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ON THE STRUCTURE OF DUAL SEMIGROUPS

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Let S be a semigroup with zero 0. If A is a non-empty subset of S we define the left annihilator $l(A)$ and the right annihilator $r(A)$ of A by

$$l(A) = \{x \mid x \in S,\ xA = 0\}, \quad r(A) = \{x \mid x \in S,\ Ax = 0\}.$$

It is easy to see that $r(A)$ and $l(A)$ are right and left ideals of S respectively.

The following properties can be easily proved:

a) $A \subset r[l(A)]$, $A \subset l[r(A)]$.

b) $A \subset B$ implies $r(B) \subset r(A)$ and $l(B) \subset l(A)$.

c) If A_α, $\alpha \in \Lambda$, is a collection of subsets of S, then $l(\bigcup_\alpha A_\alpha) = \bigcap_\alpha l(A_\alpha)$, $r(\bigcup_\alpha A_\alpha) = \bigcap_\alpha r(A_\alpha)$.

A semigroup is called dual if for every left ideal L of S we have $l[r(L)] = L$ and for every right ideal R of S we have $r[l(R)] = R$.

To exclude trivialities we shall suppose throughout of the whole paper $S \neq 0$.

Ten years ago I developed a structure theory for dual semigroups under the assumption of some minimal conditions. (See [5].) In the meantime NUMAKURA [4] proved that in the case of dual semigroups without nilpotent ideals some of these minimal conditions can be omited.

It is the first purpose of this paper to show that the majority of the results of [5] even in the case that S contains nilpotent ideals holds without any minimal condition. As a matter of fact it turns out that the condition for S to be dual implies itself some rather strong minimal conditions. For instance, any left ideal of a dual semigroup S contains a 0-minimal left ideal of S. Also any dual semigroup S contains non-zero idempotents.

The second purpose of this paper is to give a number of new results, in particular, some decomposition theorems which enable to reduce the study of a dual semigroup to some special types of dual semigroups.

Though the results of this paper denote a considerable step forward in the study of dual semigroups it remains open (even in the commutative case) the problem to find a suitable type of their representation or otherwise expressed to find methods how to construct all dual semigroups.

For completely 0-simple dual semigroups this problem has been solved in [5]. Since the structure of any dual semigroup has (at least formally) some resemblence with the structure of completely 0-simple dual semigroups the problem mentioned seems not to be quite hopeless.

I was led to these investigations by seeking (for a rather long time) for some countreexamples which finally turned out to be impossible.

The methods of treatment and proofs given in this paper differ rather essentially from those given in [4] and [5]. The problems treated are in a loose connection with a recent paper of Hotzel [2].

Besides of the first two Lemmas (which are explicitly proved in [5]) the paper is independent of the results of [5].

The terminology used is the usual one (see [1]).

1. THE EXISTENCE OF O-MINIMAL IDEALS

Lemma 1,1. *In a dual semigroup S we have:*

a) $l(\bigcap_\alpha R_\alpha) = \bigcup_\alpha l(R_\alpha)$, $r(\bigcap_\alpha L_\alpha) = \bigcup_\alpha r(L_\alpha)$, *$L_\alpha$, R_α being left and right ideals respectively.*

b) $l(S) = r(S) = 0$.

c) *If L is a 0-minimal left ideal of S, then $r(L)$ is a maximal right ideal of S.*

d) *If m is a 0-minimal two-sided ideal of S, then $r(m)$ and $l(m)$ are maximal two-sided ideals of S.*

Note that $r(m)$ and $l(m)$ need not coincide (see Example 5,1 below).

Lemma 1,2. *If S is dual, then $a \in Sa$, $a \in aS$ for every $a \in S$. In particular, $S^2 = S$.*

Note explicitly that in a dual semigroup $Sa = 0$ implies $a = 0$.

We first exclude a trivial case in order to simplify some proofs in the following.

Lemma 1,3. *A dual semigroup S in which $aS = S$ for every $a \in S$, $a \neq 0$, is a group with zero.*

Proof. The supposition implies that S contains only the trivial right ideals 0 and S itself. Therefore for the left ideal Sa we have either $r(Sa) = S$ or $r(Sa) = 0$. The first possibility implies $Sa = 0$, a contradiction with $a \neq 0$. Hence $r(Sa) = 0$ and $Sa = S$. S is a semigroup in which $ax = b$ and $ya = b$ have a solution in S for any $a, b \in S$, $a \neq 0$. It is well known that such a semigroup is a group with zero.

Conversely it is obvious that a group with zero is a dual semigroup.

Theorem 1,4. *In a dual semigroup S every non-zero right ideal of S contains a 0-minimal right ideal of S.*

Proof. Let $R \neq 0$ be a right ideal of S and $0 \neq a \in R$. We have $0 \neq a \in aS \subset R$. If $R = S$ we may and we shall suppose that $a \in S$ is chosen so that $aS \neq S$. [For if $aS = S$ for every $a \in S$, $a \neq 0$, Lemma 1,3 implies that S is itself a 0-minimal right ideal of S and there is nothing more to prove.]

Lemma 1,2 implies that there is an $y \in S$ such that $a = ya$. Since $yaS = aS$, y is a left unit for every element $\in aS$ and we certainly have $y \notin l(aS)$. Further since $aS \neq S$, we have $l(aS) \neq 0$.

Denote by L_0 the largest left ideal of S which does not contain y. Clearly $l(aS) \subset \subset L_0$ and $L_0 \neq S$. This implies $r(L_0) \subset aS \subset R$ and $r(L_0) \neq 0$.

We prove that $r(L_0)$ is a 0-minimal right ideal of S. Suppose for an indirect proof that there is a right ideal R_1 of S such that $0 \neq R_1 \subsetneqq r(L_0)$. The duality implies $L_0 \subsetneqq l(R_1) \subsetneqq S$. Since $l(R_1)$ is larger than L_0, we have $y \in l(R_1)$, and $l(R_1)\, R_1 = 0$ implies $yR_1 = 0$. On the other side $R_1 \subset aS$ implies $yR_1 = R_1$. This contradiction proves our Theorem.

Analogously we can prove that any non-zero left ideal of S contains a 0-minimal left ideal of S.

Corollary 1,5. *In a dual semigroup every right ideal R of S, $R \neq S$, is contained in a maximal right ideal of S.*

Proof. Consider the left ideal $l(R)$. By Theorem 1,4 it contains a 0-minimal left ideal L_0, $0 \neq L_0 \subset l(R)$. By duality $r(L_0)$ is a maximal right ideal of S and $R \subset \subset r(L_0) \subsetneqq S$.

Theorem 1,6. *In a dual semigroup S every 0-minimal left ideal of S is contained in a 0-minimal two-sided ideal of S.*

Proof. Let L_0 be a 0-minimal left ideal of S. By Lemma 1,2 we have $L_0 \subset L_0S$. We prove that $M_0 = L_0S$ is a 0-minimal two-sided ideal of S.

Note that for any $a \in S$ the set L_0a is either zero or a 0-minimal left ideal of S. The two-sided ideal $M_0 = L_0S$ can be written as a union of 0-minimal left ideals

$$M_0 = L_0S = L_0\{r(L_0) \cup Z\} = L_0Z = \bigcup_{\alpha\in\Lambda} [L_0z_\alpha]\,,$$

where $Z = \{z_\alpha \mid \alpha \in \Lambda\}$ is the complement of $r(L_0)$ in S.

Since $L_0 \subset M_0$, there is a $z_0 \in Z$ such that $L_0z_0 = L_0$.

Let $M \neq 0$ be a two-sided ideal of S and $M \subset M_0$. We first prove that $L_0 \subset M$. Suppose for an indirect proof that L_0 is not contained in M, so that $M = \bigcup_{\alpha\in\Lambda_1} [L_0z_\alpha]$, where $\{z_\alpha \mid \alpha \in \Lambda_1\}$ does not contain z_0. Since $MS \subset M$, we have $\bigcup_{\alpha\in\Lambda_1} [Lz_\alpha S] \subset \subset \bigcup_{\alpha\in\Lambda_1} [Lz_\alpha]$. For any $\alpha \in \Lambda_1$ the right ideal $z_\alpha S$ cannot contain z_0 (since L_0z_0 is not contained in M). Now $r(L_0)$ is a maximal right ideal of S and z_α is not contained in $r(L_0)$. With respect to the maximality of $r(L_0)$ we have $z_\alpha S \cup r(L_0) = S$. This gives an apparent contradiction, since z_0 is contained neither in $z_\alpha S$ nor in $r(L_0)$. We have proved $L_0 \subset M$.

Now $L_0 \subset M \subset L_0 S$ implies $L_0 S \subset MS \subset L_0 S^2 = L_0 S$. With respect to $M = MS$ we have $M = L_0 S = M_0$, so that M_0 is the 0-minimal two-sided ideal of S containing L_0. This proves our Theorem.

Corollary 1,7. *In a dual semigroup S every two-sided ideal of S contains (at least one) 0-minimal two-sided ideal of S.*

Corollary 1,8. *In a dual semigroup S every maximal left ideal of S contains a maximal two-sided ideal of S.*

Proof. Let L_1 be a maximal left ideal of S. By Theorem 1,6 the 0-minimal right ideal $r(L_1)$ is contained in the 0-minimal two-sided ideal $M = Sr(L_1)$. We have $r(L_1) \subset M \subset S$. Hence $0 \subset r(M) \subset L_1$. By Lemma 1,1 $r(M)$ is a maximal two-sided ideal of S.

2. A LEMMA ON MAXIMAL IDEALS

In the following we shall use some facts concerning semigroups containing maximal ideals. The following Lemma 2,1 has been proved in [6]. It is also contained in a somewhat other form in a recent paper of P. A. Grillet [3].

Lemma 2,1. *Let $\{M_\alpha \mid \alpha \in \Lambda\}$ be the set of all different maximal ideals of a semigroup S. Suppose that* card $\Lambda \geqq 2$ *and denote $P_\alpha = S - M_\alpha$ and $M^* = \bigcap_{\alpha \in \Lambda} M_\alpha$. We then have:*

a) *$P_\alpha \cap P_\beta = \emptyset$ for $\alpha \neq \beta$.*

b) *$S = [\bigcup_{\alpha \in \Lambda} P_\alpha] \cup M^*$.*

c) *We have $P_\alpha \subset M_\nu$ for $\nu \neq \alpha$.*

d) *If J is a two-sided ideal of S and $J \cap P_\alpha \neq \emptyset$, then $P_\alpha \subset J$.*

e) *For $\alpha \neq \beta$ we have $P_\alpha P_\beta \subset M^*$, so that M^* is not empty.*

Remark. For card $\Lambda = 1$ the Lemma is trivial.

We adjoin some remarks to this Lemma. If $a \in S$, we denote by $J(a) = \{a, Sa, aS, SaS\}$ the principal two-sided ideal generated by a. By J_a we denote the set of all generators of $J(a)$, i.e. $J_a = \{x \mid x \in S, J(x) = J(a)\}$. The set J_a is called the J-class containing a.

Lemma 2,2. *With the notations of Lemma* 2,1 *we have*:

a) *Every* P_α *is an* J-*class.*

b) *If* $a \in P_\alpha$, *then* $J(a) \cap P_\beta = \emptyset$ *for every* $\alpha \neq \beta$.

Proof. a) If $a \in P_\mu$, we shall write $P_\mu = P_a$ and $M_\mu = M_a$. Since M_a is a maximal two-sided ideal we have $M_a \cup J(a) = S$. Hence $P_a \subset J(a)$. For any $b \in P_a$ we have $J(b) \subset J(a)$. Now $b \in P_a$ implies also $M_a \cup J(b) = S$, hence $P_a \subset J(b)$ and $J(a) \subset \subset J(b)$. Therefore $J(a) = J(b)$. It follows that P_a is contained in an J-class. This J-class is not contained in the ideal M_a, hence it is disjoint with M_a, so that P_a is itself an J-class.

b) We have seen that $P_a \subset J(a)$. Suppose that there is a $P_\beta \neq P_a$ with $P_\beta \cap J(a) \neq \neq \emptyset$, hence (with respect to Lemma 2,1) $P_\beta \subset J(a)$. Let $c \in P_\beta$. This implies $J(c) \subset \subset J(a)$. Since $c \notin M_\beta$, we have $M_\beta \cup J(c) = S$. Clearly $a \notin M_\beta$, since $a \in M_\beta$ would imply $c \in J(c) \subset J(a) \subset M_\beta$, contrary to the assumption. Hence $a \in J(c)$, therefore $J(a) \subset J(c)$. We have $J(a) = J(c)$, whence $P_a = P_\beta$. This contradiction proves Lemma 2,2.

Consider now the difference semigroup (Rees factor semigroup) S/M^* and the homomorphism $S \to S/M^*$. Denote by 0^* the image of M^*. It follows immediately from Lemma 2,1 and Lemma 2,2 that S/M^* can be written as a mutually annihilating union of semigroups which are 0-simple or null of order two:

$$S/M^* = \bigcup_{\alpha \in A} \bar{P}_\alpha \,, \quad \text{where} \quad \bar{P}_\alpha = P_\alpha \cup \{0^*\} \,. \tag{1}$$

3. THE EXISTENCE OF IDEMPOTENTS AND THE MAIN DECOMPOSITION THEOREM

We now return to the case that S is dual. Our final aim in this and the next section is to prove that in this case every $\bar{P}_\alpha$ in the decomposition (1) is a completely 0-simple dual semigroup. To this end we first prove some theorems concerning the structure of dual semigroups, in particular, Theorem 3,10.

We have seen that if S is dual, S contains maximal two-sided ideals. In this case it is immediately clear that M^* (the intersection of all maximal two-sided ideals) is non-empty since it contains 0. Note also that for a dual semigroup $J(a) = \{a, Sa, aS, SaS\} = SaS$.

Lemma 3,1. *If* S *is dual, then* M^* *does not contain a non-zero idempotent.*

Proof. Suppose for an indirect proof that there is an idempotent $e \neq 0$, $e \in M^*$. By Theorem 1,4 the left ideal Se contains a 0-minimal left ideal L_0. The set $r(L_0)$ is a maximal right ideal of S and by Corollary 1,5 it contains a maximal two-sided ideal M_0 of S. We have $L_0 M_0 \subset L_0\, r(L_0) = 0$, and since $M^* \subset M_0$, we have in particular $L_0 e = 0$. But e is a right unit for every element $\in Se$, hence $L_0 e = L_0$. This contradiction proves our statement.

The existence of idempotents outside of M^* is proved in the next Lemma.

Lemma 3,2. *Let S be a dual semigroup. Then to any $a \in S$, $a \neq 0$, there exists an idempotent $e \in S$ such that $a \in Se$ and an idempotent $f \in S$ such that $a \in fS$.*

Proof. It is sufficient to prove the first statement. The second one follows analogously.

Since $a \in aS$, there is an element $z \in S$ such that $a = az$. Since $Sa = Saz$ we have $Sa \subset Sz$ and z is a right unit for every $b \in Sa$.

Let L_0 be a 0-minimal left ideal of S contained in Sa, $L_0 \subset Sa \subset Sz$. Clearly $L_0 z = L_0$. The maximal right ideal $r(L_0)$ does not contain z.

We prove that for any $b \in S - r(L_0)$ we have $bS = zS$. Since $b \in bS$, $z \in zS$ we have (with respect to the maximality of $r(L_0)$) $zS \cup r(L_0) = S$ and $bS \cup r(L_0) = S$. Hence $b \in zS$ and $z \in bS$. Therefore $bS \subset zS^2 = zS$, $zS \subset bS^2 = bS$, and finally $bS = zS$.

Now since $z \in Sz$, there is an element $y \in S$ such that $z = yz$. We have $y \notin r(L_0)$, since $L_0 y = 0$ would imply $L_0 z = (L_0 y) z = 0$, a contradiction. By the statement just proved we have therefore $yS = zS$. Now $y(zS) = zS$ implies that y is a left unit for every element $\in zS$. Hence y is also a left unit for every element $\in yS$. Since $y \in yS$, we have $y \,.\, y = y$. We have proved that S contains an idempotent $y \neq 0$.

Now $yS = zS$ implies also the existence of a $v \in S$ such that $y = zv$. Hence $z = yz = zvz$. This implies that $e = vz$ is an idempotent. We have $e \in Sz$ and therefore $Se \subset Sz$. Since $ze = z$, we also have $Sz \subset Se$. Hence $Sz = Se$. The inclusion $a \in Sa \subset Sz = Se$ concludes the proof of Lemma 3,2.

In the following if $a \in P_\alpha$ we shall often write $P_\alpha = P_a$.

We now sharpen the last result:

Lemma 3,3. *Let S be dual. If $a \in P_a$, then there is an idempotent $e \in P_a$ such that $a \in Se$ and an idempotent $f \in P_a$ such that $a \in fS$.*

Proof. It is sufficient to show that the idempotent e (the existence of which has been proved in Lemma 3,2) is contained in P_a. By Lemma 3,1 e is not contained in M^*. Suppose that $e \in P_b$. We then have

$$a \in Se \subset \{[\bigcup_{\lambda \in \Lambda} P_\lambda] \cup M^*\}\, e \subset [\bigcup_{\lambda \in \Lambda} P_\lambda]\, P_b \cup M^* \,.$$

Since $P_\lambda P_b \subset M^*$ for any $P_\lambda \neq P_b$, we obtain

$$a \in P_b^2 \cup M^* \subset [J(b)]^2 \cup M^* = [SbS]^2 \cup M^* \subset SbS \cup M^* \,.$$

Since SbS contains P_b and no other P_λ (see Lemma 2,2), we have $a \in SbS \subset P_b \cup M^*$, whence $P_b = P_a$ and $e \in P_a$.

Corollary 3,4. *In a dual semigroup every P_λ $(\lambda \in \Lambda)$ contains at least one idempotent.*

Lemma 3,5. *If S is dual and $0 \neq e = e^2 \in S$, then Se contains a unique non-zero idempotent (namely e itself).*

Proof. Suppose for an indirect proof that there is a non-zero idempotent $e_1 \neq e$, $e_1 \in Se$. Then $e_1 e = e_1$ and $Se_1 \subset Se$. Let L_0 be a 0-minimal left ideal of S contained in Se_1, $L_0 \subset Se_1 \subset Se$. The maximal right ideal $r(L_0)$ does not contain e_1 and e. Therefore $e_1 S \cup r(L_0) = S$. This implies $e \in e_1 S$ and $e_1 e = e$. Hence $e = e_1$, contrary to the assumption.

Remark. Analogously we can prove that eS contains a unique non-zero idempotent.

The next Lemma shows that the 0-minimal left ideal L_0 contained in Se is uniquely determined.

Lemma 3,6. *Let S be a dual semigroup and $e \neq 0$ an idempotent $\in S$. Then Se contains a unique 0-minimal left ideal of S.*

Proof. The existence of a 0-minimal left ideal has been proved in Theorem 1,4. Suppose that L_1 and L_2 are two different 0-minimal left ideals contained in Se. Then $L_1 e = L_1$, $L_2 e = L_2$ and $e \notin r(L_1)$, $e \notin r(L_2)$. The maximal right ideals $r(L_1)$, $r(L_2)$ are different, hence $r(L_1) \cup r(L_2) = S$. This constitutes an apparent contradiction since e is contained neither in $r(L_1)$ nor in $r(L_2)$.

Remark. An analogous result holds for the right ideal eS.

Lemma 3,6 and Lemma 3,2 imply:

Corollary 3,7. *If S is dual and $a \neq 0$, then Sa contains a unique 0-minimal left ideal of S.*

Theorem 3,8. *If S is dual and e_1, e_2 two non-zero different idempotents $\in S$, then $Se_1 \cap Se_2 = 0$.*

Proof. Suppose that there is an $a \in S$, $a \neq 0$, such that $a \in Se_1 \cap Se_2$. Since $a \in Sa$, we have $Sa \neq 0$. Denote by L_0 the 0-minimal left ideal contained in Sa. The relation $L_0 \subset Sa \subset Se_1 \cap Se_2$ implies $L_0 e_1 = L_0$, $L_0 e_2 = L_0$, so that $e_1 \notin r(L_0)$ and $e_2 \notin r(L_0)$. With respect to the maximality of $r(L_0)$ we have $e_2 S \cup r(L_0) = S$. This implies $e_1 \in e_2 S$, a contradiction to Lemma 3,6. Hence $Se_1 \cap Se_2 = 0$.

We finally sharpen Lemma 3,3:

Theorem 3,9. *If S is dual, then to any $a \in S$, $a \neq 0$, there exists a unique idempotent e and a unique idempotent f such that $a = ae$ and $a = fa$.*

Proof. It is sufficient to prove the first statement. By Lemma 3,2 to any $a \neq 0$ there is an idempotent e such that $a \in Se$, hence $ae = a$. Suppose that there are two idempotents $e \neq e'$ such that $0 \neq a = ae = ae'$. We then have $a = ae \subset Se$, $a = ae' \subset Se'$. Hence $a \in Se \cap Se'$. This contradicts Theorem 3,8.

In the following we denote by E the set of all non-zero idempotents $\in S$. Further E_α denotes the set of all non-zero idempotents $\in P_\alpha$, so that $E = \bigcup_{\alpha \in A} E_\alpha$.

Two subsets A, B of S will be called quasidisjoint if $A \cap B = 0$.

By summarising the above results we have:

Theorem 3,10. (*The main decomposition theorem.*) *Any dual semigroup can be written as a union of pairwise quasidisjoint principal left ideals generated by idempotents*:

$$S = \bigcup_{e \in E} Se\,, \quad Se_1 \cap Se_2 = 0 \quad \text{for} \quad e_1 \neq e_2\,.$$

Each of the summands contains a unique idempotent and a unique 0-*minimal left ideal of S. This decomposition is uniquely determined.*

There is also an analogous decomposition into principal right ideals generated by idempotents.

In [5] we have proved that in a completely 0-simple dual semigroup the product of any two different idempotents is 0. This result is a crucial one for the possibility to describe all completely 0-simple dual semigroups. The next Theorem shows that this property has any dual semigroup and this will be here of greatest importance for the possibility to describe the structure of the semigroups $\bar{P}_\alpha$.

Theorem 3,11. *If S is dual and $e \neq f$ two different idempotents $\in S$, then $ef = fe = 0$.*

Proof. Since $Se \cap Sf = 0$, we have $r(Se) \cup r(Sf) = S$. Now e does not belong to $r(Se)$, hence $e \in r(Sf)$, i.e. $Sf \,.\, e = 0$. This implies $fe = 0$. Analogously $f \in r(Se)$, hence $Sef = 0$ and $ef = 0$, concluding the proof of our Theorem.

The case when S contains a unique non-zero idempotent is settled by the following Lemma.

Lemma 3,12. *If S is dual and it contains a unique non-zero idempotent e, then $S = M^* \cup P_e$, where P_e is a group.*

Proof. By Theorem 3,10 we have $S = Se = eS$, so that e is a unit element of S. The semigroup Se has a unique 0-minimal left ideal L_0. Hence L_0 is the 0-minimal two-sided ideal of S and coincide with the (unique) 0-minimal right ideal of S. There is therefore a unique maximal left ideal which is equal to the maximal right ideal and both coincide with M^*.

Let a be any element $\in P_e$. The inclusion $a \in Sa$ implies that Sa is larger than M^*, hence $Sa = S$. Analogously $aS = S$. To any couple $a, b \in P_e$ there are $x, y \in S$ such that $ax = b$ and $ya = b$. Both x and y are contained in P_e, since, e.g., $x \in M^*$ would imply $b = ax \in M^*$, contrary to the assumption. The solvability of these equations proves that P_e is a group.

It is of some interest for further purposes to give an example of a dual semigroup having a unique P_e which is not a group.

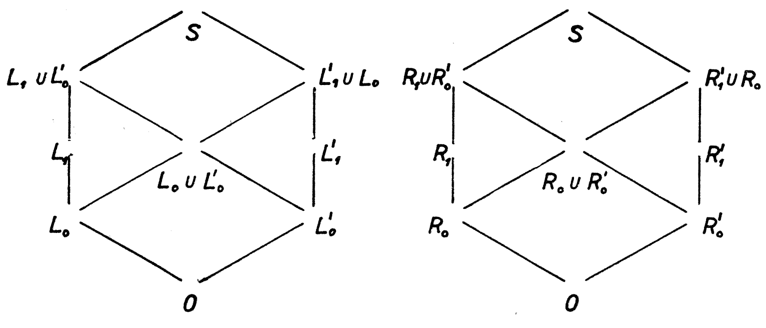
Example 3,1. Let $S = \{0, e_{11}, e_{12}, e_{21}, e_{22}, a_{11}, a_{12}, a_{21}, a_{22}\}$ be the semigroup having the following multiplication table:

	0	e_{11}	e_{12}	e_{21}	e_{22}	a_{11}	a_{12}	a_{21}	a_{22}
0	0	0	0	0	0	0	0	0	0
e_{11}	0	e_{11}	e_{12}	0	0	a_{11}	a_{12}	0	0
e_{12}	0	0	0	e_{11}	e_{12}	0	0	a_{11}	a_{12}
e_{21}	0	e_{21}	e_{22}	0	0	a_{21}	a_{22}	0	0
e_{22}	0	0	0	e_{21}	e_{22}	0	0	a_{21}	a_{22}
a_{11}	0	a_{11}	a_{12}	0	0	0	0	0	0
a_{12}	0	0	0	a_{11}	a_{12}	0	0	0	0
a_{21}	0	a_{21}	a_{22}	0	0	0	0	0	0
a_{22}	0	0	0	a_{21}	a_{22}	0	0	0	0

Denote

$$L_0 = \{0, a_{11}, a_{21}\}, \qquad R_0 = \{0, a_{11}, a_{12}\},$$
$$L_1 = \{0, a_{11}, a_{21}, e_{11}, e_{21}\}, \qquad R_1 = \{0, e_{11}, e_{12}, a_{11}, a_{12}\},$$
$$L'_0 = \{0, a_{12}, a_{22}\}, \qquad R'_0 = \{0, a_{21}, a_{22}\},$$
$$L'_1 = \{0, a_{12}, a_{22}, e_{12}, e_{22}\}, \qquad R'_1 = \{0, a_{21}, a_{22}, e_{21}, e_{22}\}.$$

The lattices of left and right ideals are given below:



It is easy to verify that S is a dual semigroup. There exists a unique P_α, namely $P_\alpha = \{e_{11}, e_{12}, e_{21}, e_{22}\}$, hence a unique maximal two-sided ideal $M^* = \{0, a_{11}, a_{12}, a_{21}, a_{22}\}$ which is at the same time the 0-minimal two-sided ideal of S. The principal left ideals generated by idempotents are $L_1 = Se_{11}$, $L'_1 = Se_{22}$ and we have $S =$

$= L_1 \cup L'_1$. The principal right ideals generated by idempotents are $R_1 = e_{11}S$, $R'_1 = e_{22}S$ and we have $S = R_1 \cup R'_1$.

We also strengthen Theorem 3,11:

Theorem 3,12. *If S is dual and $P_\alpha \neq P_\beta$, then $P_\alpha P_\beta = 0$.*

Proof. By Lemma 3,3 $P_\alpha \subset SE_\alpha$, $P_\beta \subset E_\beta S$. Since $E_\alpha E_\beta = 0$, we have $P_\alpha P_\beta \subset$ $\subset SE_\alpha E_\beta S = 0$.

We finally prove:

Lemma 3,13. *Let S be dual and $e \in S$. Then*

$$r(Se) = \{\bigcup_f fS \mid f \in E,\ f \neq e\}\,, \quad l(eS) = \{\bigcup_f Sf \mid f \in E,\ f \neq e\}\,.$$

Proof. It is sufficient to prove the first statement. If $f \neq e$, we have $Se\,.\,fS = 0$, so that $r(Se)$ certainly contains the set $\{\bigcup fS \mid f \in E, f \neq e\}$. Let $u \in eS$ and $u \in r(Se)$. Then $Seu = 0$. This implies $eu = 0$. Since e is a left unit of eS, we have $eu = u$, i.e. $u = 0$. None of the non-zero elements $\in eS$ is contained in $r(eS)$. This proves our statement.

4. THE STRUCTURE OF THE DIFFERENCE SEMIGROUP

In accordance with [1] we shall say that the semigroup S having a zero element is a 0-direct union of semigroups S_i, $i \in H$, if $S = \bigcup_{i \in H} S_i$ and $S_i \cap S_j = S_i\,.\,S_j = 0$ for $i \neq j$.

Clearly each S_i is a two-sided ideal of S so that we can speak about the 0-direct union of ideals. Note also that any left (right) ideal of S_i is a left (right) ideal of S.

We shall need the following Lemma:

Lemma 4,1. *Let $S = \bigcup_{i \in H} S_i$ be a 0-direct union of ideals S_i. Then S is dual if and only if each summand S_i is dual.*

Proof. In the following we denote the left and right annihilators in S by $l(\ldots)$ and $r(\ldots)$, the annihilators in S_i by $l_i(\ldots)$ and $r_i(\ldots)$. Further let us denote $S^i =$ $= \{\bigcup S_j \mid j \in H, j \neq i\}$.

a) Suppose that S is dual. Let R be a right ideal of S_i.

Clearly $l(R) = l_i(R) \cup S^i$. This implies

$$r[l(R)] = r[l_i(R)] \cap r(S^i)\,. \tag{2}$$

Now, since $l_i(R) \subset S_i$, we have $r[l_i(R)] = r_i[l_i(R)] \cup S^i$, so that (2) implies

$$R = \{r_i[l_i(R)] \cup S^i\} \cap r(S^i),$$
$$R = \{r_i[l_i(R)] \cap r(S^i)\} \cup \{S^i \cap r(S^i)\}.$$

Since $r_i[l_i(R)] \subset S_i \subset r(S^i)$, the first summand to the right is $r_i[l_i(R)]$. The second summand is 0, since $S^i \cap r(S^i) \subset S^i$ and $S^i \cap R = 0$. Hence $R = r_i[l_i(R)]$. Analogously we prove that $L = l_i[r_i(L)]$ for any left ideal L of S_i. We have proved that S_i is dual.

b) Suppose conversely that each S_i is dual. Let L be any left ideal of S. Denote $L_i = S_i \cap L$, so that $L = \bigcup_{i \in H} L_i$. We have

$$r(L) = r[\bigcup_{i \in H} L_i] = \bigcap_{i \in H} r(L_i) = \bigcap_{i \in H} [r_i(L_i) \cup S^i].$$

The last intersection is exactly $\bigcup_{i \in H} r_i(L_i)$. For $r_i(L_i)$ is contained in each term $r_j(L_j) \cup \cup S^j$. On the other side if $x \in R(L)$ does not belong to $\bigcup_{i \in H} r_i(L_i)$ it must be contained in $\bigcap_{i \in H} S^i$, but this intersection is 0.

Hence we have $r(L) = \bigcup_{i \in H} r_i(L_i)$, and this implies

$$l[r(L)] = \bigcap_{i \in H} l[r_i(L_i)] = \bigcap_{i \in H} \{l_i[r_i(L_i)] \cup S^i\} = \bigcap_{i \in H} \{L_i \cup S^i\}.$$

By the same argument as above we conclude that the last intersection is exactly $\bigcup_{i \in H} L_i = L$. Hence $l[r(L)] = L$. Analogously we can prove that $r[l(R)] = R$ for any right ideal R of S. Hence S is dual.

Remark. We now prove that $l(S_i) = r(S_i) = S^i$ and $l(S^i) = r(S^i) = S_i$. Clearly $S^i \subset r(S_i)$. Suppose that $S^i \neq r(S_i)$. We then have $m = r(S_i) \cap S_i \neq 0$. Now $S_i\, r(S_i) = 0$ implies $S_i m = 0$. Since S_i is dual, we have $m = 0$, a contradiction. Hence $S^i = r(S_i)$ and by duality $l(S^i) = S_i$. Analogously we can prove the two remaining relations.

We now return to the decomposition (1) and suppose that S is dual. We have seen that each of the sets P_α $(\alpha \in \Lambda)$ contains at least one idempotent.

If P_α contains a unique idempotent, say e, we have with respect to Lemma 3,2 $P_\alpha \subset Se$, $P_\alpha \subset eS$, hence $P_\alpha \subset Se \cap eS$. The 0-simple semigroup $\bar{P}_\alpha$ contains a two-sided unit element. Such a semigroup is known to be a group with zero. Hence $\bar{P}_\alpha$ is dual.

Suppose next that card $E_\alpha \geqq 2$. Then for any two idempotents $e_1 \neq e_2$ contained in $\bar{P}_\alpha$ we have $e_1 \cdot e_2 = 0^*$. Hence any non-zero idempotent $\in \bar{P}_\alpha$ is primitive. This implies that $\bar{P}_\alpha$ is a completely 0-simple semigroup.

It remains to show that $\bar{P}_\alpha$ is dual.

Any left ideal L of $\bar{P}_\alpha$ can be written in the form $L = \bigcup_{e \in H} \bar{P}_\alpha . e$, where H is a subset of E_α. We denote the left and right annihilators in $\bar{P}_\alpha$ by $\bar{l}(\ldots)$ and $\bar{r}(\ldots)$. If $E_\alpha - H \neq \emptyset$, then $\bar{r}(L)$ contains $\bigcup_{f \in E_\alpha - H} f\bar{P}_\alpha$ and cannot contain $f . \bar{P}_\alpha$ with $f \in H$ [since $L . f\bar{P}_\alpha = (\bigcup_{e \in H} \bar{P}_\alpha e) . f\bar{P}_\alpha$ contains f and it is therefore $\neq 0$]. Hence $\bar{r}(L) = \bigcup_{f \in E_\alpha - H} f\bar{P}_\alpha$. By an analogous argument $\bar{l}[\bar{r}(L)] = \bar{l}[\bigcup_{f \in E_\alpha - H} f\bar{P}_\alpha]$ is clearly equal to $\bigcup_{f \in H} \bar{P}_\alpha f$. Hence $\bar{l}[\bar{r}(L)] = L$. If $E_\alpha = H$, $L = \bar{P}_\alpha$, we have $\bar{r}(L) = 0^*$ [since for any $f \in E_\alpha$ we then have $Lf = \bar{P}_\alpha f \neq 0$]. This implies $\bar{l}[\bar{r}(\bar{P}_\alpha)] = \bar{l}(0^*) = \bar{P}_\alpha$.

Analogously we can prove that $\bar{r}[\bar{l}(R)] = R$ holds for every right ideal R of $\bar{P}_\alpha$.

We have proved:

Theorem 4,2. *Let S be a dual semigroup and M^* the intersection of all maximal two-sided ideals of S. Then the difference semigroup S/M^* is either a completely 0-simple dual semigroup or a 0-direct union of such semigroups.*

In formulae:

$$S/M^* = \bigcup_{\alpha \in A} \bar{P}_\alpha, \quad \bar{P}_\alpha \cap \bar{P}_\beta = \bar{P}_\alpha . \bar{P}_\beta = 0^*,$$

where each $\bar{P}_\alpha (\alpha \in A)$ is a completely 0-simple dual semigroup.

Remark. Note that this result has been obtained without requiring any of the usual types of minimal conditions for left (right or two-sided) ideals.

It is intuitively clear that the 0-minimal right ideals of the semigroup $\bar{P}_\alpha$ are intimately connected with the maximal right ideals of S and – a fortiori – with the 0-minimal left ideals of S. We shall now clarify this connection.

In the following we denote by L_0 the 0-minimal left ideal of S contained in Se. By Z_e we denote the complement of $r(L_0)$ in S, so that $S = Z_e \cup r(L_0)$ and $Z_e \cap r(L_0) = \emptyset$.

Clearly, Z_e depends only on e. Since $L_0 e = L_0$, we have $e \in Z_e$. Since (by Lemma 3,13) $r(L_0) \supset \bigcup_{f \neq e} fS$, we have $Z_e \subset eS$, hence $eZ_e = Z_e$.

The following Lemma is implicitly contained in the proof of Theorem 3,2.

Lemma 4,3. *For any $b \in Z_e$, we have $bS = eS$.*

Proof. Since $e \notin r(L_0)$ and $b \notin r(L_0)$ we have $eS \cup r(L_0) = S$ and $bS \cup r(L_0) = S$. Hence $b \in eS$, $e \in bS$, whence immediately $bS = eS$.

Lemma 4,4. *For any $e \in E$ we have $e\, r(L_0) \subset M^*$.*

Proof. Note first that for any $P_\beta \neq P_e$ we have $eS \cap P_\beta = \emptyset$. This follows immediately from Lemma 2,2 since

$$eS \cap P_\beta \subset SeS \cap P_\beta = J(e) \cap P_\beta = \emptyset .$$

To prove our statement it is therefore sufficient to show that we have also $P_e \cap \cap e\, r(L_0) = \emptyset$.

Suppose for an indirect proof that there is an $a \neq 0$ such that $a \in P_e \cap e\, r(L_0)$. $a \in P_e$ implies $J(a) = SaS = SeS$. Further $a \in e\, r(L_0)$ implies $aS \subset e\, r(L_0)\, S \subset \subset e\, r(L_0)$. Hence $e \in SeS = S(aS) \subset Se\, r(L_0)$. There exist therefore elements x and u, $x \in S$, $u \in r(L_0)$ such that $e = x(eu)$. Since e is an idempotent, we have

$$e = (exe)(eu)\,. \tag{3}$$

Write $S = P_e \cup M_e$, $P_e \cap M_e = \emptyset$ (where M_e is the maximal two-sided ideal of S which does not contain e). We cannot have $exe \in M_e$ since this would imply $e \in \in M_e\,.\,(eu) \subset M_e$, contrary to the definition of M_e.

We have $exe \in P_e$. Since $\bar{P}_e \cong S/M_e$ is completely 0-simple, exe belongs to a group G_e containing e as unit element [see, e. g., [1], Vol. I, pp. 77–78]. Hence there is a $v \in G_e$ such that $v\,.\,exe = e$. Multiplying (3) by v we get $ve = e(eu)$ hence $eu = = ve = e$. The right ideal $er(L_0)$ contains $eu = v$. A right ideal containing an element of a group contains the whole group, so that $e \in er(L_0)$. This gives a contradiction, since $L_0 e = L_0$, while $L_0 e\, r(L_0) = L_0\, r(L_0) = 0$. The proof of Lemma 4,3 is completed.

In the following we denote by E_e the set of all idempotents contained in P_e.

Lemma 4,5. *For any $e \in E$ we have*:

a) $eS = Z_e \cup e\, r(L_0)$, *and the summands are disjoint*;

b) $Z_e = eS \cap P_e$;

c) $P_e = \bigcup_{f \in E_e} Z_f$;

d) $fP_e = Z_f \cup \{0\}$, *for any* $f \in E_e$.

Proof. a) $S = r(L_0) \cup Z_e$ implies $eS = e\, r(L_0) \cup Z_e$. We have $e\, r(L_0) \subset r(L_0)$, since $L_0 e\, r(L_0) = L_0\, r(L_0) = 0$. Therefore $e\, r(L_0) \cap Z_e \subset r(L_0) \cap Z_e = \emptyset$, so that the summands are disjoint.

b) Taking in a) the intersection with P_e we have $eS \cap P_e = Z_e \cap P_e$. To prove our statement it is sufficient to show that $Z_e \subset P_e$. By Lemma 4,3 we have $bS = eS$ for any $b \in Z_e$. Hence $SbS = SeS$. This shows that $b \in P_e$, i.e. $Z_e \subset P_e$.

c) Since $P_e \subset \bigcup_{f \in E_e} fS$, we have $P_e \subset \bigcup_{f \in E_e} [Z_f \cup f\, r(L_0)]$. By Lemma 4.4 $\bigcup_{f \in E_e} f\, r(L_0) \subset \subset M^*$, hence $P_e \subset \bigcup_{f \in E_e} Z_f$. On the other side we have [by b)] $Z_f \subset P_e$ for any $f \in E_e$, hence $\bigcup_{f \in E_e} Z_f \subset P_e$. This proves our statement.

d) For any idempotent f we have $fZ_f = Z_f$. This implies

$$eP_e = e\{\bigcup_{f \in E_e} Z_f\} = e\{Z_e \cup [\bigcup_{f \in E_e} fZ_f]\} = Z_e \cup \{0\}\,.$$

Consider now the completely 0*-simple semigroup $\bar{P}_e = P_e \cup \{0^*\}$ and denote $Z_e^* = Z_e \cup \{0^*\}$. The 0*-minimal right ideal of $\bar{P}_e$ containing e is the set $e\bar{P}_e$. With respect to $eP_e = Z_e \cup \{0\}$ we clearly have $e\bar{P}_e = Z_e \cup \{0^*\} = Z_e^*$ and $\bar{P}_e = \bigcup_{e \in E_e} e\bar{P}_e = \bigcup_{e \in E_e} Z_e^*$.

Summarily we have proved:

Theorem 4,6. *Let e be an idempotent $\in P_e$ and L_0 the 0-minimal left ideal contained in Se. Denote by Z_e the complement of $r(L_0)$ in S. Then the 0*-minimal right ideals of the completely 0*-simple semigroups $\bar{P}_e$ are exactly the sets Z_e with the zero 0^* adjoined.*

5. THE PRODUCT OF LEFT IDEALS

The decompositions from Theorem 3,10, namely

$$S = \bigcup_{e \in E} Se = \bigcup_{f \in E} fS \tag{4}$$

imply a "finer" decomposition, namely

$$S = \bigcup_{e,f \in E} [Se \cap fS] . \tag{5}$$

We have clearly $fSe \subset fS \cap Se$. To prove that $fSe = fS \cap Se$ take any element $a \in fS \cap Se$. Then e is the unique right unit of a and f the unique left unit of a, so that $a = ae$, $a = fa$ and $a = fae$. Hence $a \in fSe$, which proves our statement.

The decomposition (5) can be therefore written in the form

$$S = \bigcup_{e,f \in S} fSe , \tag{6}$$

where (with respect to Theorem 3,9) the summands are quasidisjoint.

If $e \neq f$, we have $(fSe)^2 = fS(e \, . \, f)\, Se = 0$.

If $e, f \in E$, we have $eSf = e[\bigcup_{\alpha \in A} P_\alpha \cup M^*] f = (eP_e \cup eM^*) f = eP_e f \cup eM^* f$.

a) If $e \in P_\alpha$, $f \in P_\beta$ and $P_\alpha \neq P_\beta$, then $eP_e f \subset eP_\alpha P_\beta = 0$ so that $eSf \subset eM^*f \subset M^*$.

b) If both e and f are contained in the same $P_\alpha = P_e$, we prove that $eSf \cap P_e \neq \emptyset$. First

$$eSf = eP_e f \cup eM^* f \subset eSeSf \cup M^* \subset SeS \cup M^* = J(e) \cup M^* .$$

By Lemma 2,2 $J(e) \cap P_\beta = \emptyset$ for every $P_\beta \neq P_e$, so that eSf does not meet any

$P_\beta \neq P_e$. To prove $eSf \cap P_e \neq \emptyset$ it is therefore sufficient to prove that eSf cannot be contained in M^*. Suppose for an indirect proof that this were the case. $eSf \subset M^*$ implies $(eSf)(fSe) \subset M^*$. Since $SfS = SeS$, we would have $e(SeS)\,e \subset M^*$, hence $e \in M^*$, a contradiction with Lemma 3,1.

Summarily:

Lemma 5,1. *If S is dual, S can be written in the form $S = \bigcup_{e,f \in E} eSf$, where the summands are quasidisjoint.*

If $e \neq f$, we have $(eSf)^2 = 0$.

If $e \in P_\alpha$, $f \in P_\beta$ and $P_\alpha \neq P_\beta$, then $eSf \subset M^*$.

If e, f are contained in the same P_α, we have $eSf \cap P_\alpha \neq \emptyset$.

We are now able to prove one statement concerning the product of two left (right) ideals generated by idempotents.

Theorem 5,2. *If S is dual and the idempotents e, f belong to the same P_e, then $eSfS = eS$ and $SeSf = Sf$.*

Proof. We prove the first statement. Clearly

$$eS\,.\,fS \subset eS\,. \tag{7}$$

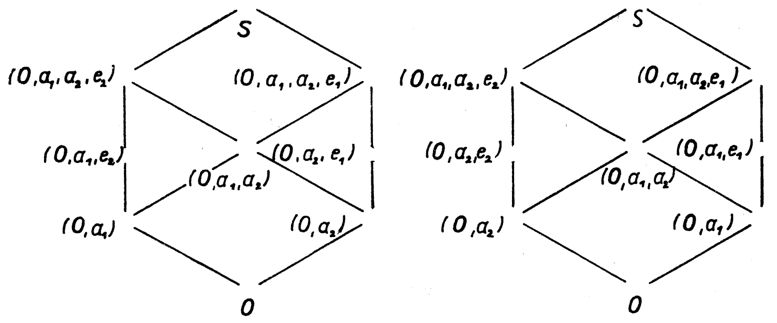
Write $eS = Z_e \cup e\,r(L_0)$, where L_0 is the 0-minimal left ideal contained in Se. By Lemma 4,4 $e\,r(L_0) \subset M^*$. By Lemma 5,1 the right ideal $eSfS$ (which contains eSf) is not contained in M^*. Hence $eSfS \cap Z_e \neq \emptyset$. Let x be any element $\in eSfS \cap Z_e$. We have certainly $x \neq 0$. By Lemma 4,3 $xS = eS$. Further $xS \subset eSfS^2 = eSfS$, therefore $eS \subset eSfS$. This together with (7) implies $eSfS = eS$.

The pertinent question arises: What can be said about the product $SeSf$ if $e \in P_\alpha$, $f \in P_\beta$ and $P_\alpha \neq P_\beta$. By Lemma 5,1 we have in this case $SeSf \subset SM^* \subset M^*$ (more precisely $SeSf \subset M^* \cap Sf$). It would be nice if it were $SeSf = 0$. But this need not be true as the following example shows.

Example 5,1. Let $S = \{0, e_1, e_2, a_1, a_2\}$ be the semigroup with the following multiplication table:

	0	e_1	e_2	a_1	a_2
0	0	0	0	0	0
e_1	0	e_1	0	a_1	0
e_2	0	0	e_2	0	0
a_1	0	0	a_1	0	0
a_2	0	a_2	0	0	0

By inspection of the lattices of left and right ideals as given below we immediately see that S is dual.



Now the left ideals generated by idempotents are $L_1 = (0, a_2, e_1)$, $L_2 = (0, a_1, e_2)$. We have

$$L_1L_2 = (0, a_1) \neq 0\,, \quad L_2L_1 = (0, a_2) \neq 0\,.$$

(Both ideals L_1, L_2 are incidentally two-sided ideals.)

If m is a 0-minimal two-sided ideal, then $r(m)$ and $l(m)$ are maximal two-sided ideals. Our example shows that $r(m)$ and $l(m)$ need not be equal. For, e.g., $m = (0, a_1)$ is a 0-minimal two-sided ideal and $l(m) = (0, a_1, a_2, e_2)$ while $r(m) = (0, a_1, a_2, e_1)$.

Note also that examples 4,1 and 5,1 show that two essentially different dual semigroups may have isomorphic lattices of ideals.

Remark. It is quite natural to consider a further decomposition which arises from the decompositions of Theorem 3,10, namely:

$$S = S^2 = \left[\bigcup_{e\in E} Se\right]\left[\bigcup_{f\in E} fS\right] = \bigcup_{e\in E} SeS = \bigcup_{e\in E} J(e) = \bigcup_{\alpha\in A} SP_\alpha S\,.$$

Here each summand contains a unique P_α $(\alpha \in A)$.

Unfortunately the summands here are not necessarily quasidisjoint and even the product $J(e)\,J(f)$ can be different from 0. [Recall that $J(e)\,J(f) \subset J(e) \cap J(f)$.] This can be again shown on Example 5,1. Here we have $Se_1S = \{0, a_1, a_2, e_1\}$, $Se_2S = \{0, a_1, a_2, e_2\}$, and

$$Se_1S\,.\,Se_2S = \{0, a_1\} \neq 0\,, \quad Se_2S\,.\,Se_1S = \{0, a_2\} \neq 0\,.$$

Nevertheless in some cases decompositions into quasidisjoint two-sided ideals are possible. This will be studied in the next section.

In this section we shall first deal with the 0-minimal two-sided ideals of a dual semigroup.

If $\{M_\alpha \mid \alpha \in \Lambda\}$ is the set of all different maximal two-sided ideals, then $\{r(M_\alpha) \mid \alpha \in \in \Lambda\}$ is the set of all 0-minimal two-sided ideals. Also, the set $\{l(M_\alpha) \mid \alpha \in \Lambda\}$, as a whole, is identical with $\{r(M_\alpha) \mid \alpha \in \Lambda\}$.

$r(M_\alpha)$ is characterised by the property $P_\alpha\, r(M_\alpha) = r(M_\alpha)$. For, $S\, r(M_\alpha) = (P_\alpha \cup \cup M_\alpha)\, r(M_\alpha) = r(M_\alpha)$, while for $P_\beta \neq P_\alpha$ we have $P_\beta\, r(M_\alpha) \subset M_\alpha\, r(M_\alpha) = 0$.

Analogously $l(M_\alpha)$ satisfies $l(M_\alpha)\, P_\alpha = l(M_\alpha)$, while $l(M_\alpha)\, P_\beta = 0$ for any $P_\beta \neq P_\alpha$.

We have $r(M_\alpha) = S\, r(M_\alpha) = SP_\alpha\, r(M_\alpha) \subset SP_\alpha S = SeS$ for any $e \in P_\alpha$. Analogously $l(M_\alpha) \subset SeS$ for any $e \in P_\alpha$.

Summarising we have:

Lemma 6,1. *To any P_α $(\alpha \in \Lambda)$ there is a unique 0-minimal two-sided ideal n_α characterised by the property $P_\alpha n_\alpha \neq 0$ and a unique 0-minimal two-sided ideal m_α characterised by the property $m_\alpha P_\alpha \neq 0$.*

A more precise description of n_α and m_α will be given in Lemma 6,5.

For a fixed idempotent $e \neq 0$ consider the two-sided ideal SeS.

If SeS itself is a 0-minimal two-sided ideal, then $SeS \cap M^* = 0$. For $SeS \cap M^* \neq \neq 0$ would imply $SeS \subset M^*$ and M^* does not contain idempotents $\neq 0$.

If SeS is not minimal, then since $SeS = SP_eS$ does not meet any $P_\beta \neq P_e$, we necessarily have $SeS \cap M^* \neq 0$ and the 0-minimal two-sided ideal contained in SeS is contained in M^*.

Let us consider the first case. Since $SeS \cap M^* = 0$, in the mapping $S \to S/M^*$ the elements of SeS are in a one-to-one correspondence with the elements $\in \bar{P}_e$ (where, in particular, $0 \leftrightarrow 0^*$ and the other elements retain in essential their identity). This implies that SeS is a completely 0-simple dual semigroup.

If SeS and SfS are two different 0-minimal two-sided ideals we have $SeS \cap SfS = = 0$, hence $SeS\,.\,SfS = 0$.

Denote by $E^{(1)}$ the set of all idempotents $\in S$ which generate a 0-minimal two-sided ideal and denote $S^{(1)} = \bigcup_{e \in E^{(1)}} SeS$ (supposing of course that $E^{(1)}$ is non-empty). If $S^{(1)}$ is non-empty, it is either a completely 0-simple dual semigroup or a 0-direct union of such semigroups. Hence $S^{(1)}$ is itself dual.

Denote by $E^{(2)}$ the set of all remaining idempotents, $E^{(2)} = E - E^{(1)}$. For every $e \in E^{(2)}$ (supposing that $E^{(2)}$ is non-empty) we have $SeS \cap M^* \neq 0$. Denote $S^{(2)} = = \bigcup_{e \in E^{(2)}} SeS$, so that $S = S^{(1)} \cup S^{(2)}$ and $S^{(1)} \cap S^{(2)} = 0$. By Lemma 4,1 $S^{(2)}$ is dual.

We have proved:

Theorem 6,2. *Any dual semigroup admits a decomposition of the form*

$$S = S^{(1)} \cup S^{(2)}\,, \quad S^{(1)}S^{(2)} = S^{(1)} \cap S^{(2)} = 0\,,$$

where the summands have the following properties:

a) $S^{(1)}$ *is either empty or it is a completely* 0*-simple dual semigroup or* 0*-direct union of completely* 0*-simple dual semigroups.*

b) $S^{(2)}$ *is either empty or it is a dual semigroup in which every two-sided ideal has a non-zero intersection with* M^*.

This implies immediately:

Theorem 6,3. *A dual semigroup* S *is a* 0*-direct union of completely* 0*-simple semigroups if and only if* $M^* = 0$.

Remark. Note that it follows from our considerations that SeS is 0-minimal if and only if $SeS \cap r(SeS) = 0$.

To prove an other decomposition theorem (formulated below as Theorem 6,8) we first give a new characterisation of the set of all 0-minimal two-sided ideals of S.

The decomposition of Theorem 3,10 can be written also in the form $S = \bigcup_{\alpha \in A} SE_\alpha$.

Let $e \in E_\alpha$ and let L_0 be the 0-minimal left ideal of S contained in Se. We know that L_0S is a 0-minimal two-sided ideal of S. Write (as in Lemma 4,5) $S = Z_e \cup r(L_0)$ and $eS = Z_e \cup e\, r(L_0)$. We then have

$$L_0S = L_0eS = L_0[e\, r(L_0) \cup Z_e] = L_0Z_e = \bigcup_{a \in Z_e} L_0a\,.$$

Since for every $a \in Z_e$ we have $L_0a \neq 0$, every L_0a is a 0-minimal left ideal of S. Since $Z_e \subset P_e = P_\alpha$, we have $Z_e \subset \bigcup_{f \in E_\alpha} Sf$ and $L_0S = L_0Z_e \subset \bigcup_{f \in E_\alpha} L_0Sf \subset \bigcup_{f \in E_\alpha} Sf = SE_\alpha$. Hence L_0S is contained in SE_α.

We next prove that L_0S contains all 0-minimal left ideals contained in $SE_\alpha = \bigcup_{f \in E_\alpha} Sf$. Let $f \in E_\alpha$. We prove that $Z_e \cap Sf \neq 0$. By Lemma 5,1 $eSf \cap P_e \neq 0$. Now $eSf = eS \cap Sf$, hence $eSf \cap P_e = eS \cap Sf \cap P_e = Sf \cap [eS \cap P_e] = Sf \cap Z_e$. Therefore $Z_e \cap Sf \neq 0$. There exists an $a \in Z_e$ such that $a \in Sf$. For this element a we have $0 \neq L_0a \subset L_0Sf \subset Sf$, hence $L_0S \cap Sf \neq 0$. This implies that Z_0S is the union of all 0-minimal left ideals contained in $\bigcup_{f \in E_\alpha} Sf$.

We have proved:

Lemma 6,4. *Let* e *be any idempotent* $\in E_\alpha$ *and* L_0 *the* 0*-minimal left ideal of* S *contained in* Se. *Then the* 0*-minimal two-sided ideal* L_0S *is contained in* SE_α *and* L_0S *is exactly the union of all* 0*-minimal left ideals contained in* SE_α.

Explicitly: If we decompose S in the form

$$S = SE_\alpha \cup SE_\beta \cup SE_\gamma \cup \dots,$$

then each of the summands SE_λ contains a unique 0-minimal two-sided ideal, say m_λ, and $\{m_\lambda / \lambda \in \Lambda\}$ is exactly the set of all 0-minimal two-sided ideals of S. m_α satisfies $m_\alpha P_\alpha \neq 0$, while $m_\alpha P_\beta = 0$ for every $P_\beta \neq P_\alpha$.

S can be decomposed also in the form

$$S = E_\alpha S \cup E_\beta S \cup E_\gamma S \cup \ldots$$

and analogously as above each $E_\lambda S$ has a unique 0-minimal two-sided ideal n_λ of S. The sets $\{m_\lambda \mid \lambda \in \Lambda\}$ and $\{n_\lambda \mid \lambda \in \Lambda\}$ coincide. We have:

Lemma 6,5. *The* 0-*minimal two-sided ideal* m_α *introduced in Lemma* 6,1 *is the unique* 0-*minimal two-sided ideal contained in* SE_α. *Analogously* n_α *is the unique* 0-*minimal two-sided ideal of* S *contained in* $E_\alpha S$.

Remark. We use Example 5,1 to show that m_α need not be equal to n_α. In this example we have the following two decompositions:

$$S = Se_1 \cup Se_2 = \{0, a_2, e_1\} \cup \{0, a_1, e_2\},$$
$$S = e_1 S \cup e_2 S = \{0, a_1, e_1\} \cup \{0, a_2, e_2\}.$$

Here $m_1 = \{0, a_2\}$, $m_2 = \{0, a_1\}$ and $n_1 = \{0, a_1\}$, $n_2 = \{0, a_2\}$, so that $n_1 \neq m_1$.

Since SE_α and $E_\alpha S$ are contained in SeS $(e \in E_\alpha)$ the two-sided ideal SeS can contain two (or more) 0-minimal two-sided ideals of S. In this connection the following Lemma clarifies the situation.

Lemma 6,6. *Let* S *be dual and* $e \in E_\alpha$. *Then* SeS *contains a unique* 0-*minimal two-sided ideal if and only if* $SE_\alpha = E_\alpha S$.

Proof. a) Suppose that SeS contains a unique 0-minimal two-sided ideal m of S. Since $SE_\alpha \subset SE_\alpha S = SeS$, we have by Lemma 6,4 $m \subset SE_\alpha$. If $f \in E - E_\alpha$, we have $SeS \cap Sf = 0$. For otherwise there would exist a 0-minimal left ideal L_1 of S such that $L_1 \subset Sf$ and $L_1 \subset SeS$. By Lemma 6,4 and our supposition $L_1 \subset m \subset SE_\alpha$. But $L_1 \subset SE_\alpha \cap Sf$ contradicts Theorem 3,8. Now $SE_\alpha S \cap Sf = 0$ for every $f \in E - E_\alpha$ implies that $SE_\alpha S \subset SE_\alpha$. Since $SE_\alpha \subset SE_\alpha S$, we have $SE_\alpha S = SE_\alpha$, i.e. $SeS = SE_\alpha$. By an analogous argument we prove that $SeS = E_\alpha S$. Hence $SE_\alpha = E_\alpha S$.

b) Conversely if $E_\alpha S = SE_\alpha$, we have $SE_\alpha S = SE_\alpha$. Since SE_α contains a unique 0-minimal two-sided ideal of S, the same holds for $SeS = SE_\alpha S$.

Lemma 6,7. *Let* S *be dual. Suppose that* $SeS = SE_\alpha S$ *contains a unique* 0-*minimal two-sided ideal of* S. *Suppose that* $E - E_\alpha \neq \emptyset$. *Then* S *can be written as a* 0-*direct union of two dual semigroups*:

$$S = S^{(0)} \cup S^{(00)}, \quad S^{(0)} S^{(00)} = S^{(0)} \cap S^{(00)} = 0,$$

where $S^{(0)} = SeS$ *and* $S^{(00)} = \bigcup_{E_\beta \neq E_\alpha} SE_\beta S$.

Proof. If $E_\beta \neq E_\alpha$, we have by Lemma 6,6

$$SE_\alpha S \,.\, SE_\beta S = (SE_\alpha S)\, E_\beta S = SE_\alpha E_\beta S = 0\,.$$

If $f \in E - E_\alpha$, then SfS cannot meet a left ideal Se_α $(e_\alpha \in E_\alpha)$. For otherwise there were a left ideal $L_0 \neq 0$, $L_0 \subset SfS \cap Se_\alpha$. Now $L_0 \subset Se_\alpha$ implies $L_0 e_\alpha = L_0$, while $L_0 \subset SfS$ implies $L_0 e_\alpha \subset SfSe_\alpha \subset Sf(SE_\alpha) = Sf(E_\alpha S) = 0$, a contradiction. Hence $SE_\alpha \cap SE_\beta S = 0$ for $E_\beta \neq E_\alpha$ and since $SE_\alpha = SE_\alpha S$, we have $SE_\alpha S \cap SE_\beta S = 0$. Finally $SE_\alpha S \cap S^{(00)} = 0$.

Since $S^{(0)} \cap S^{(00)} = 0$ and $S^{(0)} \cup S^{(00)} = S$, the union is 0-direct and Lemma 4,1 implies that both $S^{(0)}$ and $S^{(00)}$ are dual.

Let now be F the set of all idempotents e for which SeS contains a unique 0-minimal two-sided ideal of S. Suppose that F is non-empty. Using repeateadly Lemma 6.7 we can write S as a 0-direct union $S = T^{(0)} \cup T^{(00)}$. Here $T^{(0)} = \bigcup_{E_\alpha \subset F} SE_\alpha S$ itself is a 0-direct union of dual semigroups each of which contains a unique 0-minimal two-sided ideal. $T^{(00)} = \bigcup_{e \in E - F} SeS$ has the property that if it is non-empty each summand has at least two different 0-minimal two-sided ideals of S. We state this result as a Theorem.

Theorem 6,8. *Any dual semigroup admits a decomposition of the form*

$$S = T^{(0)} \cup T^{(00)}\,, \quad T^{(0)} \,.\, T^{(00)} = T^{(0)} \cap T^{(00)} = 0\,,$$

where the summands have the following properties:

a) $T^{(0)}$ *is either empty or it is a dual semigroup containing a unique* 0-*minimal two-sided ideal or it is a* 0-*direct union of such semigroups.*

b) $T^{(00)}$ *is either empty or a dual semigroup in which every two-sided ideal generated by an idempotent contains at least two* 0-*minimal two-sided ideals of S.*

Remark. This decomposition is clearly a refinement of the decomposition given in Theorem 6,2. For, $T^{(0)}$ contains all two-sided ideals which are completely 0-simple and may contain also some further summands (each of which contains a unique 0-minimal two-sided ideal).

It follows immediately from the foregoing considerations:

Theorem 6,9. *A dual semigroup is a* 0-*direct union of dual semigroups each of which contains a unique* 0-*minimal two-sided ideal if and only if for any two idempotents* $e, f \in E$ *we have either* $SeS = SfS$ *or* $SeS \cap SfS = 0$.

Remark 1. The situation mentioned in Theorem 6,9 takes place, in particular, for every commutative dual semigroup.

Remark 2. We use Example 5,1 to show that that $T^{(00)}$ can be non-empty. In this example we have $Se_1S = \{0, a_1, a_2, e_1\}$, $Se_2S = \{0, a_1, a_2, e_2\}$, so that both principal ideals Se_1S, Se_2S contain exactly two 0-minimal two-sided ideals, namely $\{0, a_1\}$, $\{0, a_2\}$. (Here $T^{(0)}$ is empty.)

Remark 3. As remarked in the introduction the structure of a completely 0-simple dual semigroup S has been fully described in [5], where a matrix representation of S by means of matrices over a group with zero has been given. As a next step it would be desirable to find some kind of representation of dual semigroups containing a unique 0-minimal two-sided ideal. Even this (rather special) problem is far from to be easy. Roughly speaking this is due to the fact that we have to deal with ideals contained in M^* and the obvious difficulties with nilpotent ideals arise. We shall illustrate this on some further examples in the next section.

7. SOME FURTHER EXAMPLES

In this section we give some examples of commutative dual semigroups which are useful to get some idea about various possibilities which can occur.

If S is commutative each of the sets P_e is a group. Each Se, $e \in E$ is a two-sided ideal containing e as a unit element. Theorem 3,10 (or Theorem 6,9) implies that S can be written as a 0-direct union of dual semigroups each of which contains a unique idempotent:

$$S = \bigcup_{e \in E} Se\,, \quad Se\,.\,Sf = Se \cap Sf = 0 \quad (e \neq f)\,.$$

Hence to get some further informations about the structure of such semigroups it is sufficient to study commutative dual semigroups containing a unit element e. S can be then written in the form $S = M^* \cup P_e$, where M^* is the unique maximal ideal and P_e is a group with e as unit element.

Even this case can be simplified. Suppose that $a, b \in S$, $a \neq 0$, $b \neq 0$. Then either $aP_e = bP_e$ or $aP_e \cap bP_e = 0$. [For if $x \in aP_e \cap bP_e$, there are two elements $u, v \in P_e$ such that $x = ua = vb$. Denote by u^{-1} the inverse of u in P_e. We have $a = u^{-1}vb$ and $P_ea = (u^{-1}v)\,P_eb = P_eb$.] Hence we can write $S = \bigcup_{a \in S} aP_e$. Consider the semigroup S_1 the elements of which are the classes $aP_e(a \in S)$. The mapping $a \to aP_e$ is clearly a homomorphism of S onto S_1. Since an ideal of S containing an element $\in aP_e$ contains the whole set aP_e, we see that the lattice of ideals of S is isomorphic to the lattice of ideals of S_1. If S is dual, S_1 is dual. Hence there is no much loss of generality if we restrict our considerations to semigroups in which P_e reduces to the trivial group $\{e\}$.

Remark. There is also a converse proceeding which enables to construct from such a dual semigroup a "larger" one by enlarging the group without changing the lattice of ideals. Analogous proceeding is also possible in the non-commutative case. We shall not deal with these constructions here.

Example 7,1. Denote by S a semigroup having a zero element 0, a unit element e and one further generating element p with the defining relation $p^n = 0$. Explicitly: $S = \{0, e, p, p^2, \ldots, p^{n-1}\}$ with an obvious multiplication.

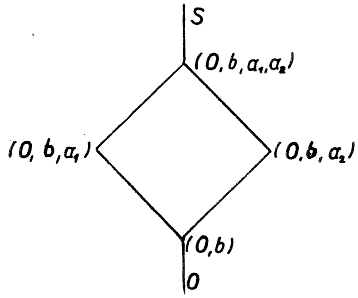
The maximal ideal is $M^* = \{0, p, p^2, \ldots, p^{n-1}\}$. The 0-minimal ideal is $\{0, p^{n-1}\}$. The ideals form a chain

$$0 \subset \{0, p^{n-1}\} \subset \{0, p^{n-1}, p^{n-2}\} \subset \ldots \subset M^* \subset S .$$

This is in some sense the simplest type of a dual semigroup of the kind required.

Example 7,2. The ideals of a commutative dual semigroup with a unit element need not form necessarily a chain. The semigroup $S = \{0, b, e, a_1, a_2\}$ with the multiplication table and lattice of ideals as given below is dual.

	0	e	b	a_1	a_2
0	0	0	0	0	0
e	0	e	b	a_1	a_2
b	0	b	0	0	0
a_1	0	a_1	0	b	0
a_2	0	a_2	0	0	b



Example 7,3. Consider now the semigroup $S = \{0, e, b, a_1, a_2, a_3, \ldots\}$ having countably infinite elements, where the multiplication is defined as follows: e is the unit element, $b^2 = 0$, $a_k^2 = b$ (for $k = 1, 2, 3, \ldots$) and $a_i \,.\, a_k = 0$ for $i \neq k$.

	0	e	b	a_1	a_2	a_3	$\ldots$
0	0	0	0	0	0	0	
e	0	e	b	a_1	a_2	a_3	
b	0	b	0	0	0	0	
a_1	0	a_1	0	b	0	0	
a_2	0	a_2	0	0	b	0	
a_3	0	a_3	0	0	0	b	
$\vdots$							

The 0-minimal ideal is $n = \{0, b\}$, the maximal ideal is $M^* = S - \{e\}$. All principal ideals, besides of n and S, are of the form $m_k = \{0, b, a_k\}$ and any ideal $N \neq S$, $N \neq n$, is a union of such principal ideals. Denote by K a set of positive integers and by K' the complement of K in $\{1, 2, 3, \ldots\}$. If $N = \{0, b, \bigcup_{i \in K} a_i\}$ is an ideal of S, then $r(N) = \{0, b, \bigcup_{i \in K'} a_i\}$ and clearly $lr(N) = N$ so that S is dual.

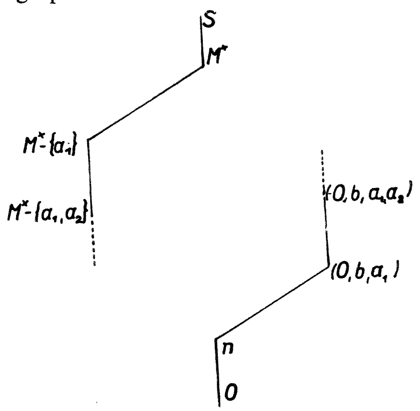
Consider, e.g., the increasing chain of ideals

$$0 \subset \{0, b\} \subset \{0, b, a_1\} \subset \{0, b, a_1, a_2\} \subset \ldots \subset M^* \subset S .$$

The corresponding annihilators form a decreasing chain

$$S \supset M^* \supset M^* - \{a_1\} \supset M^* - \{a_1, a_2\} \supset \ldots \supset n \supset 0 .$$

This is indicated on the graph:



Since there is an infinity of "smallest" ideals covering n (namely any m_k) it is clear that there is an infinity of such couples of corresponding chains. Moreover beginning with any member of any chain we can construct an infinity of different chains.

The lattice of ideals may be considered as rather complicated though the semigroup itself is very simple in the following sense: We have $M^{*3} = 0$ (since it is easy to see that the product of any three elements $\neq e$ is 0).

It is clear that this example can be modified by writing instead of $\{a_1, a_2, a_3, \ldots\}$ a set of any cardinality and defining the multiplication in an obvious way.

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