Václav Zizler

Note on separation of convex sets

Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 2, 340–343

Persistent URL: `http://dml.cz/dmlcz/101025`

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

NOTE ON SEPARATION OF CONVEX SETS

VÁCLAV ZIZLER, Praha

(Received February 27, 1970)

A statement is proved concerning separation of two convex sets by two disjoint balls.

We work in real Banach spaces. $K_r(x)$ denotes the closed ball centered at x with radius r. By a ball we shall always mean a closed one. The set of all real numbers is denoted by R, X^* means the dual space of the Banach space X with the usual supremum-norm on $K_1(0) \subset X$. $S_r(x)$ denotes the norm boundary of $K_r(x) \subset X$. $K_r^*(0) = \{f \in X^*; \|f\| \leqq r\}$, $S_r^* = \{f \in X^*; \|f\| = r\}$.

Following R. R. PHELPS ([8]), we shall call $f \in S_1^*$ the strongly exposed point of K_1^* if f attains its norm at the point $x \in S_1$ which is a point of strong (Fréchet) differentiability of $\|x\|$ of X. The set of all strongly exposed points of K_1^* will be denoted by str K_1^*. For $A \subset X$, $\delta(A)$ denotes the norm boundary of A in X. For $f \in X^*$, $A \subset X$, $f(A) \leqq c$ means $f(y) \leqq c$ for $y \in A$.

Definition 1. Let K be a convex subset of X, $z \in \delta(K)$. $f \in X^*$ is said to be *the supporting functional* of K at z if either $f(k) \geqq f(z)$ for all $k \in K$ or $f(k) \leqq f(z)$ for all $k \in K$.

Lemma. *Let $0 \neq f \in X^*$ be a supporting functional of $K_r(0)$ at $x \in \delta(K_r(0))$. Take any $p > 0$ and $z \in Rx$. Then:*

1) *f is a supporting functional of $K_p(z)$ at both points $\delta(K_p(z)) \cap Rx$.*
2) *$0 \notin K_p(z)$ implies either $f(k) > 0$ for all $k \in K_p(z)$ or $f(k) < 0$ for all $k \in K_p(z)$.*

Proof. 1) First assume $z = 0$, $p > 0$. Then it is easy to verify that f is a supporting functional of $K_p(0)$ at $\pm(p/r)\,x$. In fact, take for instance the point $(p/r)\,x$. Assume, without a loss of generality, $f(y) \geqq f(x)$ for $y \in K_r(0)$. Then if $y \in K_p(0)$, it is $(r/p)\,y \in K_r(0)$ and therefore $f((r/p)\,y) \geqq f(x)$, i.e. $f(y) \geqq f((p/r)\,x)$.

Now, take $p > 0$, $\alpha \in R$, $z = \alpha x$. $\delta(K_p(z)) \cap Rx = (\alpha \pm p/r)\,x$. Take for instance $(\alpha + p/r)\,x$. Let $Ay = y - \alpha x$ for $y \in X$. Then $AK_p(\alpha x) = K_p(0)$. Since f is a supporting functional of $K_p(0)$ at $(p/r)\,x$ we have either $f(y - \alpha x) \geqq f((p/r)\,x)$ for each $y \in K_p(\alpha x)$ or $f(y - \alpha x) \leqq f((p/r)\,x)$ for each $y \in K_p(\alpha x)$. Hence either $f(y) \geqq f((p/r)\,x + \alpha x)$ for $y \in K_p(\alpha x)$ or $f(y) \leqq f((\alpha + p/r)\,x)$ for each $y \in K_p(\alpha x)$ which means f is a supporting functional of $K_p(\alpha x)$ at $(\alpha + p/r)\,x$.

2) If $f(x) = 0$ then for all $y \in K_r(0)$ we should have either $f(y) \leqq 0$ for $y \in K_r(0)$ or $f(y) \geqq 0$ for all $y \in K_r(0)$. Both cases are obviously impossible, therefore $Rx \cap \cap f^{-1}(0) = \{0\}$. Thus the fact $0 \notin K_p(z)$ is equivalent to the fact that $K_p(z) \cap Rx \cap \cap f^{-1}(0) = \emptyset$. Denote $\delta(K_p(z)) \cap Rx = \{v, w\}$. Suppose $f(v) > 0$. Then $f(w) > 0$, otherwise there exists $\alpha_0 x \in K_p(z)$, $f(\alpha_0 x) = 0$. Then $\alpha_0 x \in f^{-1}(0) \cap Rx \cap K_p(z)$ which is a contradiction with our assumption $0 \notin K_p(z)$. We have also $f(w) \neq f(v)$, since f is one-to-one on Rx. Assume without any loss of generality $f(w) > f(v)$. Since f is a supporting functional of $K_p(z)$ at v, we have $f(y) \geqq f(v) > 0$ for all $y \in K_p(z)$. Similarly for $f(v) < 0$. The following statement was motivated by the results of S. Mazur ([7]) and R. R. Phelps ([8]):

Proposition 1. *Assume X is a Banach space such that* str $K_1^* \neq \emptyset$. *Let K be a convex closed bounded subset of X, $f \in$* str K_1^* *so that* $\inf f(K) > 0$. *Then there exists a ball $B \subset X$, $B \supset K$ so that $f(B) > 0$.*

Proof. Let $x \in S_1$ be such that $f(x) = 1$, $\|x\|$ of X is strongly differentiable at x. Let us choose $\varepsilon > 0$ so that $\inf f(K) > 2\varepsilon > 0$. Take $z = \varepsilon x$. Now, following S. Mazur ([7]), take a system $\mathscr{K}$ of balls: $K_{(r-1)\varepsilon}(rz)$ for $r > 1$.

Then it is possible to prove ([7]) that, while $0 \notin K_{(r-1)\varepsilon}(rz)$ for all $r > 1$, there exists $r_0 > 1$ so that $K \subset K_{(r_0-1)\varepsilon}(r_0 z)$. Repeat, for completeness, this proof:

The first statement is obvious.

For the second one, suppose there exist sequences $\{r_n\}$ and $\{x_n\}$ so that $r_n > 1$, $r_n \to \infty$, $\|x_n - r_n z\| > (r_n - 1)\,\varepsilon$, $x_n \in K$ for each n. Denote $y_n = -x_n/\varepsilon r_n$. Then $y_n \to 0$. We have $\|x + y_n\| - \|x\| = D\|\,.\,\|(x, y_n) + \omega(y_n)$, $\omega(y_n)/\|y_n\| \to 0$ (where $D\|\,.\,\|(x, h)$ denotes the differential of $\|\,.\,\|$ and ω the remainder), since $\|\,.\,\|$ is strongly differentiable at x. We have

$$\left\|x - \frac{x_n}{\varepsilon r_n}\right\| - 1 = f(y_n) + \omega(y_n)$$

so that $\varepsilon r_n\, \omega(y_n) = \|x_n - r_n z\| - \varepsilon r_n + f(x_n) > (r_n - 1)\,\varepsilon - \varepsilon r_n + 2\varepsilon = \varepsilon$. Hence

$$\frac{\omega(y_n)}{\|y_n\|} = \frac{\varepsilon r_n\, \omega(y_n)}{\|x_n\|} > \frac{\varepsilon}{\|x_n\|} \not\to 0$$

since $\{x_n\}$ is bounded. Therefore we have a contradiction with Fréchet differentiability of $\|\,.\,\|$ at x. Thus there exists $r_0 > 1$ such that $K \subset K_{(r_0-1)\varepsilon}(r_0 z)$. Now, we may apply our lemma on $\mathscr{K}, f$ and see that since $0 \notin K_{(r_0-1)\varepsilon}(r_0 z)$ and $f(r_0 z) > 0$ we have $f(k) > 0$ for all $k \in K_{(r_0-1)\varepsilon}(r_0 z)$.

Corollary. *Suppose a Banach space X has the property that* str K_1^* *is a norm dense in S_1^*. K_1, K_2 be closed convex bounded subsets of X, one of them being weakly compact. Then there exist balls B_1, B_2 so that $B_i \supset K_i$, $i = 1, 2$, $B_1 \cap B_2 = \emptyset$.*

Proof. By the well known Separation Theorem ([3]) there exist $f \in S_1^*$, $\varepsilon > 0$, $c \in R$ so that $f(K_1) \leqq c - \varepsilon < c < f(K_2)$.

Take $c_1 = c - \frac{1}{2}\varepsilon$. Then $\sup f(K_1) < c_1 - \frac{1}{4}\varepsilon < c_1 + \frac{1}{4}\varepsilon < \inf f(K_2)$. We may choose $\tilde{f} \in \operatorname{str} K_1^*$ so that $\sup \tilde{f}(K_1) < c_1 < \inf \tilde{f}(K_2)$. First, consider K_2. Let $z \in X$ be such that $\tilde{f}(z) = c_1$. Then consider a translation $Ay = y - z$ for $y \in X$. Denote $\tilde{K}_2 = AK_2$. $\tilde{K}_2$ is a closed convex bounded set, $\inf \tilde{f}(\tilde{K}_2) > 0$. By our proposition there exists a ball $\tilde{B} \supset \tilde{K}_2$ so that $\tilde{f}(\tilde{B}) > 0$. $A^{-1}\tilde{B} = B$ is then a ball so that $B \supset K_2$, $\tilde{f}(B) > c_1$. Analogously, dealing with $-\tilde{f}(\in \operatorname{str} K_1^*)$ we may obtain a ball $B_1 \supset K_1$, $\tilde{f}(B_1) < c_1$. Therefore $B_1 \cap B_2 = \emptyset$.

In this connection, perhaps, the following fact is worth mentioning, too:

It is almost obvious that whenever $\|.\|$ is Gâteaux differentiable at $x_0 \in S_1 \subset X$ then the limit

$$\lim_{t \to 0} \frac{\|x_0 + th\| - \|x_0\|}{t} = D\|.\|(x_0, h)$$

is uniform on $h \in K$ where K is an arbitrary norm compact subset of X. To prove it (as for example N. A. Ivanov) [3a])) suppose this is not true for some compact $K \subset X$. Then there exist $t_n \to 0$, $h_n \in K$ such that whenever we write $\|x_0 + th\| - \|x_0\| = D\|.\|(x_0, th) + \omega(x_0, th)$, then

$$\left|\frac{\omega(x_0, t_n h_n)}{t_n}\right| \geqq \varepsilon > 0\,.$$

Without any loss of generality suppose $h_n \to h \in K$. Then

$$\begin{aligned}\left|\frac{\omega(x_0, t_n h)}{t_n}\right| &= \left|\frac{\omega(x_0, t_n h_n)}{t_n} + \frac{\|x_0 + t_n h\| - \|x_0 + t_n h_n\|}{t_n} + \right.\\ &\quad \left. + D\|.\|(x_0, h_n) - D\|.\|(x_0, h)\right| \geqq \\ &\geqq \left|\frac{\omega(x_0, t_n h_n)}{t_n}\right| - \left(\left|\frac{\|x_0 + t_n h\| - \|x_0 + t_n h_n\|}{t_n}\right| + \right.\\ &\quad \left. + \left|D\|.\|(x_0, h_n) - D\|.\|(x_0, h)\right|\right) \geqq \\ &\geqq \left|\frac{\omega(x_0, t_n h_n)}{t_n}\right| - (\|h_n - h\| + \|h_n - h\|) \geqq \\ &\geqq \frac{\varepsilon}{2}\end{aligned}$$

for $n \geqq n_0$ — a contradiction with Gâteaux differentiability of $\|.\|$ at x_0.

Definition 2. Call $f \in S_1^*$ the X-exposed point of K_1^* if there exists $x \in S_1$ such that $f(x) = 1$ and $\|x\|$ is Gâteaux differentiable at x. The set of all X-exposed points of K_1^* denote by $\exp_X K_1^*$.

Analogously to Proposition 1 we may derive:

Proposition 2. *Assume X is a Banach space such that $\exp_X K_1^* \neq \emptyset$. Let K be a compact convex subset of X, $f \in \exp_X K_1^*$ so that $\inf f(K) > 0$. Then there exists a ball $B \subset X$, $B \supset K$ such that $f(B) > 0$.*

Proof. Follow the proof of Proposition 1; put further $t_n = 1/\varepsilon r_n$, $h_n = -x_n$. Then we have $t_n \to 0$,

$$\left|\frac{\omega(t_n h_n)}{t_n}\right| \geqq \frac{\varepsilon t_n}{t_n} = \varepsilon$$

– a contradiction. Therefore, we again have

Corollary. *Suppose a Banach space X has the property that $\exp_X K_1^*$ is a norm dense on S_1^*, K_1, K_2 be two disjoint compact convex sets in X. Then there exist two balls $B_i \supset K_i$, $i = 1, 2$, $B_1 \cap B_2 = \emptyset$.*

As for the assumptions of our propositions we would like to remark the following:

First, the Bishop-Phelps Theorem ([2]) says that for every Banach space X the set C of all continuous linear functionals on X which attain their norms on $S_1 \subset X$ is norm-dense in X^*. Therefore if we suppose $\|x\|$ of X is Fréchet (Gâteaux) differentiable at every $x \in S_1$ we have immediately $\operatorname{str} K_1^* = C \cap S_1^*$ ($\exp_X K_1^* = C \cap S_1^*$). Thus our assumptions as for the density of strongly exposed (X-exposed) points of K_1^* are satisfied if $\|x\|$ of X is Fréchet (Gâteaux) differentiable on $S_1 \subset X$.

Acknowledgement. The author wishes to thank the University of Washington, Seattle especially to Professor V. L. Klee for the excellent working conditions and extremal hospitality during his stay there when this paper was prepared.

References

[1] *E. Asplund:* Frèchet differentiability of convex functions, Acta Mathem. *121* (1–2), (1968), 31–48.

[2] *E. Bishop, R. R. Phelps:* A proof that every Banach space is subreflexive, Bull. Am. Math. Soc. *67*, 79–98.

[3] *N. Dunford, J. T. Schwartz:* Linear operators. Part I, Interscience, New York 1966.

[3a] *Н. А. Иванов:* О дифференциалах Гато и Фреше, Успехи Матем. наук *10* (1955), 161–166.

[4] *M. I. Kadec:* On the spaces isomorphic to locally uniformly rotund spaces (Russian), Izvestija vysš. uč. zav. Matem. 1, (1959), 6, 51–57 and 1, (1961), 6, 186–187.

[5] *J. Lindenstrauss:* On operators which attain their norm, Israel Journal Math. *1* (1963), 139–148.

[6] *A. R. Lovaglia:* Locally uniformly convex Banach spaces, Trans. Am. Math. Soc. *78* (1958) 225–238.

[7] *S. Mazur:* Über schwache Konvergenz in den Räumen L^p, Studia Math. *4* (1933) 128–133.

[8] *R. R. Phelps:* A representation theorem for bounded convex sets, Proc. Am. Math. Soc. *11* (1960), 976–983.

[9] *V. L. Šmuljan:* Sur la structure de la sphere unitaire dans l'espace de Banach, Matem. Sbornik *9* (1941) 545–572.

Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU).