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ON ALMOST COMPLETE AND ALMOST PRECOMPACT QUASI-UNIFORM SPACES

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1. INTRODUCTION

In this paper we introduce almost complete and almost precompact quasi-uniform spaces. We show that almost precompact quasi-uniform spaces and almost complete quasi-uniform spaces relate to almost realcompact, countably almost-compact, and almost-compact topological spaces in a natural way. We also show that in a regular quasi-uniform space almost completeness is equivalent to completeness. We prove that almost completeness and almost precompactness are preserved under arbitrary products and obtain a generalization of Niemytzki-Tychonoff theorem. These results may be considered as evidence that almost completeness is a reasonable generalization of completeness in uniform spaces. A question of some importance is the extent to which the relationship between almost completeness and almost realcompactness in quasi-uniform spaces mimics the relationship between completeness and realcompactness in uniform spaces. In particular we raise the question of whether or not the natural analogue of Shirota's theorem holds for quasi-uniform spaces [Conjecture 4.2]. We show that the truth of the conjecture would have impact upon the study of realcompact spaces.

Throughout this paper all spaces are considered to be Hausdorff. In particular, although it is true that every topological space is quasi-uniformizable, whenever we consider a quasi-uniform space $(X, \mathscr{U})$ we shall presuppose that its associated topology $\mathscr{T}_{\mathscr{U}}$ is Hausdorff.

2. PRELIMINARIES

Definition. Let X be a nonempty set and let $\mathscr{U}$ be a filter on $X \times X$ such that

(i) each element of $\mathscr{U}$ is a reflexive relation on X

(ii) if $U \in \mathscr{U}$, there exists $W \in \mathscr{U}$ such that $W \circ W \subset U$.

Then $\mathscr{U}$ is a *quasi-uniformity on X*.

Definition. Let X be a set and let $\mathscr{U}$ be a quasi-uniformity on X. Let $\mathscr{T}_{\mathscr{U}} = \{A \subset X$: if $a \in A$, then there exists $U \in \mathscr{U}$ such that $U(a) \subset A\}$. Then $\mathscr{T}_{\mathscr{U}}$ is the *quasi-uniform topology on X generated by $\mathscr{U}$.*

Definition. Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{U}$ be a quasi-uniformity on X. Then $\mathscr{U}$ is *compatible* if $\mathscr{T} = \mathscr{T}_{\mathscr{U}}$.

Definition. [13]. A filter $\mathscr{F}$ in a quasi-uniform space $(X, \mathscr{U})$ will be called a Cauchy filter if and only if for every $U \in \mathscr{U}$ there is a point $z \in X$ such that $U(z) \in \mathscr{F}$.

Definition [11]. A quasi-uniform space $(X, \mathscr{U})$ is *complete* if and only if every Cauchy filter on X has a cluster point.

Definition [13]. A quasi-uniform space $(X, \mathscr{U})$ is *precompact* if and only if for each $U \in \mathscr{U}$ there is a finite subset A in X such that $U(A) = X$.

Definition [3]. An open cover $\mathscr{C}$ of a topological space $(X, \mathscr{T})$ is a *Q-cover* provided that for each $x \in X$, $\bigcap\{C \in \mathscr{C} : x \in C\}$ is open.

Definition [3]. Let $(X, \mathscr{T})$ be a topological space and let $x \in A \in \mathscr{T}$. A *cover of X about* (x, A) is an open cover $\mathscr{C}$ of X such that there exists $C \in \mathscr{C}$ with $x \in C \subset A$.

Theorem 2.1 [3, Theorem 1]. *Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{A}$ be a collection of Q-covers of X such that for each $A \in \mathscr{T}$, $\mathscr{A}$ contains a cover of X about (x, A). For each $\mathscr{C} \in \mathscr{A}$ and each $x \in X$, let $A_x^{\mathscr{C}} = \bigcap\{C : x \in C \in \mathscr{C}\}$ and let $U_{\mathscr{C}} = \bigcup\{\{x\} \times A_x^{\mathscr{C}} : x \in X\}$. Let $\mathscr{B} = \{U_{\mathscr{C}} : \mathscr{C} \in \mathscr{A}\}$. Then β is a subbase for a compatible quasi-uniformity $\mathscr{U}_{\mathscr{A}}$ on X.*

If in the above theorem $\mathscr{A}$ is taken to be the collection of all finite open covers, then $\mathscr{U}_{\mathscr{A}}$ is called the *Pervin quasi-uniformity* and if $\mathscr{A}$ is taken to be the collection of all point-finite open covers, then $\mathscr{U}_{\mathscr{A}}$ is called the *point-finite covering quasi-uniformity.*

Theorem 2.2. *Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{U}$ be a compatible quasi-uniformity for $(X, \mathscr{T})$. There is a base β for $\mathscr{U}$ such that if $V \in \beta$ and $x \in X$, then $V(x) \in \mathscr{T}$.*

Proof. For each $V \in \mathscr{U}$, let $V^* = \bigcup\{\{x\} \times [\operatorname{int} V(x)] : x \in X\}$. Since $x \in \operatorname{int} V(x)$, V^* is a reflexive relation on X. Let $V \in \mathscr{U}$. There exists $W \in \mathscr{U}$ such that $W \circ W \subset V$. Let $(x, z) \in W^* \circ W^*$. Then there exists $y \in X$ such that $y \in \operatorname{int} W(x)$ and $z \in \operatorname{int} W(y)$. It follows that $W(y) \subset W \circ W(x) \subset V(x)$ so that $z \in \operatorname{int} W(y) \subset \operatorname{int} V(x) = V^*(x)$. Consequently $\beta = \{V^* : V \in \mathscr{U}\}$ is a base for a quasi-uniformily $\mathscr{U}^*$ on X. It is clear that if $V^* \in \beta$ and $x \in X$, then $V^*(x) = \operatorname{int} V(x) \in \mathscr{T}$. It remains to show that

$\mathscr{U}^* = \mathscr{U}$. Let $V \in \mathscr{V}$. Then for each $x \in X$, $V^*(x) \subset V(x)$. Thus $V^* \subset V$ and so $\mathscr{U} \subset \subset \mathscr{U}^*$. Let $U^* \in \beta$. There exist $U \in \mathscr{U}$ and $W \in \mathscr{U}$ such that $W \circ W \subset U$ and such that for each $x \in X$, $U^*(x) = \operatorname{int} U(x)$. Let $y \in W(x)$. Then $W(y) \subset W \circ W(x) \subset U(x)$. Thus for each $x \in X$, $W(x) \subset \operatorname{int} U(x) = U^*(x)$. It follows that $W \subset U^*$ and so $\mathscr{U}^* = \mathscr{U}$.

The following method of constructing a compatible quasi-uniformity for an arbitrary topological space is due to V. S. KRISHNAN who stated the theorem in terms of Fréchet V-spaces [9]. The theorem may be found stated and proved in terms of quasi-uniform spaces in [1].

Theorem 2.3 [1, Theorem 3.1]. *Let $(X, \mathscr{T})$ be a topological space and let U be the collection of all upper semi-continuous functions on X. For each $\varepsilon > 0$ and each $f \in U$, let $U_{(f,\varepsilon)} = \{(x, y) : x, y \in X$ and $f(y) - f(x) < \varepsilon\}$. Let $\mathscr{U}' = \{U_{(f,\varepsilon)} : f \in U, \varepsilon > 0\}$. Then $\mathscr{U}'$ is a subbase for a compatible quasi-uniformity on $(X, \mathscr{T})$.*

The compatible quasi-uniformity constructed in the above theorem is called the *upper semi-continuous quasi-uniformity for $(X, \mathscr{T})$.*

3. ALMOST-COMPACTNESS

A filter base in a topological space $(X, \mathscr{T})$ is an *open filter base* provided that each member of the filter base is open. An open filter is a filter which is generated by an open filter base. An open ultrafilter is a maximal element in the collection of all open filters of $(X, \mathscr{T})$. The following proposition concerning open ultrafilters may be found in [10].

Proposition 3.1. *Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{U}$ be an open filter on X. Then the following hold:*

(1) *$\mathscr{U}$ is an open ultrafilter on X if and only if for any open set G such that $G \cap U \neq \emptyset$ for lach $U \in \mathscr{U}$, then $G \in \mathscr{U}$.*

(2) *$\mathscr{U}$ is an open ultrafilter on X if and only if for any open set G such that $G \notin \mathscr{U}$, then $X - \bar{G} \in \mathscr{U}$.*

(3) *If $\mathscr{U}$ is an open ultrafilter, then p is a cluster point of $\mathscr{U}$ if and only if $\mathscr{U}$ converges to p.*

Definition [4 and 8]. A Hausdorff topological space $(X, \mathscr{T})$ is *almost-compact* (H-closed in the terminology of [8]) provided that if $\mathscr{C}$ is an open cover of X, then there is a finite subcollection $\{C_i : 1 \leqq i \leqq n\}$ of $\mathscr{C}$ such that $X = \bigcup_{i=1}^n \bar{C}_i$.

Definition. A quasi-uniform space $(X, \mathscr{U})$ is *almost complete* provided that every open Cauchy filter has a cluster point.

Definition. A quasi-uniform space $(X, \mathscr{U})$ is *almost precompact* provided that if $U \in \mathscr{U}$, then there is a finite subset F of X such that $X = \overline{U(F)}$.

It is obvious that every precompact quasi-uniform space is almost precompact and that every complete quasi-uniform space is almost complete. In particular it follows that every topological space has a compatible (almost) precompact quasi-uniformity, namely the Pervin quasi-uniformity [13, Theorem 1]; and it follows that every metacompact topological space has a compatible (almost) complete quasi-uniformity, namely the point finite covering quasi-uniformity [3, Corollary to Theorem 2].

Lemma 3.1. *A quasi-uniform space $(X, \mathscr{U})$ is almost precompact if and only if every open ultrafilter on X is a Cauchy filter.*

Proof. Let $(X, \mathscr{U})$ be an almost precompact quasi-uniform space, let $\mathscr{M}$ be an open ultrafilter and let $U \in \mathscr{U}$. By Theorem 2.2, there is $V \in \mathscr{U}$ such that $V \subset U$ and such that for each $x \in X$, $V(x) \in \mathscr{T}$. Since $(X, \mathscr{U})$ is almost precompact there is a finite subset F of X such that $X = \bigcup \overline{\{V(x) : x \in F\}} = \bigcup \{\overline{\operatorname{int} V(x)} : x \in F\}$. Suppose that for each $x \in F$, $\operatorname{int} V(x) \notin \mathscr{M}$. Then by Proposition 3.1, $\emptyset = X - \bigcup\{\overline{\operatorname{int} V(x)} : x \in F\} = \bigcap\{X - \overline{\operatorname{int} V(x)} : x \in F\} \in \mathscr{M}$ – a contradiction. It follows that $\mathscr{M}$ is a Cauchy filter.

Suppose that every open ultrafilter on X is Cauchy filter and that $(X, \mathscr{U})$ is not almost precompact. Then there is $U \in \mathscr{U}$ such that for each finite subset F of X, $X - \overline{U(F)} \neq \emptyset$. The collection $\{X - \overline{U(F)} : F$ is a finite subset of $X\}$, is an open filter base for an open filter $\mathscr{C}$ which is contained in an open ultrafilter $\mathscr{M}$. Then $\mathscr{M}$ is a Cauchy filter so that there is $x \in X$ such that $U(x) \in \mathscr{M}$ – a contradiction.

Theorem 3.1. *A Hausdorff topological space $(X, \mathscr{T})$ is almost-compact if and only if every compatible quasi-uniformity for $(X, \mathscr{T})$ is almost complete and almost precompact.*

The following theorem may be considered as a generalization of the Niemytzki-Tychonoff Theorem [12].

Theorem 3.2. *A Hausdorff space is almost compact if and only if it is almost complete with respect to every compatible quasi-uniformity.*

Example 3.1. Let $(X, \mathscr{T})$ be an almost-compact metacompact space which is not compact (such as [14, Example 6, Page 269]). Let $\mathscr{P}$ be the Pervin quasi-uniformity for $(X, \mathscr{T})$ and let $\mathscr{U}$ be the point finite covering quasi-uniformity for $(X, \mathscr{T})$. Then $(X, \mathscr{P})$ is an almost complete quasi-uniform space which is not complete and $(X, \mathscr{U})$ is an almost precompact quasi-uniform space which is not precompact.

Theorem 3.3. *Let $\{(X_\alpha, \mathscr{U}_\alpha) : \alpha \in A\}$ be a collection of quasi-uniform spaces, let $X = \prod\{X_\alpha : \alpha \in A\}$, and let $\mathscr{U}$ be the product quasi-uniformity on X. Then $(X, \mathscr{U})$ is almost precompact if and only if $(X_\alpha, \mathscr{U}_\alpha)$ is almost precompact for each $\alpha \in A$.*

Proof. Suppose first that $(X, \mathscr{U})$ is almost precompact. Let $\alpha \in \Lambda$ and let $V_\alpha \in \mathscr{U}_\alpha$. Since π_α is a quasi-uniformily continuous function, there is an entourage U of the product quasi-uniformity such that if $(x, y) \in U$, then $(\pi_\alpha(x), \pi_\alpha(y)) \in V_\alpha$. There is a finite subset F of X such that, $\overline{U(F)} = X$. Let $x_\alpha \in X_\alpha$ and let G_α be a $\mathscr{T}_{\mathscr{U}_\alpha}$-open set about x_α. Then $\pi_\alpha^{-1}(G_\alpha)$ is an open set about $x \in \pi_\alpha^{-1}(x_\alpha)$ in the product topology. Thus there is $z \in \pi_\alpha^{-1}(G_\alpha) \cap U(F)$. Then $\pi_\alpha(z) \in G_\alpha \cap [V_\alpha(\pi_\alpha(F))]$, so that $x_\alpha \in \overline{V_\alpha(\pi_\alpha(F))}$. Hence $X_\alpha = \overline{V[\pi_\alpha(F)]}$.

Now suppose that for each $\alpha \in \Lambda$, $(X_\alpha, \mathscr{U}_\alpha)$ is an almost precompact quasi-uniform space. Let $\mathscr{M}$ be an open ultrafilter on X. Let $\alpha \in \Lambda$. Then $\pi_\alpha(\mathscr{M})$ is an open filter on X_α. Let $G \in \mathscr{T}_{\mathscr{U}_\alpha} - \pi_\alpha(\mathscr{M})$. Then $\pi_\alpha^{-1}(G) \in \mathscr{T}_{\mathscr{U}} - \mathscr{M}$ so that $X - \overline{\pi_\alpha^{-1}(G)} \in \mathscr{M}$. Thus $X_\alpha - G = \pi_\alpha(X - \overline{\pi_\alpha^{-1}(G)}) \in \pi_\alpha(\mathscr{M})$. It follows that $\pi_\alpha(\mathscr{M})$ is an open ultrafilter on X_α. Thus for each $\alpha \in \Lambda$, $\pi_\alpha(\mathscr{M})$ is a Cauchy filter and so $\mathscr{M}$ is a Cauchy filter.

Theorem 3.4. *Let $\{(X_\alpha, \mathscr{U}_\alpha) : \alpha \in \Lambda\}$ be a collection of quasi-uniform spaces, let $X = \Pi\{X_\alpha : \alpha \in \Lambda\}$ and let $\mathscr{U}$ be the product quasi-uniformity on X. Then $(X, \mathscr{U})$ is almost complete if and only if $(X_\alpha, \mathscr{U}_\alpha)$ is almost complete for each $\alpha \in \Lambda$.*

Proof. Suppose that $(X, \mathscr{U})$ is almost complete. Let $\alpha \in \Lambda$ and let $\mathscr{F}_\alpha$ be an open Cauchy filter on X_α. For each $\beta \in \Lambda$ with $\beta \neq \alpha$ let $\mathscr{F}_\beta$ be an open Cauchy filter on X_β. Let $\mathscr{F} = \prod_{i \in \Lambda} \mathscr{F}_i$ be the product filter. Then $\mathscr{F}$ is an open Cauchy filter on X and so $\mathscr{F}$ has a cluster point $x \in X$. Then x_α is a cluster point of $\mathscr{F}_\alpha$.

Now suppose that for each $\alpha \in \Lambda$, $(X_\alpha, \mathscr{U}_\alpha)$ is an almost complete quasi-uniform space. Let $\mathscr{F}$ be an open Cauchy filter on X. Then $\mathscr{F}$ is contained in a Cauchy open ultrafilter $\mathscr{F}'$. For each $\alpha \in \Lambda$, $\pi_\alpha(\mathscr{F}')$ is an open Cauchy filter in X_α, and so for each $\alpha \in \Lambda$, $\pi_\alpha(\mathscr{F}')$ has a cluster point t_α. Choose one such t_α for each $\alpha \in \Lambda$ and let $t = (t_\alpha)$. For each open set N_α about t_α, $N_\alpha \cap F_\alpha \neq \emptyset$ and $F_\alpha \in \pi_\alpha(\mathscr{F}')$. Hence for each $F \in \mathscr{F}'$, $\pi_\alpha^{-1}(N_\alpha) \cap F \neq \emptyset$. Since $\pi_\alpha^{-1}(N_\alpha)$ is open and $\mathscr{F}'$ is an open ultrafilter, $\pi_\alpha^{-1}(N_\alpha) \in \mathscr{F}'$. Let U be an open neighborhood of t. Then there is a finite subset β of Λ such that $\bigcap_{\alpha \in \beta} \pi_\alpha^{-1}(N_\alpha) \subset U$. Consequently $\mathscr{F}'$ converges to t, so that t is a cluster point of $\mathscr{F}$.

Corollary. [2]. *The product of nonempty Hausdorff spaces is almost-compact if and only if each coordinate space is almost-compact.*

The next theorem is a generalization of a result due to Gál [6]. Its proof follows *mutatis mutandis* from the proof of Theorem 4.17 of [11].

Theorem 3.5. *Let $(X, \mathscr{T})$ be a regular space and let A be a dense subset of X. If every open filter on A has cluster point in X, then X is compact.*

Corollary. *Let $(X, \mathscr{U})$ be an almost precompact quasi-uniform space and let $(X^*, \mathscr{U}^*)$ be a regular completion of $(X, \mathscr{U})$. Then $(X^*, \mathscr{U}^*)$ is compact.*

Proof. Let $\mathscr{F}$ be an open filter on X. Then $\mathscr{F}$ is contained in an open ultrafilter $\mathscr{M}$ which is Cauchy, since $(X, \mathscr{U})$ is almost precompact. Let $\mathscr{M}^*$ be the extension of $\mathscr{M}$ to X^*. Then $\mathscr{M}^*$ is a Cauchy filter on X^* and so $\mathscr{M}^*$ converges. Hence $\mathscr{F}$ has a cluster point.

Let $(X, \mathscr{U})$ be a quasi-uniform space. For each $U \in \mathscr{U}$, let $W(U) = \{\operatorname{int} U(x) : x \in X\}$ and let $\alpha = \{W(U) : U \in \mathscr{U}\}$. Then α is a complete collection of open covers [4, Definition 2] if and only if $(X, \mathscr{U})$ is almost complete. Consequently, the following theorem is a restatement of [4, Lemma 2].

Theorem 3.6. *Every regular almost complete quasi-uniform space is complete.*

Every semi-regular Urysohn almost-compact space is compact [8]. Thus it seems reasonable to conjecture that Theorem 3.5, its corollary and Theorem 3.6 might remain true for Urysohn, semi-regular spaces.

4. ALMOST REALCOMPACT AND COUNTABLY ALMOST-COMPACT SPACES

Definition [4]. A topological space is said to be *countably almost-compact* (in the other terminology – *countably H-closed*) if $\bigcap \overline{\mathscr{U}} \neq \emptyset$ for every countable open filter base $\mathscr{U}$, or equivalently, if every countable open covering $\mathscr{C}$ on X has a finite subcollection $\{C_i : 1 \leqq i \leqq n\}$ such that $X = \bigcup_{i=1}^{n} \overline{C}_i$.

The proof of the following theorem is based upon the proof of analogous results in [1].

Theorem 4.1. *Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{U}$ be the upper semi-continuous quasi-uniformity for $(X, \mathscr{T})$. Then $(X, \mathscr{U})$ is almost precompact if and only if $(X, \mathscr{T})$ is countably almost-compact.*

Proof. Suppose that $(X, \mathscr{U})$ is almost precompact and let $\mathscr{C} = \{A_n : n \geqq 0\}$ be a countable open cover of X. Define $f : X \to R$ as follows: For each $x \in X$, let $f(x)$ be the least, non-negative integer n such that $x \in A_n$. Since for each positive integer n, $f^{-1}[(-\infty, n + 1)] = \bigcup_{i=1}^{n} A_i$, f is an upper semi-continuous function. Let $\varepsilon > 0$. Then $U_{(f,\varepsilon)} \in \mathscr{U}$ so there is a finite subset F of X such that $\overline{U_{(f,\varepsilon)}(F)} = X$. Let $x \in F$ such that if $y \in F$, then $f(x) \geqq f(y)$. Let n be the least non-negative integer so that $f(x) + \varepsilon < n + 1$. Then $X = \overline{U_{(f,\varepsilon)}(F)} = \overline{U_{(f,\varepsilon)}(x)} = \overline{f^{-1}[(-\infty, n + 1)]} = \overline{\bigcup_{i=1}^{n} A_i}$. Therefore $(X, \mathscr{T})$ is countably almost-compact.

Now suppose that $(X, \mathscr{T})$ is countably almost-compact and let $U \in \mathscr{U}$. There exist upper semi-continuous function f_i with $1 \leqq i \leqq n$ and $\varepsilon > 0$ such that $\bigcap_{i=1}^{n} U_{(f_i,\varepsilon)} \subset U$.

For each $x \in X$ and each integer i with $1 \leqq i \leqq n$ let t_x^i be a rational number in $(f_i(x), f_i(x) + \varepsilon)$. Let $\mathscr{C} = \{\bigcap_{i=1}^{n} f_i^{-1}[(-\infty, t_x^i)] : x \in X, \text{ and } 1 \leqq i \leqq n\}$. Then $\mathscr{C}$ is a countable open cover of X. Since $(X, \mathscr{T})$ is countably almost-compact, there is a finite subset $F = \{x_i : 1 \leqq i \leqq k\}$ of X such that $\{\bigcap_{i=1}^{n} f_i^{-1}[(-\infty, t_{x_1}^i)], \ldots, \bigcap_{i=1}^{n} f_i^{-1}[(-\infty, t_{x_k}^i)]\} \subset \subset \mathscr{C}$ and such that $\bigcup\{\overline{\bigcap_{i=1}^{n} f_i^{-1}[(-\infty, t_{x_1}^i)]}, \ldots, \overline{\bigcap_{i=1}^{n} f_i^{-1}[(-\infty, t_{x_k}^i)]}\} = X$. Let $x \in X$. Then there is an integer j with $1 \leqq j \leqq k$ such that $x \in \overline{\bigcap_{i=1}^{n} f_i^{-1}[(-\infty, t_{x_j}^i)]} \subset \subset \overline{U(x_j)} \subset \overline{U(F)}$. Thus $U(F) = X$ and so $(X, \mathscr{U})$ is almost precompact.

Definition [4]. A space $(X, \mathscr{T})$ will be called *almost realcompact* if the following condition is fulfilled.

If $\mathscr{U}$ is an open ultrafilter which does not have a cluster point, then for some countable subfamily $\mathscr{B}$ of $\mathscr{U}$, $\bigcap\{B : B \in \mathscr{B}\} = \emptyset$.

By Theorem 1 of [5], the upper semi-continuous quasi-uniformity of an almost realcompact space must be almost complete; and hence it is evident that every almost realcompact, countably almost-compact space is almost compact [4, Theorem 2]. This result, together with our Theorem 4.1, suggests that the following analogue of Shirota's Theorem might obtain:

Conjecture 4.2. *Let $(X, \mathscr{T})$ be a topological space in which every closed discrete subspace has nonmeasurable cardinal. A necessary and sufficient condition that $(X, \mathscr{T})$ has a compatible almost complete quasi-uniformity is that $(X, \mathscr{T})$ be almost realcompact.*

It is shown in [3] that every metacompact space has a compatible complete quasi-uniformity, and Z. Frolík has shown that every almost realcompact normal space is realcompact [4]. Thus if the above conjecture holds, it follows that every normal metacompact space in which every closed discrete subspace has nonmeasurable cardinal, is realcompact.[1])

R. Blair and S. Mrowka have observed that there exists a completely regular, almost realcompact space which is not realcompact and which has the property that

[1]) The authors have been informed of a counterexample to Conjecture 4.2 in a private communication from Troy Hicks and John Carlson of University of Missouri, Rolla, Missouri. Nevertheless it has recently been shown that every normal metacompact space in which every closed discrete subspace has nonmeasurable cardinal is realcompact [*W. Moran*, Measures on metacompact spaces, Proc. London Math. Soc. *20* (1970) 507—524]. Some of the results of this paper were obtained independently by John Carlson.

every closed discrete subspace has nonmeasurable cardinal. The upper semi-continuous quasi-uniformity of such a space must be complete. Nevertheless, in light of Shirota's Theorem, there does not exist a compatible complete uniformity for such a space.

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