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BOOK REVIEWS

Černý, Ilja: ZÁKLADY ANALYSY V KOMPLEXNÍM OBORU. (Fundaments of analysis in the complex domain.) Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague 1967. 1st edition. 600 pp., 52 fig.

While introductory courses and textbooks of real analysis written recently proceed usually quite rigorously, the situation in the complex domain is considerably different. In complex analysis, it is still common e.g. to use figures to deduce results which are actually applications of some more advanced topological theorem (which sometimes is not even stated) or of some special assumptions (which again may happen not to be introduced at all). The reviewed book cannot be reproached with anything of this sort. Moreover, it solves even other delicate problems. Unfortunately enough, in mathematics we meet often and at various levels with a gap "between the theory and practice". On one hand, there are rigorous textbooks whose theorems are difficult to apply in concrete cases, and on the other hand there are textbooks which solve concrete problems without making it clear which theorems are actually used. However, Černý's book is comprehensible to anybody who has mastered the elements of the real analysis and is familiar to a certain degree with mathematical reasoning; at the same time it leads the reader through the theorems on analytic functions to concrete results.

As I have already suggested, the author makes only minor demands on the reader. In the first four chapters he deals with the fundamental properties of the Gaussian plane (including some elementary topological properties). In the following two chapters he deals with the logarithm and the general power. In Chapter 7 the author proves some more advanced theorems concerning the topology of plane, namely Eilenberg, Janiszewski and Jordan theorems. Moreover, he proves e.g. the following important assertion: If φ is a continuous mapping of an interval $\langle\alpha, \beta\rangle$ into the complex plane, $\varphi(\alpha) \neq \varphi(\beta)$, then there exists a one-to-one continuous function ψ such that $\psi(\langle\alpha, \beta\rangle) \subset \varphi(\langle\alpha, \beta\rangle)$, $\psi(\alpha) = \varphi(\alpha)$, $\psi(\beta) = \varphi(\beta)$. In the next chapter curves are discussed (by a curve the author means a continuous mapping of a one-dimensional compact interval into the complex plane), especially the index of a point with respect to a closed curve not passing through it. A method is given which often makes it possible to evaluate the index (without referring to a figure). This completes the necessary topological apparatus. In the following two chapters definitions are given of the derivative with respect to the complex variable, of Stieltjes and curvilinear integrals and some theorems concerning these concepts are deduced.

Further the author proceeds to the proper study of functions of the complex variable. At the beginning of Chapter 11 he proves Cauchy Theorem in the following form: *If Ω is an open set in the open complex plane not separating the closed complex plane and if F is a holomorphic function in Ω, then $\int_f F \, dz = 0$ for any curve φ of finite length, belonging to Ω.* The proof is carried out by approximating the curve φ by a piecewise linear curve, which is "decomposed" into Jordan (piecewise linear) curves, each of which is triangulized, and finally, Cauchy Theorem for the perimeter of a triangle is proved in the usual way. This theorem serves as a starting point to prove Liouville and Morera theorems and Schwarz reflection principle. In Chapter 12 the author tackles Laurent series and the classification of functions in an annular neighbourhood of a given point. In Chapter 13 the residue theorem "with the index" is proved for an arbitrary curve of a finite length and an arbitrary holomorphic function in $\Omega - M$, Ω being an open subset of the open complex plane not separating the closed complex plane, and M an isolated set in Ω. A number of methods for the evaluation of residues as well as numerous applications of the residue theorem are presented. E.g. the author discusses the integrals

$$\int_0^\infty R(t)\, t^a \,\mathrm{d}t\,, \qquad \int_{-\infty}^\infty R(t) \sin t \,\mathrm{d}t\,, \qquad \int_0^\infty R(t) \ln t \,\mathrm{d}t$$

and the series $\sum_{n=-\infty}^{\infty} R(n)$ (R being a rational function). The principle of argument, Rouché theorem, theorem on preserving the region (when mapped by means of a non-constant meromorphic function), on differentiation of the inverse function, and the maximum modulus principle are proved.

In Chapter 15 analytic functions are studied. First of all the author explains the motives of introducing the concept of something like a "multi-valued function". He introduces the analytic element and proves a theorem on the relation of notions of "a chain of elements along a curve" and "a finite chain of meromorphic functions". Further he defines fundamental notions concerning analytic functions and proves some easy statements. In Chapter 16 the author deals first with the sum and product of analytic functions. The sum $\mathscr{F}_1 + \mathscr{F}_2$ of analytic functions $\mathscr{F}_1, \mathscr{F}_2$ in a region Ω is the system of all analytic functions in Ω, determined by the elements $\mathscr{E}_1 + \mathscr{E}_2$, $\mathscr{E}_j$ being an element of the function $\mathscr{F}_j$ ($j = 1, 2$); this system may be empty, but it may as well include several (even infinitely many) analytic functions. If $\mathscr{F}$ is an analytic function, it is evident what is meant by $-\mathscr{F}$; provided $\mathscr{F}$ is not identically zero, it is again evident what is meant by $1/\mathscr{F}$. The definition of the difference and the ratio of two analytic functions does not constitute any difficulties. A little more complicated situation arises in defining the inverse analytic function $\mathscr{F}_{-1}$. Here the author considers only analytic functions in the closed complex plane. He mentions the possibility of defining the analytic function inverse to a (non-constant) analytic function in an arbitrary region Ω but directs the reader's attention to the fact that $(\mathscr{F}_{-1})_{-1}) = \mathscr{F}$ (which holds in the case considered above) would not generally hold. Further the author deals with the composition of analytic functions $\mathscr{F}_1, \mathscr{F}_2$; the system of the corresponding composed analytic functions is denoted by $\mathscr{F}_2 * \mathscr{F}_1$. Under certain assumptions he finds the answer to the questions when all functions of the system $\mathscr{F}_2 * \mathscr{F}_1$ are arbitrarily continuable, when none of these functions includes other elements than "composed" ones and when this system includes only one function. Finally the author studies the derivative, the primitive function and the curvilinear integral of an analytic function.

In Chapter 17 it is proved that an arbitrarily continuable analytic function in a simply connected region is single-valued (monodromy theorem). Further the author investigates arbitrarily continuable analytic functions in the simplest double-connected regions, namely in annular neighbourhoods of a point b, and shows that any such function can be obtained by the composition of a function meromorphic in a certain half-plane $\operatorname{Re} z < c$ and the function $\log (z - b)$ (or accordingly $\sqrt[n]{(z - b)}$ provided that the given function is n-valued). Singularities of analytic functions, especially isolated ones, are discussed further in the chapter. The next Chapter 18 is devoted to basic theorems on conformal mapping. The following assertion is proved: *Let Ω_1, Ω_2 be regions each of which is either the interior or the exterior of a circumference, or a half-plane. Then there exists a linear rational mapping F, $F(\Omega_1) = \Omega_2$.* If Φ is a conformal mapping of Ω_1 onto Ω_2, then Φ is a linear rational function. A number of examples are given (e.g. the conformal mapping of a strip onto an angular sector, of an angular sector onto a half-plane, Zhukowskiĭ function $(z + 1/z)/2$, etc.). The following chapter concerns more theoretical problems. It is shown that the complement of the set $F(\Omega)$ has n components, provided that F is the conformal mapping of an open set Ω whose complement has n components (n positive integer), and that under this assumption to every component K of the complement of Ω there is precisely one component K^* of the complement of $F(\Omega)$ such that the point $F(z)$ approaches K^* if z approaches K. Further the author solves in sufficient generality the problem when the orientation of a Jordan curve is preserved or changed by a conformal mapping and proves Montel-Stieltjes-Osgood theorem on the choice of a locally uniformly convergent subsequence of holomorphic functions, Hurwitz theorem on the relation between the number of zero points of functions f_n holomorphic in a given region and the number of zero points of the function f to which $\{f_n\}$ converges locally uniformly, and Riemann theorem on the conformal mapping of a simply connected region whose complement contains at least two points onto the unit circle.

In Chapter 20 the author deals with the problem under which conditions a conformal mapping of a region may be extended continuously to its closure. In order to be able to formulate at least some of the main results, let us introduce the following definitions: α) Let Ω be a region and b a point of its boundary. Point b is said to be arbitrarily accessible from Ω if for any sequence of points of Ω which converges to b there is a curve starting from point b into Ω and passing through an infinite number of these points. β) Let Ω be a region and b a point of its boundary. Let φ, ψ be curves starting from the point b into Ω. The curves φ, ψ are said to belong to the same bundle if for any neighbourhood $\mathcal{U}$ of point b there is a curve λ passing through the set $\mathcal{U} \cap \Omega - \{b\}$ with the initial point on φ and the terminal point on ψ. Thus the set of all curves starting from b into Ω is decomposed into classes; each of these classes is called a bundle of curves (starting from b into Ω). The main results of Chapter 20 may be now formulated as follows: a) *Let Φ be a conformal mapping of an open circle $\mathcal{U}$ onto the region Ω. The mapping Φ may be continuously extended to $\overline{\mathcal{U}}$ iff all points of the boundary of Ω are arbitrarily accessible from Ω.* b) *Let Ω be a non-empty region whose complement contains at least two points. Then the following three conditions are equivalent:* 1) *The boundary of the set Ω is a topological circumference.* 2) *Each point of the boundary of the set Ω is arbitrarily accessible from Ω and to each such point there exists precisely one bundle of curves starting from it into Ω.* 3) *Any conformal mapping of an open circle $\mathcal{U}$ onto Ω may be extended to a homeomorphic mapping of the closed circle $\overline{\mathcal{U}}$ onto Ω.* Finally the theorem on the approximation of a Jordan curve by smooth Jordan curves lying inside the given curve is proved. This theorem is used in the next (and last) chapter to prove the following theorem: *Let F be a holomorphic function in a bounded region whose boundary is a topological circumference, and let F be continuous and finite on $\overline{\Omega}$. Then to every $\varepsilon > 0$ there is a polynomial p such that $|F(z) - p(z)| < \varepsilon$ for all $z \in \overline{\Omega}$.* Hence the following generalization of Cauchy Theorem is easily deduced: *Let φ be a Jordan curve of a finite length. Let F be a function holomorphic on the interior Ω of the curve φ and continuous and finite on $\overline{\Omega}$. Then $\int_\varphi F \, \mathrm{d}z = 0$.* Analogous generalizations of Cauchy formula, residue theorem, principle of argument and Rouché theorem are given. In the Appendix the author proves some theorems on Newton integral (which is defined as the difference of limits of the generalized primitive function in the terminal and initial points of the interval).

In conclusion I should like to make some remarks on the method of the book. Above all, carefulness and reliability should be emphasized. For example, the author never attempts to elude a difficulty by asserting that something is "easy to see". In my opinion, he succeeded also in finding the right proportion as for the details of the proofs; easy steps are often left to the reader. There are no exercises in the form we are accustomed to from other textbooks. However, this is compensated by a considerable number of examples (which I consider to be chosen really suitably). The exposition is clear and in many places it is supplied with figures that contribute to the intelligibility. The misprints are negligibly few.

Considering all that was said above, it would be certainly needless to add more praise here. The usefulness and significance of the matter dealt with for both the pure and applied mathematics need not be stressed. Probably most of the readers of the book will be students of mathematics. Nevertheless, in my opinion, the book may discover new and interesting facts to anybody; if he is a specialist in mathematics, this is true especially of its concluding chapters. However, as no special preliminary knowledge is required, the book will be useful even to many research workers in technical sciences.

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