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ON THE NUMBER OF POLYNOMIALS IN ORDERED ALGEBRA

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Let $\mathfrak{A} = (A; \boldsymbol{F})$ be an algebra in the sense of MARCZEWSKI ([2]). Let the set A be ordered (an order is considered as a reflexive, antisymmetric and transitive relation) and every f belonging to $\boldsymbol{F}$ be isotone in all variables. Then $\mathfrak{A}$ will be called an ordered algebra (see [3]). In [3] bidirected algebras were studied, in particular the sets $\mathscr{S}(\mathfrak{A}) = \{n : p_n(\mathfrak{A}) \neq 0\}$ were described ($p_n(\mathfrak{A})$ is the number of essentially n-ary polynomials over $\mathfrak{A}$). This paper deals with the possibilities of extension of the results and constructions given in [1] to the case of ordered algebras. As a corollary, the $\mathscr{S}(\mathfrak{A})$'s for directed algebras will be described. Let us emphazise that, for card $A > 1$, we take identity mapping as an essentially unary polynomial (see a different point of view in [1]). It turns out that the only condition on $\mathscr{S}(\mathfrak{A})$, where $\mathfrak{A}$ is at least two-element algebra, is $1 \in \mathscr{S}(\mathfrak{A})$.

The paper [1] will be considered as known, nevertheless the most important facts concerning our paper are recalled.

In our considerations we shall need the constructions of the algebras given in the cases (i) and (iv) of the Theorem from [1].

It may be convenient for the reader to state the relevant statements here (in our notation concerning $p_1(\mathfrak{A})$).

Let $\langle p_0, \ldots, p_n, \ldots \rangle$ be a sequence elements of which are cardinal numbers. Such a sequence is called *representable* if there exists an algebra $\mathfrak{A}$ such that $p_n(\mathfrak{A}) = p_n$ for all n.

In the sequel α is the smallest ordinal with $p_i < \bar{\alpha}$ for all i. For every $i < \alpha$ we take a countable set A_i such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\text{(i)}\quad \text{If } p_0 > 0 \text{ then } \langle p_0, p_1 + 1, \ldots, p_n, \ldots \rangle \text{ is representable}\,.$$

The algebra $\mathfrak{A}$ representing the sequence $\langle p_0, \ldots, p_n, \ldots \rangle$ is constructed in the following way:

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Let card $K = p_0 + 1$, $k_0, k_1 \in K$, $k_0 \neq k_1$, let K be disjoint to the A_i's. Let $A = K \cup \bigcup\{A_i : i < \alpha\}$.

Operations are defined as follows.

For every $k \in K$, $k \neq k_0$, f_k is the nullary operation with the value k. For $n > 0$, $0 \leqq i < p_n$, f_i^n is defined on A by

$$f_i^n(a_0, \ldots, a_{n-1}) = \begin{cases} k_0, & \text{if} \quad a_0, \ldots, a_{n-1} \in A_i, \quad \text{card}\{a_0, \ldots, a_{n-1}\} = n. \\ k_1 & \text{otherwise}. \end{cases}$$

Put

$$\mathfrak{A} = (A; \{f_k : k \in K, \ k \neq k_0\} \cup \{f_i^n : 0 \leqq i < p_n\}).$$

(iv) If $p_0 = 0$, $p_1 > 0$ and n divides p_n, then $\langle p_0, p_1 + 1, \ldots, p_n, \ldots \rangle$ is representable.

The algebra $\mathfrak{A}$ representing this sequence is constructed in the following way: $A = \bigcup_{i<\alpha} A_i \cup \{t_0, t_1, t_2\}$, where t_0, t_1, t_2 are three objects not belonging to $\bigcup_{i<\alpha} A_i$.

Operations are defined as follows:

$$g^1(a) = \begin{cases} t_0 & \text{if} \quad a = t_0. \\ t_2 & \text{otherwise}. \end{cases}$$

For $n \geqq 1$, $i < n$, $j < p_n/n$

$$h_{i,j}^n(a_0, \ldots, a_{n-1}) = \begin{cases} t_0, & \text{if} \quad a_i = t_0. \\ t_1, & \text{if} \quad a_0, \ldots, a_{n-1} \in A_j, \quad \text{card}\{a_0, \ldots, a_{n-1}\} = n. \\ t_2 & \text{otherwise}. \end{cases}$$

Put

$$\mathfrak{A} = \left(A; \{g^1\} \cup \bigcup_{n=1}^{\infty} \left\{h_{i,j}^n : i < n, \ j < \frac{p_n}{n} \text{ for } n > 1 \text{ and } j < p_1 - 1 \text{ for } n = 1\right\}\right).$$

Theorem 1. *Let $\mathfrak{A} = (A; \mathbf{F})$ be the algebra from Theorem, case (i) in [1]. There exists an ordering $\leqq$ on A such that $(A; \leqq)$ is an upper semilattice and in respect to this order $(A; \mathbf{F})$ is an ordered algebra.*

Proof. It is sufficient to order $\bigcup_{i<\alpha} A_i \cup (K - \{k_1\})$ trivially (i.e. any two distinct elements are incomparable) and put $(A; \leqq) = (\bigcup_{i<\alpha} A_i \cup (K - \{k_1\}) \oplus \{k_1\}$ ($\oplus$ means the ordinal sum).

Remark 1. As it is evident from results in [3], one cannot demand in general $\leqq$ to be a bidirected order for $\mathfrak{A}$.

Remark 2. Similar result to that one of Theorem 1 can be deduced in the case (ii) of Theorem from [1]. But in this case we have $\mathscr{S}(\mathfrak{A}) = \{1, 2, 3, \ldots\}$ so a priori there is a possibility to construct bidirected algebras, representing a given sequence with $p_0 = 0$. However, a simple argument concerning t_1 and t_2 in the construction dealing with the case (ii) shows that the algebra constructed in that case cannot be provided with a compatible bidirected order. Nevertheless, constructing another type of algebras we shall find the corresponding result for bidirected algebras. Let us state the relevant theorem in a rather different form than that one used in [1]. In this formulation, the universality of the construction is emphazised. Theorem in [1] could be refrazed in an analogous way.

Theorem 2. *There exists an algebra $\mathfrak{A} = (A; \mathbf{F})$, a lattice order $\leqq$ on A and, for every $n \geqq 1$, an essentially n-ary polynomial $g^n(x_1, \ldots, x_n)$ over $\mathfrak{A}$ with the following properties:*

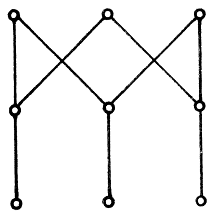
1) *$\mathfrak{A}$ is an ordered algebra in respect to $\leqq$.*
2) *There exist $\aleph_0$ essentially n-ary polynomials over $\mathfrak{A}$ for every n.*
3) *There is no constant in $\mathfrak{A}$.*
4) *Let $\mathbf{F}_1$ be a system of polynomials over $\mathfrak{A}$. Then $\mathbf{F}_1 \cup \{g^n : 1 \leqq n\}$ and projections are the only polynomials over $(A; \mathbf{F}_1 \cup \{g^n : 1 \leqq n\})$.*

Remark 3. A trivial modification of the construction given below presents a similar result where $\aleph_0$ is replaced by an arbitrary infinite cardinal number.

Proof of Theorem 2. The construction of A and of the order $\leqq$. For $n > 1$, let A^n be the set of nonvoid subsets of the set $\{1, \ldots, n\}$ different from the whole set $\{1, \ldots, n\}$. Order A^n by inclusion.

A^1 be an one-element set.

Let A_i^n $(i = 1, 2, 3, \ldots)$ be a copy of A^n, where two-element chains are imbedded instead of minimal elements. For instance, if $n = 3$, we get the ordered sets A_i^3 of the form



All A_i^n are supposed mutually disjoint.

Let $a, b, c \notin \bigcup_{i,n} A_i^n$. Then

$(A, \leqq) = \{a\} \oplus \sum_{n,i} A_i^n \oplus \{b\} \oplus \{c\}$ ($\oplus$ means the ordinal sum, $\sum$ the cardinal sum).

$(A, \leqq)$ is clearly a complete lattice.

Definition of an algebra on A. We shall define the functions $g^n(x_1, \ldots, x_n), f_i^n(x_1, \ldots \ldots, x_n)$, $n, i = 1, 2, 3, \ldots$

Define $g^1(x)$ as follows:

$g^1(b) = c.$

Let x be minimal in $\sum_{n,i} A_i^n$, x^* be the upper neighbor of x in A. Then $g^1(x) = x^*$.

Put $g^1(x) = x$ otherwise.

For $n \geqq 2$, define g^n as $g^n(x_1, \ldots, x_n) = g^1(x_1 \vee \ldots \vee x_n)$.

Let $A_i^1 = \{a_i, b_i\}$, $a_i < b_i$. Put $f_i^1(a_i) = a_i$, $f_i^1(x) = g^1(x)$ otherwise.

For $n \geqq 2$, let $a_1^{n,i}, \ldots, a_n^{n,i}$ be the minimal elements in A_i^n. Define $f_i^n(x_1, \ldots, x_n)$ by the following equations:

$f_i^n(a_1^{n,i}, \ldots, a_n^{n,i}) = b.$

$f_i^n(x_1, \ldots, x_n) = g^n(x_1, \ldots, x_n)$ otherwise.

All operations are clearly symmetrical.

g^n's are apparently isotone in all their variables. So are f_i^1's. We shall prove the same for f_i^n for $n > 1$. Isotonicity must be checked in respect to the n-tuple $a_1^{n,i}, \ldots, a_n^{n,i}$, in other cases the isotonicity of f_i^n is a consequence of the isotonicity of g^n. Replace $a_1^{n,i}$ by a, then $f_i^n(a, a_2^{n,i}, \ldots, a_n^{n,i}) \in A_i^n$, so $f_i^n(a, a_2^{n,i}, \ldots, a_n^{n,i}) < b = f_i^n(a_1^{n,i}, \ldots, a_n^{n,i})$. Replace $a_1^{n,i}$ by some u, $u > a_1^{n,i}$. Then

$$f_i^n(u, a_2^{n,i}, \ldots, a_n^{n,i}) = g^n(u, \ldots, a_n^{n,i}) = g^1(u \vee \ldots \vee a_n^{n,i}) = \\ = g^1(b) = c > b = f_i^n(a_1^{n,i}, \ldots, a_n^{n,i}).$$

Now, we prove that every identification in any function gives some $g^n(x_1, \ldots, x_n)$. In particular (for $n \geqq 2$)

$$g^n(x, x, x_3, \ldots, x_n) = g^1(x \vee x \vee \ldots \vee x_n) = g^1(x \vee x_3 \vee \ldots \vee x_n) = \\ = g^{n-1}(x, x_3, \ldots, x_n).$$

$$f_i^n(x, x, x_3, \ldots, x_n) = g^n(x, x, x_3, \ldots, x_n) = g^{n-1}(x, x_3, \ldots, x_n).$$

Superpositions yield no other polynomial, as well. Let us prove that.

α) Clearly

$$g^1(g^n(x_1, \ldots, x_n)) = g^n(x_1, \ldots, x_n).$$
$$g^1(f_i^n(x_1, \ldots, x_n)) = g^n(x_1, \ldots, x_n).$$

Let $n > 1$.

We shall prove

$$g^n(g^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = g^{n+m-1}(x_1, \ldots, x_m, y_2, \ldots, y_n).$$

By the definition

$$g^n(g^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = g^1(g^1(x_1 \vee \ldots \vee x_m) \vee y_2 \vee \ldots \vee y_{n_1}) .$$

If $g^1(x_1 \vee \ldots \vee x_m) = x_1 \vee \ldots \vee x_m$, the formula is clearly valid.

a) Let $x_1 \vee \ldots \vee x_m = b$. In this case, both sides of our formula are c.

b) Let $x_1 \vee \ldots \vee x_m$ be minimal in $\sum A_i^n$ and x^* be the upper neighbor of $x_1 \vee \ldots \ldots \vee x_m$.

b_1) Let $y_2, \ldots, y_n \leqq x_1 \vee \ldots \vee x_m$. Then both sides equal to x^*.

b_2) Let $x_1 \vee \ldots \vee x_m$ be not the greatest element among $y_2, \ldots, y_n, x_1 \vee \ldots \vee x_m$. Then

$$g^n(g^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = g^1(x^* \vee y_2 \vee \ldots \vee y_n) .$$

Under our conditions

$$g^1(x_1 \vee \ldots \vee x_m \vee y_2 \vee \ldots \vee y_n) = g^1((x_1 \vee \ldots \vee x_m) \vee (y_2 \vee \ldots \vee y_n)) = \\ = g^1(x^* \vee y_2 \vee \ldots \vee y_n) .$$

The formula is proved.

It is

$$g^n(f_i^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = g^{n+m-1}(x_1, \ldots, x_m, y_2, \ldots, y_n) .$$

Let $m > 1$.

If

$$\{x_1, \ldots, x_m\} \neq \{a_1^{m,i}, \ldots, a_m^{m,i}\}$$

then

$$f_i^m(x_1, \ldots, x_m) = g^m(x_1, \ldots, x_m) .$$

The validity of the formula follows from the previous case.

Let $\{x_1, \ldots, x_m\} = \{a_1^{m,i}, \ldots, a_m^{m,i}\}$. Then both sides are equal to c.

If $m = 1$

$$g^n(f_i^1(x_1), y_2, \ldots, y_n) = g^n(x_1, y_2, \ldots, y_n) .$$

β) Clearly

$$f_i^n(g^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = g^n(g^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = \\ = g^{m+n-1}(x_1, \ldots, x_m, y_2, \ldots, y_n) .$$

If $n \neq 1$ or $m \neq 1$

$$f_i^n(f_j^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = g^n(f_j^m(x_1, \ldots, x_m), y_2, \ldots, y_n) = \\ = g^{m+n-1}(x_1, \ldots, x_m, y_2, \ldots, y_n) .$$

Let $n = 1 = m$. Let $i \neq j$. We have $f_i^1(f_j^1(x)) = g^1(x)$. If $i = j$, we get $f_i^1(f_i^1(x)) = f_i^1(x)$.

Now, it is already easy to finish the proof of our theorem. We have to prove 2), 3), 4).

g^n, f_i^n are evidently essentially n-ary, so 2) is true.

Considerations on compositions of f_i^m and g^n, on identifications together with the symmetry of all operations give 3) and 4).

The result of the case (iii) of [1] cannot be extended for ordered algebras, for which the order is a semilattice order. First of all, we shall prove the following Lemma.

Lemma. *Let $\mathfrak{A} = (A; \mathsf{F})$ be an algebra with $p_0(\mathfrak{A}) = 0$, $p_1(\mathfrak{A}) = 2$ and $p_{2n}(\mathfrak{A}) = 0$, $p_{2n+1}(\mathfrak{A}) = 1$ for $n = 1, 2, 3, \ldots$ Let $g(x)$ be the unary polynomial over $\mathfrak{A}$ different from the identity mapping. The $g(g(x)) = g(x)$.*

Proof. Let us suppose that our assertion is not true. Then $g(g(x)) = x$. Let us choose two elements $a, b \in A$ such that $g(a) = b$, $g(b) = a$, $a \neq b$. They exist as $g(x) \neq x$. Let $f^3(x, y, z)$ be the essentially ternary polynomial over $\mathfrak{A}$. As $p_3(\mathfrak{A}) = 1$, there is just one such a polynomial. As $p_2(\mathfrak{A}) = 0 = p_0(\mathfrak{A})$ there are the following possibilities.

a) $f^3(x, x, y) = x$.

b) $f^3(x, x, y) = y$.

c) $f^3(x, x, y) = g(x)$.

d) $f^3(x, x, y) = g(y)$.

Ad a). It is $f^3(a, b, g(b)) = a, f^3(b, b, g(b)) = b$. Hence $f^3(x, y, g(z))$ depends on x, by symmetry of $f^3(x, y, g(z))$ in x, y, we get the dependence on y and for $p_2(\mathfrak{A}) = 0, f^3(x, y, g(z))$ depends on z, too. At the same time $f^3(a, b, b) = b \neq a = f^3(a, b, g(b))$. Hence $f^3(x, y, z) \neq f^3(x, y, g(z))$, a contradiction to $p_3(\mathfrak{A}) = 1$.

In the remaining cases similar conclusions can be drawn from the following equations:

Ad b) $f^3(a, b, g(b)) = b$, $f^3(b, b, g(b)) = a$.

Ad c) $f^3(a, b, g(b)) = b$, $f^3(b, b, g(b)) = a$.

Ad d) $f^3(a, b, g(b)) = a$, $f^3(b, b, g(b)) = b$.

The proof is finished.

Now, we can prove the following proposition.

Proposition. *Let $\mathfrak{A} = (A; \mathsf{F})$ be an ordered algebra with $p_0(\mathfrak{A}) = 0$, $p_1(\mathfrak{A}) = 2$ and $p_{2n}(\mathfrak{A}) = 0, p_{2n+1}(\mathfrak{A}) = 1$ for $n = 1, 2, 3, \ldots$ Then the order of A is not a semilattice order.*

Proof. Let again $g(x)$ be the only essentially unary polynomial over $\mathfrak{A}$ different from the identity mapping. Then, by Lemma, $g(g(x)) = g(x)$. Put

$$B = \{x : g(x) = x\} .$$

B is a subalgebra in $\mathfrak{A}$. Let us prove it. Let f^n be the essentially n-ary polynomial over $\mathfrak{A}$ for $n > 1$. Then $g(f^n(x_1, \ldots, x_n))$ is essentially n-ary over $\mathfrak{A}$. As $p_n(\mathfrak{A}) = 0$ or 1 we have $g(f^n(x_1, \ldots, x_n)) = f^n(x_1, \ldots, x_n)$, i.e. $f^n(x_1, \ldots, x_n) \in B$ for every $x_1, \ldots, x_n \in A$.

Let $\bar{f}^n$ be the restriction of f^n to B. $\bar{f}^n$ is not a constant, because the $f^n(g(x), \ldots, g(x))$ would be a constant over $\mathfrak{A}$, which contandicts to $p_0(\mathfrak{A}) = 0$.

As $f^n(x_1, \ldots, x_n)$ is symmetrical, $\bar{f}^n(x_1, \ldots, x_n)$ is symmetrical and therefore essentially n-ary.

Put $\mathfrak{B} = (B; \{\bar{f}^n : n = 3, 5, \ldots\})$.

We shall prove that $\bar{f}^3, \bar{f}^5, \ldots$ are the only nontrivial polynomials over $\mathfrak{B}$. Let $F(x_1, \ldots, x_n)$ be some polynomial symbol over $\mathfrak{B}$. Let $f(x_1, \ldots, x_n)$ be the corresponding polynomial over $\mathfrak{B}$. Replace $\bar{f}^m$ in $F(x_1, \ldots, x_n)$ by f^m for every m. We obtain a polynomial symbol $H(x_1, \ldots, x_n)$ over $\mathfrak{A}$. Let $h(x_1, \ldots, x_n)$ be the corresponding polynomial over $\mathfrak{A}$. Clearly $\bar{h} = f$. If h is trivial, f is trivial, too. Therefore nontrivial polynomial f over $\mathfrak{B}$ is of the form $\bar{h}$, where h is a nontrivial polynomial over $\mathfrak{A}$. g is identity on $\mathfrak{B}$, hence $\bar{f}^3, \bar{f}^5, \ldots$ are the only nontrivial polynomials over $\mathfrak{B}$. It is $p_{2n+1}(\mathfrak{B}) = 1$, $p_{2n}(\mathfrak{B}) = 0$.

Hence, $\mathfrak{B}$ is an idempotent algebra without constants and it is by Urbanik's Theorem 2.2 in [4] a reduct of an at least two-element Boolean group $(G, +)$, in which $x + y + z$ is taken as the fundamental operation.

Suppose $(A, \leqq)$ is a semilattice: At the same time, B is a semilattice in regard to $\leqq$. To prove this take $a, b \in B$. Let $c = a \vee b$ in A. Then $g(c) \in B$, $g(c) \geqq a$, $g(c) \geqq b$. If $d \geqq a$, $d \geqq b$, $d \in B$, then $d \geqq c$ and $d \geqq g(c)$.

On the other hand, such an algebra cannot exist. In particular, take some $b \in B$, for which $b < 0$. Then $0 = 0 + 0 + 0 > b + 0 + 0 \geqq b + b + 0 = 0$, a contradiction. If such a b does not exist in B, we have for some b $b > 0$ and $0 = 0 + 0 + 0 < b + 0 + 0 \leqq b + b + 0 = 0$, again a contradiction.

We see that $(A, \leqq)$ is not a semilattice and this concludes the proof.

Theorem 3. *Let $\mathfrak{A} = (A; \mathsf{F})$ be the algebra from Theorem, case* (iv) *in* [1]. *Then there exists an ordering $\leqq$ on A such that $(A, \leqq)$ is an upper semilattice and $\mathfrak{A}$ is an ordered algebra in respect to this order.*

Proof. Let $\bigcup_{i<\alpha} A_i$ be ordered as an antichain. Put

$$(A, \leqq) = \bigcup_{i<\alpha} A_i \oplus \{t_1\} \oplus \{t_2\} \oplus \{t_0\} .$$

Easy calculations show that $h^n_{i,j}$ and g^1 are isotone in respect to $\leqq$.

From theorems 1 and 3 we conclude

Theorem 4. *Let $k_0, k_2, \ldots, k_n, \ldots$ be an arbitrary sequence consisting of zeroes and ones (so $k_n = 0$ or 1). There exists an ordered algebra $\mathfrak{A}$ with the following properties:*

1) *The order on $\mathfrak{A}$ is a semilattice order.*
2) *If $n \neq 1$, $p_n(\mathfrak{A}) \neq 0$ exactly when $k_n = 1$.*

Remark 4. $n = 1$ is excluded as, by our agreement on identity mapping, $p_1(\mathfrak{A}) \neq 0$ for every $\mathfrak{A} = (A; \mathbf{F})$ with card $A > 1$.

Remark 5. In [3] the sequences $k_0, k_2, \ldots, k_n, \ldots$ have been described for bidirected algebras. It turned out that only quite special sequences are representable (in the sense of Theorem 4) by bidirected algebras.

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