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LIMITS OF APPROXIMATELY CONTINUOUS FUNCTIONS

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In the paper [1] it is proved that any function of the second class of Baire is the limit of a sequence of derivatives. But it does not follow from this proof that any such function is the limit of a sequence of bounded derivatives. In this paper it is proved that any function of the second class of Baire is the limit of a sequence of approximately continuous functions, consequently any such function is the limit of a sequence of bounded derivatives.

We denote by R the real line and if $M \subset R$ we denote by c_M the characteristic function of M.

At first we prove the following.

Lemma. *Let M be a G_δ and F_σ set. Let H be a G_δ set of measure zero. Let H contain all points of M which are not points of density of M and all points of $R - M$ which are not points of density of $R - M$. Let G be an open set, $G \supset H$.*

Then there exists an approximately continuous function φ such that $\{x \in R, \varphi(x) \neq c_M(x)\} \subset G - H$.

Proof. We put

$$E_1 = M \cap [H \cup (R - G)], \quad E_2 = (R - M) \cap [H \cup (R - G)],$$
$$N = R - (E_1 \cup E_2).$$

Then E_1, N, E_2 are disjoint sets, $E_1 \cup N \cup E_2 = R$ and E_1, E_2 are G_δ sets. Further

$$E_1 \cup N = R - E_2 = M \cup [(R - H) \cap G]$$
$$E_2 \cup N = R - E_1 = (R - M) \cup [(R - H) \cap G].$$

Hence it follows that $E_1 \cup N$, $E_2 \cup N$ are sets of the class M_5 (see [2]).

This implies that there exists an approximately continuous function φ such that

$$\begin{aligned} \varphi(x) &= 0 \quad \text{for} \quad x \in E_2 \\ 0 < \varphi(x) &< 1 \quad \text{for} \quad x \in N \\ \varphi(x) &= 1 \quad \text{for} \quad x \in E_1 \end{aligned}$$

(see lemma 12 in [2]).

This function φ clearly satisfies the statement of the lemma.

Theorem. *A function f (possibly infinite) defined on R is an element of the second class of Baire if and only if f is the limit of a sequence of approximately continuous functions.*

Proof. From the fact that every approximately continuous function is of the first class of Baire it follows that if f is the limit of a sequence of approximately continuous functions then f is an element of the second class of Baire.

Now let f be an element of the second class of Baire. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of functions of the first class of Baire such that

$$\lim_{n \to \infty} g_n = f$$

and

$$g_n = \sum_{k=1}^{m_n} c_{k,n} h_{k,n}$$

where $c_{k,n}$ are real numbers and $h_{k,n}$ is the characteristic function of a set $H_{k,n}$ which is at the same time G_δ and F_σ set (see [3]).

Let $H^*_{k,n}$ be the set of all points of $H_{k,n}$ which are not points of density of the set $H_{k,n}$ and all points of $R - H_{k,n}$ which are not points of density of $R - H_{k,n}$. We put

$$H^* = \bigcup_{k,n} H^*_{k,n} .$$

Then H^* is a set of measure zero. Let $H \supset H^*$ be a G_δ set of measure zero. Let G_l be open sets such that

$$G_l \supset G_{l+1} , \quad \bigcap_{l=1}^{\infty} G_l = H .$$

According to the lemma there exist approximately continuous functions $\varphi_{k,n}$ such that

$$\{x \in R,\ \varphi_{k,n}(x) \neq h_{k,n}(x)\} \subset G_n - H .$$

We put $f_n = \sum_{k=1}^{m_n} c_{k,n} \varphi_{k,n}$.

The functions f_n are clearly approximately continuous and the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f.

References

[1] *D. Preiss:* Limits of derivatives and Darboux-Baire functions, Rev. Roum. Math. Pures Appl. *14* (1969), 1201—1206.

[2] *Z. Zahorski:* Sur la première derivée, Trans. Amer. Math. Soc. *69* (1950), 1—54.

[3] *K. Kuratowski:* Topologie I, Warszava 1952.

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