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# SOME RELATIONS AMONG INVARIANTS OF GRAPHS

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## 1. INTRODUCTION

R. E. NETTLETON [2] investigated relations among number of vertices, connectivity, diameter, degrees of vertices and chromatic number of graphs. We shall use or improve his results to present a complete survey of the best possible bounds for the last two invariants in terms of the first three.

Our notation follows HARARY [1]. Particularly, we reserve: a letter $p$ for a number of points, $\varkappa$ for connectivity, $d$ for diameter, $\chi$ for chromatic number of a graph and $d(v)$ for a degree of a point $v$.

## 2. RESULTS

Nettleton has proved among others the following theorems:

N. 1. $\chi \geqq p/(p - \varkappa)$.
N. 2. *If for any positive integer* $j$, $d \geqq 2j + 1$, *then* $\chi \leqq p - 2(\varkappa j - \varkappa + 1)$.
N. 3. *If* $d = j + 1$ *for* $j \geqq 3$, *no point of* $G$ *can have degree* $> p - \varkappa(j - 3) - 3$.
N. 4. $d - 4 \leqq (p - \varkappa - 3)/\varkappa$.

Later on, M. E. WATKINS [3] improved N. 4. into the following best possible form: $p \geqq \varkappa(d - 1) + 2$.

We shall prove two theorems:

**Theorem 1.**

$$p/(p - \varkappa) \leqq \chi \leqq p - 1 \quad \textit{if} \quad d = 2 \tag{1a}$$

$$2 \leqq \chi \leqq p - \varkappa(d - 3) - 2 \quad \textit{if} \quad 3 \leqq d < \infty \tag{1b}$$

*and these bounds are the best possible.*

**Theorem 2.** *Given any point v one has*

(2a) $$\varkappa \leqq d(v) \leqq p - 1 \qquad \text{if} \quad d = 2\,,$$

(2b) $$\varkappa \leqq d(v) \leqq p - 2 \qquad \text{if} \quad d = 3\,,$$

(2c) $$\varkappa \leqq d(v) \leqq p - \varkappa(d - 4) - 3 \quad \text{if} \quad 4 \leqq d < \infty$$

*and these bounds are the best possible.*

These theorems show that N. 1. and N. 3. are best possible while N. 2. is not.

## 3. SOME SPECIAL GRAPHS

In this paragraph, we shall construct graphs employed as examples in proofs.

**1.** $G_1(p, \varkappa)$ where $\varkappa \leqq p - 2$ is a complete $(a + 1)$-partite graph $K(p - \varkappa, p - \varkappa, \ldots, p - \varkappa, b)$ where $p = a(p - \varkappa) + b$, $0 \leqq b < p - \varkappa$. It is easy to see that $G_1$ is $\varkappa$-connected, its diameter is 2 and its chromatic number $a$ (if $b = 0$), resp. $a + 1$ (if $b > 0$). In both cases, the last quantity is the least integer $\geqq p/(p - \varkappa)$.

**2.** $G_2(p, \varkappa)$ where $\varkappa \leqq p - 2$ has a set of points $V_1 \cup V_2 \cup \{w\}$ $V_1, V_2, \{w\}$ are pairwise disjoint, $|V_1| = p - \varkappa - 1$, $|V_2| = \varkappa$. A couple $uv$ is not a line of $G_2$ iff $u \in V_1$, $v = w$. $G_2$ is $\varkappa$-connected, its diameter is 2 and its chromatic number is $p - 1$.

**3.** $G_3(p, \varkappa, d)$ where $d \geqq 3$, $\varkappa(d - 1) \leqq p - 2$ has a set of points $V = V_0 \cup V_1 \cup \ldots \cup V_d$ where $|V_0| = |V_d| = 1$, $|V_1| = p - \varkappa(d - 2) - 2$, $|V_i| = \varkappa$ for $i = 2, 3, \ldots, d - 1$, $V_i \cap V_j = \emptyset$ for $i \neq j$. A couple $uv$ is a line of $G_3$ iff $u \in V_i$, $b \in V_j$, $|i - j| = 1$. $G_3$ is $\varkappa$-connected, has a diameter $d$ and a chromatic number 2.

**4.** $G_4(p, \varkappa, d)$ where $d \geqq 3$, $\varkappa(d - 1) \leqq p - 2$ has the same set of points as $G_3$. A couple $uv$ is a line of $G_4$ iff $u \in V_j$, $u \in V_j$, $|u - v| \leqq 1$. $G_4$ is $\varkappa$-connected, has a diameter $d$ and a chromatic number $p - \varkappa(d - 3) - 2$.

## 4. PROOFS

Proof of Theorem 1: The left side of (1a) is Nettleton's result N. 1., the right side of (1a) and the left side of (1b) are trivial. Now, let $G$ be a graph with set of points $V$ and a diameter $d \geqq 3$. There are $u, v \in V$ such that $d(u, v) = d$. Put $V_i = \{w \in V; d(u, w) = i\}$ for $i = 0, 1, \ldots, d$. A graph $G - V_i$ is disconnected whenever $1 \leqq i < d$. Hence, $|V_i| \geqq \varkappa$ and one can find $\{v_i^1, v_i^2, \ldots, v_i^\varkappa\} \subset V_i(1 \leqq i \leqq d)$. Write $v_0^1$ for $u$ and $v_d^1$ for $v$. Remark that $xy$ is not a line if $x \in V_i$, $y \in V_j$, $|i - j| \geqq 2$ and define color classes $O_1, O_2, \ldots, O_\varkappa, E_1, E_2, \ldots, E_\varkappa$ by

$$O_j = \{v_i^j;\ i \text{ odd}\}\,, \quad E_j = \{v_i^j;\ i \text{ even}\}\,.$$

Consider each of the remaining $p - (\varkappa(d - 1) + 2)$ vertices as a color class consisting of a single point. Altogether, $2\varkappa + p - (\varkappa(d - 1) + 2) = p - \varkappa(d - 3) + 2$ color classes were introduced and (1b) is proved.

To show that these bounds are best possible remember the result of Watkins and use the graphs $G_1(p, \varkappa)$, $G_2(p, \varkappa)$, $G_3(p, \varkappa, d)$ and $G_4(p, \varkappa, d)$ as examples.

Proof of Theorem 2: The left sides are trivial (cf. [1], Theorem 5.1) and so is the right side of (2a). The right side of (2b) follows by the fact that $d(v) = p - 1$ implies $d \leqq 2$. The right side of (2c) is Nettleton's result N. 3.

To see that the bounds are best possible consider graphs $G_2(p, \varkappa)$ and $G_4(p, \varkappa, d)$.

*References*


[1] *F. Harary:* Graph Theory, Addison-Wesley, Reading, Mass. 1969.

[2] *R. E. Nettleton:* Some Generalized Theorems on Connectivity, Canad. J. Math. *XII* (1960), 546—554.

[3] *M. E. Watkins:* A Lower Bound for the Number of Vertices of a Graph, Amer. Math. Monthly *74* (1967), 297.



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