$, $0 < \varepsilon^* \leqq \varepsilon_0$ such that the problem $(\mathscr{M})$ has a unique solution $U(\varphi, \psi)(\varepsilon) \in \mathscr{U}$ for each $r = (\varphi, \psi) \in B(\tilde{r}; \delta)$ and $\varepsilon \in \langle 0, \varepsilon^* \rangle$. This solution is continuous in r and ε and has a G-derivative $U_r'(r)(\varepsilon)$ continuous in r and ε.*

Remark. In the formulation of the problem $(\mathscr{M})$ we have now omitted the function g on the right hand side in (5.2.1), since putting $u = u_0 + v$, where u_0 is the solution to the problem $(\mathscr{M}^0)$ we get for v the problem $(\mathscr{M})$ of the form introduced above.

§6. PERIODIC SOLUTIONS IN THE WEAKLY NONLINEAR CASE

6.1. General situation. Let $\omega > 0$ be given. Let the problem $(\mathscr{P}_\omega)$ be given by

(6.1.1) $$u_{tt} + u_{xxxx} = \varepsilon f(t, x, u, u_x, u_{xx}, u_t, \varepsilon)$$

(6.1.2) $$u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0$$

(6.1.3) $$u(0, x) - u(\omega, x) = u_t(0, x) - u_t(\omega, x) = 0$$

In all the paragraph we suppose that the function f satisfies minimally the assumption $(\mathscr{A}_1)$ and that it is ω-periodic in t, i.e.

(6.1.4) $$f(t + \omega, x, u_0, u_1, u_2, u_3, \varepsilon) = f(t, x, u_0, u_1, u_2, u_3, \varepsilon)$$

for $(t, x, u_0, u_1, u_2, u_3, \varepsilon) \in \mathscr{Q}_1$.

Remark 6.1.1. The more general problem with $g(t, x) + \varepsilon f(t, x, u_0, u_1, u_2, u_3, \varepsilon)$ with g sufficiently smooth and ω-periodic in t may be reduced easily to the precceding one. If the limit case (for $\varepsilon = 0$) does not admit any ω-periodic solution, then evidently neither the given problem admits any. In the opposite case if the limit problem has a solution $u_0(t, x)$, performing the substitution $u = u_0 + v$ we get for v the problem $(\mathscr{P}_\omega)$.

By Theorem 5.2.1 to every couple $(\tilde{\varphi}, \tilde{\psi}) \in \mathscr{H}_0^5 \times \mathscr{H}_0^3$ there exist numbers δ and ε_1, $\delta > 0$, $0 < \varepsilon_1 \leqq \varepsilon_0$ such that the associated problem $(\mathscr{M})$ has a unique solution $U(\varphi, \psi)(\varepsilon) \in \mathscr{U}$ for each $(\varphi, \psi) \in B((\tilde{\varphi}, \tilde{\psi}); \delta)$ and $0 \leqq \varepsilon \leqq \varepsilon_1$, having the property stated in Theorem 5.2.1 and satisfying the identity

$$(6.1.5) \qquad U(\varphi, \psi)(\varepsilon)(t, x) = L(\varphi, \psi)(t, x) + \varepsilon R(U(\varphi, \psi)(\varepsilon))(\varepsilon)(t, x) .$$

Now inserting $U(\varphi, \psi)(\varepsilon)$, expressed by the right-hand side in (6.1.5), into (6.1.3) we get neccessary and sufficient conditions which φ and ψ must satisfy that a solution to $(\mathscr{P}_\omega)$ may exist. Analogously as in paragraph 3 we shall distinguish three different cases according to the character of the number ω.

6.2. The case (A). Performing the same considerations and arrangements as in section 3.2 we find that $(\mathscr{P}_{2\pi n})$ has a solution $u \in \mathscr{U}$ if and only if the equations (the so called determining or bifurcation equations)

$$(6.2.1) \qquad G_1(\varphi, \psi)(\varepsilon)(x) \equiv$$

$$\equiv \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, f(U(\varphi, \psi))(\varepsilon))(\varepsilon)(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$G_2(\varphi, \psi)(\varepsilon)(x) \equiv$$

$$\equiv \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

have a solution $(\varphi, \psi) \in \mathscr{H}_0^5 \times \mathscr{H}_0^3$. To get from this some conditions applicable to concrete problems we can make use, for instance, of Lemma 5.1.2. Putting $\varepsilon = 0$ in (6.2.1) we obtain immediately the necessary conditions for $r_0 = (\varphi_0, \psi_0)$:

$$(6.2.2) \qquad G_1(r_0)(0)(x) \equiv$$

$$\equiv \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, f(L(\varphi_0, \psi_0))(0)(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0\,,$$

$$G_2(r_0)(0)(x) \equiv$$

$$\equiv \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, f(L(\varphi_0, \psi_0))(0)(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0\,.$$

Assuming that (6.2.2) have a convenient solution $(\varphi_0^*, \psi_0^*) = r_0^*$ let us look for some sufficient conditions by investigating the existence of an inverse operator $H = [G_r'(r_0^*)(0)]^{-1}$ $(G = (G_1, G_2))$. Since the smootheness properties of the space into which $G(r)(\varepsilon)$ maps $\mathscr{H}_0^5 \times \mathscr{H}_0^3$ depend on the type of the function f and since

(6.2.3) $$G'_{1r}(r)(\varepsilon)(\bar{r})(x) =$$

$$= \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\to} \cos \lambda^2 . f'_u(U(r)(\varepsilon))(\varepsilon)(U'_r(r)(\varepsilon)(\bar{r}))(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\lambda) \, d\lambda \, d\vartheta$$

$$G'_{2r}(r)(\varepsilon)(\bar{r})(x) =$$

$$= \int_0^{2\pi n} \int_{(-\infty,+\infty)}^{\to} \sin \lambda^2 . f'_u(U(r)(\varepsilon))(\varepsilon)(U'_r(r)(\varepsilon)(\bar{r}))(\vartheta, x - 2(\sqrt{(2\pi n - \vartheta)})\lambda) \, d\lambda \, d\vartheta$$

where

$$f'_u(u)(\varepsilon)(\bar{u})(t, x) = \sum_{i=0}^{3} \frac{\partial f}{\partial u_i}(t, x, u_0(t, x), u_1(t, x), u_2(t, x), u_3(t, x), \varepsilon) . \bar{u}_i(t, x) .$$

we have to distinguish several cases according to the type of the function f.

Below, $\tilde{\mathscr{S}}$ or $\hat{\mathscr{S}}$ denote always a subspace of the corresponding space $\mathscr{S}$.

Theorem 6.2.1. *Let the problem $(\mathscr{P}_{2\pi n})$ be given. Let the function f be $2\pi n$-periodic in t and let it satisfy $(\mathscr{A}_1)$ with $T = 2\pi n$ and $|\partial f/\partial u_2| + |\partial f/\partial u_3| \not\equiv 0$.*

Then the problem $(\mathscr{P}_{2\pi n})$ has a solution $u \in \mathscr{U}$ only if

(i) *the equations* (6.2.2) *have a solution $r_0^* = (\varphi_0^*, \psi_0^*) \in \tilde{\mathscr{H}}_0^5 \times \tilde{\mathscr{H}}_0^3$.*

If, moreover,

(ii) *there exists*

$$H = [G'_r(r_0^*)(0)]^{-1} \in [\hat{\mathscr{H}}_0^3 \times \hat{\mathscr{H}}_0^3 \to \tilde{\mathscr{H}}_0^5 \times \tilde{\mathscr{H}}_0^3] ,$$

where $G(\tilde{\mathscr{H}}_0^5 \times \tilde{\mathscr{H}}_0^3) \subset \hat{\mathscr{H}}_0^3 \times \hat{\mathscr{H}}_0^3$, then there exists $\varepsilon^* \in (0, \varepsilon_1\rangle$ such that for all $\varepsilon \in (0, \varepsilon^*\rangle$ there exists a unique solution $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$, with $\varphi^*(\varepsilon)$, $\psi^*(\varepsilon)$ continuous in ε and $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$.

Proof. Let us verify that all assumptions of Lemma 5.1.2 are satisfied. Putting $\mathscr{R}_1 = \tilde{\mathscr{H}}_0^5 \times \tilde{\mathscr{H}}_0^3$, $\mathscr{R}_2 = \hat{\mathscr{H}}_0^3 \times \hat{\mathscr{H}}_0^3$ we see that the assumptions (i), (ii) of our theorem ensure that the assumptions (i), (iii) of Lemma 5.1.2 are fulfilled. The assumption (ii) is an immediate consequence of the assumption $(\mathscr{A}_1)$ and of Theorem 5.2.1. Hence the existence of $(\varphi^*(\varepsilon), \psi^*(\varepsilon)) \in \tilde{\mathscr{H}}_0^5 \times \tilde{\mathscr{H}}_0^3$ continuous in ε for $\varepsilon \in \langle 0, \varepsilon_2 \rangle$, $0 < \varepsilon_2 \leqq \varepsilon_1$, is guaranteed. Reducing ε, if necessary, into the interval $\langle 0, \varepsilon^* \rangle$, $0 < \varepsilon^* \leqq \varepsilon_2$ to have $(\varphi^*(\varepsilon), \psi^*(\varepsilon)) \in B(r_0^*; \delta)$, δ defined in Theorem 5.2.1, we have the assertion of the theorem.

Proofs of the following theorems are quite analogous.

Theorem 6.2.2. *Let the problem $(\mathscr{P}_{2\pi n})$ be given. Let the function f be $2\pi n$-periodic in t and let it satisfy $(\mathscr{A}_2)$ with $T = 2\pi n$ and $|\partial f/\partial u_1| \not\equiv 0$.*

Then the problem $(\mathscr{P}_{2\pi n})$ *has a solution* $u \in \mathscr{U}$ *only if*

(i) *the equations* (6.2.2) *have a solution* $r_0^* = (\varphi_0^*, \psi_0^*) \in \mathscr{H}_0^5 \times \mathscr{H}_0^3$.

If, moreover,

(ii) *there exists* $H = [G_r'(r_0^*)(0)]^{-1} \in [\mathscr{H}_0^4 \times \mathscr{H}_0^4 \to \mathscr{H}_0^5 \times \mathscr{H}_0^3]$, *where* $G(\mathscr{H}_0^5 \times \times \mathscr{H}_0^3) \subset \mathscr{H}_0^4 \times \mathscr{H}_0^4$, *then there exists* $\varepsilon^* \in (0, \varepsilon_1\rangle$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$, *with* $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ *continuous in* ε *and* $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$.

Theorem 6.2.3. *Let the problem* $(\mathscr{P}_{2\pi n})$ *be given. Let the function* f *be* $2\pi n$*-periodic in* t *and let it satisfy* $(\mathscr{A}_3)$ *with* $T = 2\pi n$ *and* $|\partial f/\partial u_0| \not\equiv 0$.

Then the problem $(\mathscr{P}_{2\pi n})$ *has a solution* $u \in \mathscr{U}$ *only if*

(i) *the equations* (6.2.2) *have a solution* $r_0^* = (\varphi_0^*, \psi_0^*) \in \mathscr{H}_0^5 \times \mathscr{H}_0^3$.

If, moreover,

(ii) *there exists* $H = [G_r'(r_0^*)(0)]^{-1} \in [\mathscr{H}_0^5 \times \mathscr{H}_0^5 \to \mathscr{H}_0^5 \times \mathscr{H}_0^3]$, *where* $G(\mathscr{H}_0^5 \times \times \mathscr{H}_0^3) \subset \mathscr{H}_0^5 \times \mathscr{H}_0^5$, *then there exists* $\varepsilon^* \in (0, \varepsilon_1\rangle$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$ *with* $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ *continuous in* ε *and* $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$.

The fact that the spaces $\mathscr{H}_0^m$ are isometric and isomorfic with the spaces $\mathfrak{h}^m$ may be used to another formulation of necessary or sufficient conditions for the existence of a solution to $(\mathscr{P}_{2\pi n})$. We show it e.g. under assumptions of Theorem 6.2.1.

Theorem 6.2.1′. *Let the* $(\mathscr{P}_{2\pi n})$ *be given. Let the function* f *be* $2\pi n$*-periodic in* t *and let it satisfy* $(\mathscr{A}_1)$ *with* $T = 2\pi n$ *and* $|\partial f/\partial u_2| + |\partial f/\partial u_3| \not\equiv 0$.

Then the problem $(\mathscr{P}_{2\pi n})$ *has a solution* $u \in \mathscr{U}$ *only if*

(i) *The equations*

$$\bar{G}_1(\{\varphi_j\}, \{\psi_j\})(\varepsilon) \equiv \left\{\int_0^{2\pi n} \int_0^{2\pi} f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(t, x) \,.\, \cos k^2 t \,.\, \sin kx \, dx \, dt\right\}_{k=1}^{\infty} = 0$$

$$\bar{G}_2(\{\varphi_j\}, \{\psi_j\})(\varepsilon) \equiv \left\{\int_0^{2\pi n} \int_0^{2\pi} f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(t, x) \,.\, \sin k^2 t \,.\, \sin kx \, dx \, dt\right\}_{k=1}^{\infty} = 0$$

have for $\varepsilon = 0$ *the solution* $r_0^* = (\{\varphi_{0,j}^*\}, \{\psi_{0,j}^*\}) \in \tilde{\mathfrak{h}}^5 \times \tilde{\mathfrak{h}}^3$.

If, moreover,

(ii) *there exists*

$$\bar{H} = [\bar{G}_r'(r_0^*)(0)]^{-1} \in [\hat{\mathfrak{h}}^3 \times \hat{\mathfrak{h}}^3 \to \tilde{\mathfrak{h}}^5 \times \tilde{\mathfrak{h}}^3]$$

where $\bar{G} = (\bar{G}_1, \bar{G}_2)$ *and* $\bar{G}(\tilde{\mathfrak{h}}^5 \times \tilde{\mathfrak{h}}^3) \subset \hat{\mathfrak{h}}^3 \times \hat{\mathfrak{h}}^3$, *then there exists* $\varepsilon^* \in (0, \varepsilon_1\rangle$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$ *with* $(\varphi^*(\varepsilon), \psi^*(\varepsilon))$ *continuous in* ε *and corresponding to* $(\{\varphi_j^*(\varepsilon)\}_{j=1}^{\infty}; \{\psi_j^*(\varepsilon)\}_{j=1}^{\infty})$ *such that* $\varphi_j^*(0) = \varphi_{0,j}^*$, $\psi_j^*(0) = \psi_{0,j}^*$.

6.3. The case (B). Repeating the considerations and arrangements of section 3.3. we find that the problem $(\mathscr{P}_{2\pi p/q})$ has a solution $u \in \mathscr{U}$ if and only if the equations

$$(6.3.1) \qquad G_{11}(\varphi, \psi)(\varepsilon)(x) \equiv$$
$$\equiv \int_0^{\omega q} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x - 2(\sqrt{(\omega q - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$
$$G_{12}(\varphi, \psi)(\varepsilon)(x) \equiv$$
$$\equiv \int_0^{\omega q} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x - 2(\sqrt{(\omega q - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$(6.3.2) \quad G_{21}(\varphi, \psi)(\varepsilon)(x) \equiv -{}^2\varphi(x) + \frac{\varepsilon}{\sqrt{(2\pi)}} \int_0^{\omega} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 - \sin \lambda^2)\,.$$
$$.\sum_{j=1}^{q-1} \frac{q-j}{q} F(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x - 2(\sqrt{(j\omega - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$G_{22}(\varphi, \psi)(\varepsilon)(x) \equiv -{}^2\psi(x) - \frac{\varepsilon}{\sqrt{(2\pi)}} \int_0^{\omega} \int_{(-\infty,+\infty)}^{\rightarrow} (\cos \lambda^2 + \sin \lambda^2)\,.$$
$$.\sum_{j=1}^{q-1} f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x - 2(\sqrt{(j\omega - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

have a solution (φ, ψ), $\varphi = {}^1\varphi + {}^2\varphi$, $\psi = {}^1\psi + {}^2\psi$, ${}^1\varphi \in \mathscr{H}^5_{0,q}$, ${}^2\varphi \in (\mathscr{H}^5_{0,q})^\perp$, ${}^1\psi \in \mathscr{H}^3_{0,q}$, ${}^2\psi \in (\mathscr{H}^3_{0,q})^\perp$.

We want to apply again Lemma 5.1.2 and therefore we write down this system of equations for $\varepsilon = 0$:

$$(6.3.3) \qquad G_{11}({}^1\varphi_0, {}^1\psi_0)(0)(x) =$$
$$= \int_0^{\omega q} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, f(U({}^1\varphi_0, {}^1\psi_0)(0))(0)(\vartheta, x - 2(\sqrt{(\omega q - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$
$$G_{12}({}^1\varphi_0, {}^1\psi_0)(0)(x) =$$
$$= \int_0^{\omega q} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, f(U({}^1\varphi_0, {}^1\psi_0)(0))(0)(\vartheta, x - 2(\sqrt{(\omega q - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 0$$

$$(6.3.4) \qquad G_{21}({}^2\varphi_0, {}^2\psi_0)(0)(x) = -{}^2\varphi(x) = 0$$
$$G_{22}({}^2\varphi_0, {}^2\psi_0)(0)(x) = -{}^2\psi(x) = 0$$

Similarly as theorems in the preceding section it may be proved

Theorem 6.3.1. *Let the problem* $(\mathscr{P}_{2\pi\,p/q})$ *be given. Let the function* f *be* $2\pi\,p/q$*-periodic in* t *and let it satisfy one of the following assumptions* α) *the assumption* $(\mathscr{A}_1)$ *with* $T = 2\pi\, p/q$ *and* $|\partial f/\partial u_2| + |\partial f/\partial u_3| \not\equiv 0$,

β) *the assumption* $(\mathscr{A}_2)$ *with* $T = 2\pi\, p/q$ *and* $\partial f/\partial u_1 \not\equiv 0$,

γ) *the assumption* $(\mathscr{A}_3)$ *with* $T = 2\pi\, p/q$ *and* $\partial f/\partial u_0 \not\equiv 0$.

Then the problem $(\mathscr{P}_{2\pi\, p/q})$ *has a solution only if*

(i) *the equations* (6.3.3) *have a solution*

$$r_0^* = ({}^1\varphi_0^*, {}^1\psi_0^*)\,, \quad {}^1\varphi_0^* \in \mathscr{H}^5_{0,q}\,, \quad {}^1\psi_0^* \in \mathscr{H}^3_{0,q}\,.$$

If, moreover, there exists (denoting $G_1 = (G_{11}, G_{12})$*)*

(ii) α) $H = [G'_{1r}(r_0^*)(0)]^{-1} \in [\mathscr{H}^3_{0,q} \times \mathscr{H}^3_{0,q} \to \tilde{\mathscr{H}}^5_{0,q} \times \tilde{\mathscr{H}}^3_{0,q}]$

or β) $H = [G'_{1r}(r_0^*)(0)]^{-1} \in [\mathscr{H}^4_{0,q} \times \mathscr{H}^4_{0,q} \to \tilde{\mathscr{H}}^5_{0,q} \times \tilde{\mathscr{H}}^3_{0,q}]$

or γ) $H = [G'_{1r}(r_0^*)(0)]^{-1} \in [\mathscr{H}^5_{0,q} \times \mathscr{H}^5_{0,q} \to \tilde{\mathscr{H}}^5_{0,q} \times \tilde{\mathscr{H}}^3_{0,q}]$ *respectively,*

then there exists $\varepsilon^* \in (0, \varepsilon_1\rangle$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$, $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ *continuous in* ε *and* $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$.

Now let us consider the particular case $\omega = 2\pi(2r-1)/2s$, r, s natural numbers, which has been studied several times by Soviet mathematicians for $s = 1$ and under various assumptions on f ([6]–[10]). The theorem stated below generalizes all of them. Let us suppose that the function f satisfies the relation

$$f(t, \pi - x, u_0, -u_1, u_2, u_3, \varepsilon) = f(t, x, u_0, u_1, u_2, u_3, \varepsilon) \tag{6.3.5}$$

for $(t, x, u_0, u_1, u_2, u_3, \varepsilon) \in \mathscr{Q}_1$. We shall seek a solution in the subspace $\tilde{\mathscr{U}}$ which is formed by functions from $\mathscr{U}$ fulfilling the relation

$$u(t, \pi - x) = u(t, x)\,. \tag{6.3.6}$$

Obviously if f fulfils (6.3.5) and $u \in \tilde{\mathscr{U}}$, then $f(u)(t, x)$ satisfies (6.3.6). Then by the proof of Theorem 3.3.2 the conditions (6.3.1) are satisfied for every $U(r)(\varepsilon) \in \tilde{\mathscr{U}}$ and we see that we have to put ${}^1\varphi \equiv 0$, ${}^1\psi \equiv 0$ (cf. (3.3.1)). It may be verified that also the second primitive function $F(U)(\varepsilon)$ defined by (5.1.8) has the property (6.3.6) if f satisfies (6.3.5) and $U \in \tilde{\mathscr{U}}$. Thus the integral in (6.3.2$_1$) maps $(\mathscr{H}^5_{0,q})^\perp \times (\mathscr{H}^3_{0,q})^\perp$ into $(\mathscr{H}^5_{0,q})^\perp$ and the integral (6.3.2$_2$) maps $(\mathscr{H}^5_{0,q})^\perp \times (\mathscr{H}^3_{0,q})^\perp$ into $(\mathscr{H}^3_{0,q})^\perp$. Hence by Lemma 5.1.2 (setting $r = ({}^2\varphi, {}^2\psi)$ and $\mathscr{R}_1 = \mathscr{R}_2 = (\mathscr{H}^5_{0,q})^\perp \times (\mathscr{H}^3_{0,q})^\perp$) we get

Theorem 6.3.2. *Let the problem* $(\mathscr{P}_{2\pi p/q})$ *be given with* $p = 2r - 1$, $q = 2s$, r, s *natural. Let the function f be* $2\pi\, p/q$*-periodic in t and let it satisfy the assumption* $(\mathscr{A}_1)$ *with* $T = 2\pi\, p/q$ *and* (6.3.5).

Then there exists $\varepsilon^* \in (0, \varepsilon_1\rangle$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \tilde{\mathscr{U}}$ *with* $\varphi^*(\varepsilon)$, $\psi^*(\varepsilon)$ *continuous in* ε *and* $\varphi^*(0) = 0$, $\psi^*(0) = 0$.

Remark 6.3.1. Using to the resolution of the system (6.3.2) the method of successive approximations, the conditions imposed on the smootheness of f could be weakened.

6.4. The case (C). Having in mind the results of section 3.4 we see that in nonlinear case only those α can be considered for which the function f and hence in nonlinear case $f(u)(\varepsilon)(t, x)$ need not be smoother than $\mathscr{C}(\langle 0, 2\pi\alpha\rangle; \mathscr{H}_0^5)$ as otherwise we cannot insert $u \in \mathscr{U}$ into it. Thus only the cases $\varrho = 3, 4$ may be considered. Using now the representation of functions φ, ψ and u in the form of a Fourier series and proceeding as in section 3.4 we get for $\{\varphi_k\}$ and $\{\psi_k\}$ corresponding to the periodic solution, the following system of equations:

(6.4.1)

$$G_{1,k}(\varphi, \psi)(\varepsilon) \equiv \varphi_k - \frac{\varepsilon}{\pi k^2}\left[\frac{\cos(k^2\omega/2)}{\sin(k^2\omega/2)}\int_0^\omega\int_0^\pi f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x) \,.\, \sin kx \,.\right.$$

$$\left. .\, \cos k^2\vartheta \,\mathrm{d}x\,\mathrm{d}\vartheta + \int_0^\omega\int_0^\pi f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x) \,.\, \sin kx \,.\, \sin k^2\vartheta \,\mathrm{d}x\,\mathrm{d}\vartheta\right] = 0$$

$$G_{2,k}(\varphi, \psi)(\varepsilon) \equiv \psi_k - \frac{\varepsilon}{\pi}\left[\frac{\cos(k^2\omega/2)}{\sin(k^2\omega/2)}\int_0^\omega\int_0^\pi f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x) \,.\, \sin kx \,.\right.$$

$$\left. .\, \sin k^2\vartheta \,\mathrm{d}x\,\mathrm{d}\vartheta - \int_0^\omega\int_0^\pi f(U(\varphi, \psi)(\varepsilon))(\varepsilon)(\vartheta, x) \,.\, \sin kx \,.\, \cos k^2\vartheta \,\mathrm{d}x\,\mathrm{d}\vartheta\right] = 0\,.$$

Setting $r = (\{\varphi_k\}, \{\psi_k\})$, $\mathscr{R}_1 = \mathscr{R}_2 = \mathfrak{h}^5 \times \mathfrak{h}^3$, $G = (\{G_{1,k}\}, \{G_{2,k}\})$ and using Lemma 5.1.2 we easily obtain

Theorem 6.4.1. *Let the problem* $(\mathscr{P}_{2\pi\alpha})$ *be given. Let* f *be* $2\pi\alpha$*-periodic in* t *and let it satisfy one of following two assumptions*:

(a) f *fulfils* $(\mathscr{A}_2)$ *and* α *fulfils* (3.4.2) *with* $\varrho = 3$,

(b) f *fulfils* $(\mathscr{A}_3)$ *and* α *fulfils* (3.4.2) *with* $\varrho = 3, 4$.

Then there exists $\varepsilon^* \in (0, \varepsilon_1\rangle$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$, *with* $\varphi^*(\varepsilon)$, $\psi^*(\varepsilon)$ *continuous in* ε *and* $\varphi^*(0) = 0$, $\psi^*(0) = 0$.

§7. SEVERAL SIMPLE EXAMPLES

In this paragraph we formulate existence theorems to the problem $(\mathscr{P}_\omega)$ for some particular ω and particular perturbations f.

Theorem 7.1. *Let problem* $(\mathscr{P}_{2\pi})$ *be given with*

(7.1) $\qquad f(u)(\varepsilon)(t, x) = h(t, x) + \alpha u + \beta u_t + \varepsilon f_1(t, x, u, u_x, u_{xx}, u_t, \varepsilon)$

where $\alpha\beta \neq 0$, $h \in \mathscr{C}(\langle 0, 2\pi\rangle; \mathscr{H}_0^3)$ and f_1 satisfies the assumption $(\mathscr{A}_1)$ with $T = 2\pi$ and h, f_1 are 2π-periodic in t.

Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*\rangle$ there exists a unique solution $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$ with $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ continuous in ε and $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$, where φ_0^*, ψ_0^* are defined by (7.3).

Proof. Clearly the function f satisfies the assumption $(\mathscr{A}_1)$ with $n = 1$. Inserting

$$U(\varphi, \psi)(0)(t, x) = \sum_{k=1}^{\infty} (\varphi_k \cos k^2 t + \psi_k k^{-2} \sin k^2 t) \sin kx$$

and making use of the formulae (2.7) the necessary conditions (6.2.2) in our case take the form

$$\begin{aligned} &G_1(\varphi_0, \psi_0)(0)(x) = \\ &= \pi \left(\frac{\pi}{2}\right)^{1/2} \{\alpha(\varphi_0(x) + \Psi_0(x)) + \beta(-\varphi_0''(x) + \Psi_0''(x)) + h_1(x)\} = 0 \end{aligned} \tag{7.2}$$

$$\begin{aligned} &G_2(\varphi_0, \psi_0)(0)(x) = \\ &= \pi \left(\frac{\pi}{2}\right)^{1/2} \{\alpha(\varphi_0(x) - \Psi_0(x)) + \beta(\varphi_0''(x) + \Psi_0''(x)) + h_2(x)\} = 0 \end{aligned}$$

where

$$h_1(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2} \int_0^{2\pi} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \, . \, h(\vartheta, x - 2(\sqrt{(2\pi - \vartheta)})\, \lambda) \, d\lambda \, d\vartheta$$

$$h_2(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2} \int_0^{2\pi} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \, . \, h(\vartheta, x - 2(\sqrt{(2\pi - \vartheta)})\, \lambda) \, d\lambda \, d\vartheta \, .$$

By the Fredholm alternative this system has only one 2π-periodic solution. This solution has the form

$$\begin{aligned} \varphi_0^*(x) = {} & c_1^* \, . \exp\left(\frac{\gamma}{\sqrt{2}} x\right) . \sin\left(\frac{\gamma}{\sqrt{2}} x\right) + c_2^* \, . \exp\left(\frac{\gamma}{\sqrt{2}} x\right) . \cos\left(\frac{\gamma}{\sqrt{2}} x\right) + \\ & + c_3^* \, . \exp\left(\frac{-\gamma}{\sqrt{2}} x\right) . \sin\left(\frac{\gamma}{\sqrt{2}} x\right) + c_4^* \, . \exp\left(\frac{-\gamma}{\sqrt{2}} x\right) . \cos\left(\frac{\gamma}{\sqrt{2}} x\right) + \\ & + \frac{1}{\gamma\sqrt{2}} \int_0^x \left\{\cosh\left[\frac{\gamma(x - \xi)}{\sqrt{2}}\right] . \sin\left[\frac{\gamma(x - \xi)}{\sqrt{2}}\right] . (H_1(\xi) + H_2(\xi)) + \right. \\ & \left. + \sinh\left[\frac{\gamma(x - \xi)}{\sqrt{2}}\right] . \cos\left[\frac{\gamma(x - \xi)}{\sqrt{2}}\right] . (H_1(\xi) - H_2(\xi))\right\} d\xi \end{aligned} \tag{7.3$_1$}$$

$$(7.3_2)\quad \Psi_0^*(x) = c_1^* \exp\left(\frac{\gamma}{\sqrt{2}}\,x\right).\cos\left(\frac{\gamma}{\sqrt{2}}\,x\right) - c_2^* \exp\left(\frac{\gamma}{\sqrt{2}}\,x\right).\sin\left(\frac{\gamma}{\sqrt{2}}\,x\right) -$$

$$- c_3^* \exp\left(\frac{-\gamma}{\sqrt{2}}\,x\right).\cos\left(\frac{\gamma}{\sqrt{2}}\,x\right) + c_4^* \exp\left(\frac{-\gamma}{\sqrt{2}}\,x\right).\sin\left(\frac{\gamma}{\sqrt{2}}\,x\right) +$$

$$+ \frac{1}{\gamma\sqrt{2}}\int_0^x \left\{\sinh\left[\frac{\gamma(x-\xi)}{\sqrt{2}}\right].\cos\left[\frac{\gamma(x-\xi)}{\sqrt{2}}\right].(H_1(\xi) + H_2(\xi)) -\right.$$

$$\left. - \cosh\left[\frac{\gamma(x-\xi)}{\sqrt{2}}\right].\sin\left[\frac{\gamma(x-\xi)}{\sqrt{2}}\right].(H_1(\xi) - H_2(\xi))\right\} d\xi$$

where

$$\gamma = \left(\frac{\alpha}{\beta}\right)^{1/2}$$

$$H_1(x) = \frac{1}{2\beta}(h_1(x) - h_2(x)), \quad H_2(x) = \frac{-1}{2\beta}(h_1(x) + h_2(x))$$

and c_r^*, $r = 1, 2, 3, 4$, are determined so that φ_0^*, Ψ_0^* are 2π-periodic in x. Clearly $\varphi_0^*, \Psi_0^* \in \mathscr{H}^5$ because of $H_1, H_2 \in \mathscr{H}_0^3$. It may be verified easily that every uniquely determined solution φ, Ψ to (7.2) is odd in x and hence (according to 2π-periodicity of φ_0^*, Ψ_0^*) $\varphi_0^*, \Psi_0^* \in \mathscr{H}_0^5$. Thus the assumption (i) of Theorem 6.2.1 is fulfilled. The existence of an inverse operator

$$H = [G'_{r0}(\varphi_0^*, \psi_0^*)(0)]^{-1} \in [\mathscr{H}_0^3 \times \mathscr{H}_0^3 \to \mathscr{H}_0^5 \times \mathscr{H}_0^5]$$

may be shown in the same manner. This completes the proof.

Theorem 7.2. *Let the problem $(\mathscr{P}_{2\pi})$ be given with f defined by* (7.1), *where $\alpha = 0$, $\beta \neq 0$ and h, f_1 satisfy the same assumptions as in Theorem* 7.1.

Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*\rangle$ there exists a unique solution $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$ with $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ continuous in ε and $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$, where φ_0^*, ψ_0^* are defined by (7.5).

Proof. Now the necessary conditions (6.2.2) have the form

$$(7.4)\qquad \begin{aligned} \beta\,\varphi_0''(x) &= \tfrac{1}{2}(h_1(x) - h_2(x)),\\ \beta\,\Psi_0''(x) &= -\tfrac{1}{2}(h_1(x) + h_2(x)). \end{aligned}$$

By the Fredholm alternative these two equations have a 2π-periodic solution if and only if

$$\int_0^{2\pi}(h_1(\xi) - h_2(\xi))\,d\xi = 0, \quad \int_0^{2\pi}(h_1(\xi) + h_2(\xi))\,d\xi = 0.$$

But according to $h_1, h_2 \in \mathscr{H}_0^3$, these conditions are fulfilled, indeed. The unique solutions of (7.4) belonging to $\mathscr{H}_0^5$ reads

$$(7.5) \qquad \varphi_0^*(x) = \frac{1}{2\beta}\int_0^x \int_0^\xi (h_1(\eta) - h_2(\eta))\, d\eta\, d\xi - \\ - \frac{x}{4\pi\beta}\int_0^{2\pi}\int_0^\xi (h_1(\eta) - h_2(\eta))\, d\eta\, d\xi$$

$$\Psi_0^*(x) = -\frac{1}{2\beta}\int_0^x \int_0^\xi (h_1(\eta) + h_2(\eta))\, d\eta\, d\xi + \\ + \frac{x}{4\pi\beta}\int_0^{2\pi}\int_0^\xi (h_1(\eta) + h_2(\eta))\, d\eta\, d\xi\,.$$

The proof of existence of an inverse operator

$$H = [G'_{r0}(\varphi_0^*, \psi_0^*)(0)]^{-1} \in [\mathscr{H}_0^3 \times \mathscr{H}_0^3 \to \mathscr{H}_0^5 \times \mathscr{H}_0^5]$$

is similar. This by Theorem 6.2.1 completes the proof.

Theorem 7.3. *Let the problem* $(\mathscr{P}_{2\pi})$ *be given with*

$$(7.6) \qquad f(u)(t, x) = h(t, x) + \alpha u + \varepsilon f_2(t, x, u, \varepsilon)$$

where $\alpha \neq 0$, $h \in \mathscr{C}(\langle 0, 2\pi\rangle; \mathscr{H}_0^5)$ *and* f_2 *satisfies the assumption* $(\mathscr{A}_3)$ *with* $T = 2\pi$ *and* h *and* f_2 *are* 2π*-periodic in* t.

Then there exists $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*\rangle$ *there exists a unique solution* $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$ *with* $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ *continuous in* ε *and* $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$, *where* φ_0^*, ψ_0^* *are defined by* (7.7).

Proof. Clearly the function f satisfies the assumption $(\mathscr{A}_3)$ with $n = 1$. The necessary conditions (6.2.2) have now the unique solution

$$(7.7) \quad \varphi_0^*(x) = \frac{-1}{2\alpha}(h_1(x) + h_2(x)) \in \mathscr{H}_0^5, \qquad \Psi_0^*(x) = \frac{-1}{2\alpha}(h_1(x) - h_2(x)) \in \mathscr{H}_0^5.$$

The existence of an inverse operator

$$H = [G'_r(\varphi_0^*, \psi_0^*)]^{-1} \in [\mathscr{H}_0^5 \times \mathscr{H}_0^5 \to \mathscr{H}_0^5 \times \mathscr{H}_0^5]$$

is proved readily and by Theorem 6.2.3. This completes the proof.

Theorem 7.4. *Let the problem* $(\mathscr{P}_{2\pi/3})$ *be given with* f *given by* (7.6) *where* $\alpha \neq 0$, $h \in \mathscr{C}(\langle 0, 2\pi/3\rangle; \mathscr{H}_0^5)$ *and* f_2 *satisfies the assumption* $(\mathscr{A}_3)$ *with* $T = 2\pi/3$ *and* h *and* f_2 *are* $2\pi/3$*-periodic in* t.

Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*\rangle$ there exists a unique solution $U(\varphi^*(\varepsilon), \psi^*(\varepsilon))(\varepsilon) \in \mathscr{U}$ with $\varphi^*(\varepsilon), \psi^*(\varepsilon)$ continuous in ε and $\varphi^*(0) = \varphi_0^*$, $\psi^*(0) = \psi_0^*$, where φ_0^*, ψ_0^* are defined by (7.8).

Proof. Clearly the function f satisfies the assumption $(\mathscr{A}_3)$ with $p = 1$, $q = 3$. In our case the necessary conditions (6.3.3) read

$$^1\varphi(x) + {}^1\psi(x) + h_1(x) = 0\,, \quad {}^1\varphi(x) - {}^1\psi(x) + h_2(x) = 0$$

where

$$h_1(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2} \int_0^{2\pi} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, h(\vartheta, x - 2(\sqrt{(2\pi - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta$$

$$h_2(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2} \int_0^{2\pi} \int_{(-\infty,+\infty)}^{\rightarrow} \sin \lambda^2 \,.\, h(\vartheta, x - 2(\sqrt{(2\pi - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta$$

and the solution is obviously given by

$$(7.8) \quad {}^1\varphi_0^*(x) = \frac{-1}{2}\,(h_1(x) + h_2(x)) \in \mathscr{H}_0^5\,, \quad {}^1\Psi_0^*(x) = \frac{-1}{2}\,(h_1(x) - h_2(x)) \in \mathscr{H}_0^5\,.$$

Since for any $g(t) \in \mathscr{C}(-\infty, +\infty)$ of period $2\pi/3$

$$\int_0^{2\pi} g(t)\,.\cos kt\,\mathrm{d}t = \int_0^{2\pi} g(t)\,.\sin kt\,\mathrm{d}t = 0$$

for every $k \neq 3l$, k, l natural numbers, writing $h(t, x) = \sum_{k=1}^{\infty} h_k(t) \sin kx$, we have

$$h_1(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2} \int_0^{2\pi} \int_{(-\infty,+\infty)}^{\rightarrow} \cos \lambda^2 \,.\, h(\vartheta, x - 2(\sqrt{(2\pi - \vartheta)})\,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\vartheta =$$

$$= \frac{1}{\pi} \sum_{k=1}^{\infty} \left[\int_0^{2\pi} h_k(\vartheta)\,.\,(\cos k^2\vartheta - \sin k^2\vartheta)\,\mathrm{d}\vartheta\right].\sin kx$$

and analogously for $h_2(x)$ so that $h_1(x)$, $h_2(x)$ belong to $\mathscr{H}_{0,q}^5$ and hence also ${}^1\varphi_0^*(x)$, ${}^1\Psi_0^*(x)$ belong to $\mathscr{H}_{0,q}^5$, too. As G_1 maps $\mathscr{H}_{0,q}^5 \times \mathscr{H}_{0,q}^3$ into $\mathscr{H}_{0,q}^5 \times \mathscr{H}_{0,q}^5$ the existence of the inverse operator

$$H = [G'_{1r0}(\varphi_0^*, \psi_0^*)(0)]^{-1} \in [\mathscr{H}_{0,q}^5 \times \mathscr{H}_{0,q}^5 \rightarrow \mathscr{H}_{0,q}^5 \times \mathscr{H}_{0,q}^3]$$

may be proved in the same way. By Theorem 6.3.1 this completes the proof.

Of course the theorems analogous to Theorems 7.1 and 7.2 could be also proved for the problem $(\mathscr{P}_{2\pi/3})$.

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