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HOMOGENEOUS DIGRAPHS

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At the Czechoslovak conference on graph theory in Smolenice in 1966 M. FIEDLER and V. KNICHAL suggested to study the so-called homogeneous graphs.

A homogeneous undirected graph is an undirected graph G satisfying the following conditions:

(α) To any two vertices u, v of G an automorphism φ of G exists such that $v = \varphi(u)$.

(β) If u is a vertex of G and $h_1, \ldots, h_k$ the edges incident with it, then to each permutation p of the number set $\{1, \ldots, k\}$ there exists an automorphism ψ_p of G such that $\psi_p(h_i) = h_{p(i)}$ for $i = 1, \ldots, k$.

A weaker concept, weakly homogeneous graphs, is studied in [1]. Analogously to the case of homogeneous undirected graphs also homogeneous digraphs can be defined.

A homogeneous digraph is a directed graph G satisfying the following conditions:

(α) To any two vertices u, v of G an automorphism φ of G exists such that $v = \varphi(u)$.

(β_1) If u is a vertex of G and $h_1, \ldots, h_k$ the edges outgoing from it, then to each permutation p of the number set $\{1, \ldots, k\}$ there exists an automorphism ψ_p of G such that $\psi_p(h_i) = h_{p(i)}$ for $i = 1, \ldots, k$.

(β_2) If u is a vertex of G and $e_1, \ldots, e_l'$ the edges incoming into it, then to each permutation q of the number set $\{1, \ldots, l\}$ there exists an automorphism ψ_q' of G such that $\psi_q'(e_i) = e_{q(i)}$ for $i = 1, \ldots, l$.

We shall try to investigate homogeneous digraphs with help of the concept of alternating connectivity introduced in [2]. From this paper we take also the symbols which will be used in the following.

In [2] the concepts of $(+-)$-connectivity and of $(-+)$-connectivity of two vertices of a digraph G are introduced. Each of the relations of being $(+-)$-connected and of being $(-+)$-connected is an equivalence on the vertex set of the digraph G. The equivalence class of the relation of being $(+-)$-connected or $(-+)$-connected which contains the vertex a is denoted by $C^{+-}(a)$ or $C^{-+}(a)$ respectively. If there exists an edge outgoing from a vertex of $C^{+-}(a)$ and incoming into a vertex of $C^{-+}(b)$, we say that the classes $C^{+-}(a)$ and $C^{-+}(b)$ are associated to each other. In [2] it is proved that in a digraph G without sources and sinks exactly one class $C^{-+}(b)$ is associated to any class $C^{+-}(a)$ and exactly one class $C^{+-}(a)$ is associated to any class $C^{-+}(b)$. If $C^{+-}(a)$ and $C^{-+}(b)$ are associated to each other, we denote

$$C(a, b) = C^{+-}(a) \cup C^{-+}(b), \quad C_1(a, b) = C^{+-}(a) \dot{-} C^{-+}(b),$$
$$C_0(a, b) = C^{+-}(a) \cap C^{-+}(b), \quad C_2(a, b) = C^{-+}(b) \dot{-} C^{+-}(a).$$

Now let $C(a, b)$ be one of the above defined classes. The subgraph of G formed by the vertices of $C(a, b)$ and the edges outgoing from $C^{+-}(a)$ (and at the same time incoming into $C^{-+}(b)$) will be denoted by $G(a, b)$.

Lemma 1. *A homogeneous digraph G with at least one edge contains no sources.*

Proof. Assume that there exists a source u in G. Let v be another vertex of G. There exists an automorphism φ of G such that $\varphi(u) = v$. Therefore v must be also a source. As v was chosen arbitrarily, all vertices of G must be sources. But if G contains an edge, its terminal vertex cannot be a source, which is a contradiction.

Lemma 2. *A homogeneous digraph G with at least one edge contains no sinks.*

Proof is analogous to that of Lemma 1.

Lemma 3. *Let $C(a, b)$, $C(c, d)$ be two above defined classes of a homogeneous digraph G. Then the graphs $G(a, b)$, $G(c, d)$ are isomorphic to each other.*

Proof. As G is homogeneous, there must exist an automorphism φ of G which maps a onto c. An automorphism of G evidently preserves the relations of being $(+-)$-connected and of being $(-+)$-connected. Thus φ maps $C(a, b)$ onto $C(c, d)$ and $G(a, b)$ onto $G(c, d)$ and these two subgraphs are isomorphic to each other.

Lemma 4. *Let G be a connected homogeneous digraph. G is alternatingly connected, if and only if in any set $C(a, b)$ we have $C_1(a, b) = C_2(a, b) = \emptyset$, thus $C(a, b) = C_0(a, b)$. G is not alternatingly connected, if and only if in any set $C(a, b)$ we have $C_0(a, b) = \emptyset$, $C_1(a, b) \neq \emptyset$, $C_2(a, b) \neq \emptyset$.*

Proof. Assume that $C_0(a, b) \neq \emptyset$, $C_1(a, b) \neq \emptyset$. Let $u \in C_0(a, b)$, $v \in C_1(a, b)$. We have $u \in C^{+-}(a)$, thus any edge outgoing from u must come into a vertex of $C^{-+}(b)$ and belong to $G(a, b)$. Since $u \in C^{-+}(b)$ as well, any edge incoming into u must go out from a vertex of $C^{+-}(a)$ and belong to $G(a, b)$, too. On the other hand,

$v \in C^{+-}(a)$, thus any edge outgoing from v must come into a vertex of $C^{-+}(b)$ and belong to $G(a, b)$. But $v \notin C^{-+}(b)$, thus no edge incoming into v goes out from a vertex of $C^{+-}(a)$ and belongs to $G(a, b)$. Let φ be an automorphism of G which maps u onto v. As any automorphism of G preserves the relations of being $(+-)$-connected and of being $(-+)$-connected, φ maps $C^{+-}(a)$ onto itself (as both u and v belong to it) and $C^{-+}(b)$ onto itself (because also the associated classes are evidently mapped onto associated classes by any automorphism of G). Thus φ maps $C(a, b)$ onto itself and also $G(a, b)$ onto itself. As G is homogeneous, u cannot be a source, therefore there exists at least one edge incoming into u. According to what was proved above, such an edge belongs to $G(a, b)$ and by φ it is mapped onto an edge incoming into v and also belonging to $G(a, b)$. But we have proved above that no edge incoming into v belongs to $G(a, b)$, thus we have obtained a contradiction. We have proved that at least one of the sets $C_0(a, b)$, $C_1(a, b)$ must be empty. Dually we can prove that at least one of the sets $C_0(a, b)$, $C_2(a, b)$ must be empty. Therefore either $C_0(a, b)$, or both $C_1(a, b)$ and $C_2(a, b)$ are empty.

At first consider the case $C_0(a, b) \neq \emptyset$. Then $C_1(a, b) = C_2(a, b) = \emptyset$ and $C_0(a, b) = C(a, b)$. As we have seen, any edge incoming into a vertex of $C_0(a, b)$ must go out from a vertex of $C^{+-}(a)$, thus also of $C_0(a, b)$; any edge outgoing from a vertex of $C_0(a, b)$ must come into a vertex of $C^{-+}(b)$, thus also of $C(a, b)$. Therefore any edge incident with a vertex of $C(a, b)$ is not incident with any vertex not belonging to $C(a, b)$. The graph generated by $C(a, b)$ is alternatingly connected [2], therefore also connected. Thus this subgraph is a connected component of G. As G is connected, it may contain only one connected component and the subgraph of G generated by $C(a, b)$ is equal to G. We have proved that G is alternatingly connected.

Now consider the case $C_0(a, b) = \emptyset$. In this case both $C_1(a, b)$, $C_2(a, b)$ are non-empty, because $C_1(a, b) = \emptyset$ would imply $C^{+-}(a) = C_0(a, b) \cup C_1(a, b) = \emptyset$ and $C_2(a, b) = \emptyset$ would imply $C^{-+}(b) = C_0(a, b) \cup C_2(a, b) = \emptyset$, which is both impossible since $a \in C^{+-}(a)$, $b \in C^{-+}(b)$. Each edge of $G(a, b)$ goes out from a vertex of $C_1(a, b)$ and comes into a vertex of $C_2(a, b)$. These two sets are disjoint, therefore $G(a, b)$ is a bipartite digraph. As $C_1(a, b) \neq \emptyset$, we have $C^{+-}(a) \dot{-} C^{-+}(b) \neq \emptyset$ and any element of this set is not $(+-)$-connected with a. Thus we have proved that G is not alternatingly connected. The inverse implication evidently follows from the above proved dichotomy.

Now we shall investigate homogeneous digraphs which are not alternatingly connected. Denote by $\mathfrak{C}(G)$ the system of all sets $C(a, b)$ in G.

Lemma 5. *Let G be a homogeneous digraph which is connected but not alternatingly connected. Then all non-empty intersections of two different sets of $\mathfrak{C}(G)$ have equal cardinalities.*

Proof. Any automorphism of G preserves the relations of being $(+-)$-connected and of being $(-+)$-connected. Thus any automorphism of G maps any set of $\mathfrak{C}(G)$ onto a set of $\mathfrak{C}(G)$ and also any non-empty intersection of two of these sets onto such an intersection. Let M_1, M_2 be two of such intersections. We have (according to [2]) $M_1 = C_1(a_1, b_1) \cap C_2(c_1, d_1)$, $M_2 = C_1(a_2, b_2) \cap C_2(c_2, d_2)$ for some vertices $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$, where $C(a_1, b_1) \neq C(c_1, d_1)$, $C(a_2, b_2) \neq C(c_2, d_2)$ and either $C(a_1, b_1) \neq C(a_2, b_2)$ or $C(c_1, d_1) \neq C(c_2, d_2)$. If $M_1 \cap M_2 \neq \emptyset$, then $C_1(a_1, b_1) \cap C_2(c_1, d_1) \cap C_1(a_2, b_2) \cap C_2(c_2, d_2) \neq \emptyset$, thus $C_1(a_1, b_1) \cap C_1(a_2, b_2) \neq \emptyset$, $C_2(c_1, d_1) \cap C_2(c_2, d_2) \neq \emptyset$. As $C_1(a_1, b_1) \subset C^{+-}(a_1)$, $C_1(a_2, b_2) \subset C^{+-}(a_2)$, we have $C^{+-}(a_1) \cap C^{+-}(a_2) \neq \emptyset$. As $C^{+-}(a_1)$ and $C^{+-}(a_2)$ are equivalence classes, $C^{+-}(a_1) = C^{+-}(a_2)$. But as the associated classes are uniquely determined, we have $C(a_1, b_1) = C(a_2, b_2)$. Analogously from the relation $C_2(c_1, d_1) \cap C_2(c_2, d_2) \neq \emptyset$ we can derive the equality $C(c_1, d_1) = C(c_2, d_2)$. But this is a contradiction with the assumption. Thus $M_1 \cap M_2 = \emptyset$ and we have proved that any two intersections described above are disjoint. Let us again have M_1, M_2 and choose $u \in M_1$, $v \in M_2$. The set M_1 and M_2 is the unique intersection of two sets of $\mathfrak{C}(G)$ containing u, v respectively. There exists an automorphism of G which maps u onto v; it must map M_1 onto M_2 and thus M_1 and M_2 have equal cardinalities. As M_1, M_2 were chosen arbitrarily, all of such intersections have equal cardinalities.

Now we shall construct a digraph $\tilde{G}$ assigned to G. Its vertices are the sets of $\mathfrak{C}(G)$. If $C(a, b)$, $C(c, d)$ are two sets of $\mathfrak{C}(G)$, there exists an edge outgoing from $C(a, b)$ and incoming into $C(c, d)$ in $\tilde{G}$ if and only if $C_2(a, b) \cap C_1(c, d) \neq \emptyset$.

Now we shall define the concept of the weakly homogeneous digraph analogously to the concept of the weakly homogeneous undirected graph which was introduced in [1].

A digraph G is called weakly homogeneous, if and only if the following conditions are satisfied:

(α) To any two vertices u, v of G an automorphism φ of G exists such that $v = \varphi(u)$.

(β) If u is a vertex of G and h_1, h_2 two of the edges outgoing from u, then there exists an automorphism ψ of G such that $\psi(h_1) = h_2$.

(γ) If u is a vertex of G and e_1, e_2 two of the edges incoming into u, then there exists an automorphism ψ' of G such that $\psi'(e_1) = e_2$.

Lemma 6. *Let G be a connected homogeneous digraph which is not alternatingly connected. Then $\tilde{G}$ is weakly homogeneous.*

Proof. As we have shown above, any set of $\mathfrak{C}(G)$ can be mapped onto another set of $\mathfrak{C}(G)$ by an automorphism of G. Now let $C(a, b)$, $C(c_1, d_1)$, $C(c_2, d_2)$ be sets of $\mathfrak{C}(G)$ and let there exist two edges h_1, h_2 in G whose common initial vertex is $C(a, b)$ and whose terminal vertices are consequently $C(c_1, d_1)$, $C(c_2, d_2)$. This means that $C_2(a, b) \cap C_1(c_1, d_1) \neq \emptyset$, $C_2(a, b) \cap C_1(c_2, d_2) \neq \emptyset$. As we have seen in the proof of the preceding lemma, there exists an automorphism φ of G which maps $C_2(a, b) \cap C_1(c_1, d_1)$ onto $C_2(a, b) \cap C_1(c_2, d_2)$. Let $\tilde{\varphi}$ be an automorphism of $\tilde{G}$ such that the image of any vertex of $\tilde{G}$ corresponds to the set of $\mathfrak{C}(G)$ which is the image of the set of $\mathfrak{C}(G)$ corresponding to the original vertex of $\tilde{G}$ in the mapping $\tilde{\varphi}$. Then $\tilde{\varphi}(h_1) =$

$= h_2$. Analogously we can proceed if the two edges of $\tilde{G}$ have the same terminal vertex. Therefore $\tilde{G}$ is weakly homogeneous.

Let us have an undirected bipartite graph H, let A, B be its dominating sets. The graph H will be called semihomogeneous, if and only if the following conditions are satisfied:

(α) If $u_1 \in A$, $u_2 \in A$, there exists an automorphism φ of H such that $\varphi(u_1) = u_2$.

(β) If $v_1 \in B$, $v_2 \in B$, there exists an automorphism φ' of H such that $\varphi'(v_1) = v_2$.

(γ) If u is a vertex of H and $h_1, \ldots, h_k$ the edges incident with it, then to each permutation p of the number set $\{1, \ldots, k\}$ there exists an automorphism ψ_p of H such that $\psi_p(h_i) = h_{p(i)}$ for $i = 1, \ldots, k$.

Lemma 7. *In a connected homogeneous digraph G which is not alternatingly connected any subgraph $G(a, b)$ considered as an undirected graph is a semihomogeneous bipartite graph.*

The validity of this lemma is evident.

Lemma 8. *Let G be a finite connected homogeneous digraph which is not alternatingly connected. In each subgraph $G(a, b)$ the number of sources (i. e. elements of $C_1(a, b)$) is equal to the number of sinks (i.e. elements of $C_2(a, b)$).*

Proof. The graph $\tilde{G}$ is weakly homogeneous, therefore it is regular. As it is finite, the indegrees of all vertices are equal to their outdegrees. Let the indegree of any vertex be r. Thus to any set $C(a, b)$ of $\mathfrak{C}(G)$ there exist exactly r sets $C(c, d) \in \mathfrak{C}(G)$ such that $C_2(a, b) \cap C_1(c, d) \neq \emptyset$. As we have shown above, these intersections are pairwise disjoint and have equal cardinalities. Let the cardinality of any of these intersections be s. Each element of $C_2(a, b)$ is contained in some $C_1(c, d)$, thus the cardinality of $C_2(a, b)$ is rs. Analogously we can prove that the cardinality of $C_1(a, b)$ is also rs.

Lemma 9. *Let G be a connected homogeneous digraph which is not alternatingly connected. If $C(a, b)$, $C(c, d)$ are two sets of $\mathfrak{C}(G)$ such that $C_2(a, b) \cap C_1(c, d) \neq \emptyset$, then either from any vertex of $C_1(a, b)$ at most one edge goes into a vertex of $C_2(a, b) \cap C_1(c, d)$ or any vertex of $C_1(a, b)$ is joined with all vertices of $C_2(c, d)$.*

Proof. Let two edges go from a vertex $u \in C_1(a, b)$ into vertices of $C_2(a, b) \cap$ $\cap\, C_1(c, d)$, let their terminal vertices be v, w. At first assume that all edges outgoing from u have their terminal vertices in $C_2(a, b) \cap C_1(c, d)$. As G is homogeneous and each each of its automorphisms maps an intersection of two sets of $\mathfrak{C}(G)$ again onto an intersection of two sets of $\mathfrak{C}(G)$, any vertex of G has the property that any two edges outgoing from it have their terminal vertices in the same intersection of two sets of $\mathfrak{C}(G)$. Let N be the set of initial vertices of edges whose terminal vertices are in $C_2(a, b) \cap C_1(c, d)$. Evidently $u \in N$, $N \subset C_1(a, b)$ and all edges outgoing from vertices of N have their terminal vertices in $C_2(a, b) \cap C_1(c, d)$. Let $P = [u_1, e_1, v_1, h_1, u_2, \ldots, u_{k-1}, e_{k-1}, v_{k-1}, h_{k-1}, u_k]$ be a $(+-)$-path such that $u_1 = u$. From the facts mentioned above it follows that $u_i \in N$ for $i = 1, \ldots, k$, $v_i \in C_2(a, b) \cap \cap C_1(c, d)$ for $i = 1, \ldots, k-1$. Therefore $C^{+-}(u) \subset N$. But as $N \subset C_1(a, b) = = C^{+-}(a)$, we have $C^{+-}(u) \subset C^{+-}(a)$ and thus $C^{+-}(u) = C^{+-}(a)$, because $C^{+-}(u)$, $C^{+-}(a)$ are equivalence classes. We obtain $C^{+-}(a) \subset N \subset C^{+-}(a)$, which means $N = C^{+-}(a)$. This implies $C_2(a, b) \cap C_1(c, d) = C_2(a, b)$, i.e. $C_2(a, b) \subset \subset C_1(c, d)$. In this case $C_2(a, b)$ has a non-empty intersection with no other set of $\mathfrak{C}(G)$ than $C(c, d)$ and the outdegree of $C(a, b)$ in $\tilde{G}$ is 1. As $\tilde{G}$ is regular, it must be a cycle or a two-way infinite arc.

Now assume that there exists at least one edge outgoing from u whose terminal vertex is not in $C_2(a, b) \cap C_1(c, d)$. Let $h_1, \ldots, h_k$, $k \geqq 3$, be the edges outgoing from u; without any loss of generality assume that terminal vertices of h_1 and h_2 are v and w (therefore they are in $C_2(a, b) \cap C_1(c, d)$) and the terminal vertex of h_3 is not in $C_2(a, b) \cap C_1(c, d)$. Let p be a permutation of the number set $\{1, \ldots, k\}$ such that $p(1) = 1$, $p(2) = 3$, $p(3) = 2$, $p(i) = i$ for $i = 4, \ldots, k$. There exists an automorphism ψ_p of G such that $\psi_p(h_i) = h_{p(i)}$ for $i = 1, \ldots, k$. Thus $\psi_p(h_1) = h_1$, $\psi_p(h_2) = h_3$, $\psi_p(h_3) = h_2$. The edges h_1, h_2 have terminal vertices in the same intersection of two sets of $\mathfrak{C}(G)$. As these intersections are pairwise disjoint and any automorphism of G maps any of them again onto some of them, the images h_1, h_3 of the edges h_1, h_2 in ψ_p must be contained in the same intersection of two sets of $\mathfrak{C}(G)$, which is a contradiction.

Lemma 10. *If in a connected homogeneous digraph G which is not alternatingly connected a subgraph $G(a, b)$ is a complete bipartite graph (considered as an undirected graph), then G is either a cycle or a two-way infinite arc.*

This is an immediate consequence of Lemma 9.

Lemma 11. Let G be a connected homogeneous digraph which is not alternatingly connected. If $C(a, b)$, $C(c, d)$ are two sets of $\mathfrak{C}(G)$ such that $C_2(a, b) \cap C_1(c, d) \neq \emptyset$, then either at most one edge comes from a vertex of $C_2(a, b) \cap C_1(c, d)$ into any vertex of $C_2(c, d)$, or any vertex of $C_2(c, d)$ is joined with all vertices of $C_1(a, b)$.

Proof is dual to that of Lemma 9.

The results obtained will be summarized in a theorem.

Theorem 1. *Let G be a connected homogeneous digraph which is not alternatingly connected. Then the digraph G has the following properties:*

(A) G is the union of a system $\mathfrak{G}$ of pairwise isomorphic connected bipartite subgraphs (i.e. digraphs whose vertex sets consist only of sources and sinks). Each of these subgraphs is semihomogeneous (considered as an undirected graph) and its set of sources has the same cardinality as its set of sinks.

(*B*) *Any two different digraphs of the system* $\mathfrak{G}$ *have disjoint source sets and disjoint sink sets.*

(*C*) *All non-empty intersections of source sets of graphs of* $\mathfrak{G}$ *with sink sets of these graphs have the same cardinality.*

(*D*) *Either from a source of one graph of* $\mathfrak{G}$ *at most one edge goes into a source of another graph of* $\mathfrak{G}$, *or any sink set of a graph of* $\mathfrak{G}$ *is a source set of another and vice versa.*

(*E*) *Either into a sink of one graph of* $\mathfrak{G}$ *at most one edge comes from a sink of another graph of* $\mathfrak{G}$, *or any source set of a graph of* $\mathfrak{G}$ *is a sink set of another and vice versa.*

(*F*) *Let* $\tilde{G}$ *be a digraph whose vertices are the digraphs of* $\mathfrak{G}$ *such that from a vertex* u *into a vertex* v *an edge goes if and only if the sink set of the graph corresponding to* u *has a non-empty intersection with the source set of the graph of* $\mathfrak{G}$ *corresponding to* v. *Then* $\tilde{G}$ *is weakly homogeneous.*

Now we shall prove some constructive theorems. At first we shall give the definition of the strongly homogeneous digraph.

A strongly homogeneous digraph is a directed graph G satisfying the following conditions:

(α) To any two vertices u, v of G an automorphism φ of G exists such that $v=\varphi(u)$.

(β) If u is a vertex of G and $h_1, \ldots, h_k$ the edges outgoing from it, $e_1, \ldots, e_l$ the edges incoming into it, then to each pair $[p, q]$, p being a permutation of the number set $\{1, \ldots, k\}$ and q a permutation of the number set $\{1, \ldots, l\}$ there exists an automorphism ψ_{pq} of G such that $\psi_{pq}(h_i) = h_{p(i)}$, $\psi_{pq}(e_j) = e_{q(j)}$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$.

Theorem 2. *Let a strongly homogeneous connected digraph* G_0 *be given, let the indegrees and the outdegrees of its vertices be equal to* r. *Then there exists a connected homogeneous digraph* G *which is not alternatingly connected such that* $\tilde{G} \cong G_0$, *any graph of* $\mathfrak{G}$ *has* r *sources and* r *sinks and from any of its sources edges go into all its sinks.*

Proof. To any vertex u of G_0 we assign a graph $G(u)$ of $\mathfrak{G}$ described above. In any $G(u)$ we choose a one-to-one correspondence between its sinks and the edges outgoing from u in G_0 and a one-to-one correspondence between its sources and the edges incoming into u in G_0. After identifying the vertices assigned to the same edges we obtain the graph required. Now let φ_0 be an automorphism of G_0. Let us define the mapping of the vertex set of G onto itself so that the image of a vertex a of G is the vertex corresponding to the edge of G_0 which is the image of the edge of G_0 corresponding to the vertex a. In G, an edge goes from a vertex a into a vertex b if and only if the terminal vertex of the edge of G_0 corresponding to a coincides with the initial vertex of the edge of G_0 corresponding to b. Therefore the defined mapping induces an automorphism φ of the graph G. Let a, b be two sources of the same subgraph of G. The edges of G_0 corresponding to these vertices have the same terminal vertex, therefore there exists an automorphism of G_0 which maps one of them onto another. The corresponding automorphism of G maps a onto b. We proceed analogously if a and b are two sinks of the same graph of G. Now let u, v be two vertices of G_0. There exists an automorphism φ_0 of G_0 such that $\varphi_0(u) = v$. The automorphism φ_0 maps all edges incoming into or outgoing from u onto the edges incoming into v or outgoing from v respectively, thus the corresponding mapping φ of the vertex set of G onto itself maps the sources or sinks of $G(u)$ onto sources or sinks respectively of $G(v)$. Thus if c is a source of $G(u)$ and d a source of $G(v)$, the vertex c is mapped by this automorphism onto a source of $G(v)$ and this source can be mapped by another automorphism onto d. Now let a be a vertex of G, $h_1, \ldots, h_r$ the edges outgoing from it, $b_1, \ldots, b_r$ their terminal vertices. Let $e(a), e(b_1), \ldots, e(b_r)$ be the edges of G_0 corresponding to the vertices $a, b_1, \ldots, b_r$. The terminal vertex of $e(a)$ is the initial vertex of all $e(b_i)$ for $i = 1, \ldots, r$. Let p be a permutation of the number set $\{1, \ldots, r\}$, let q be an identical permutation of this set. Let us take the automorphism ψ^0_{pq} of G_0. We have $\psi^0_{pq}(e(a)) = e(a)$, $\psi^0_{pq}(e(b_i)) = b_{p(i)}$ for $i = 1, \ldots, r$. In the corresponding mapping ψ_{pq} of G we have $\psi_{pq}(a) = a$, $\psi_{pq}(b_i) = b_{p(i)}$, thus $\psi_{pq}(h_i) = h_{p(i)}$. Analogously we proceed for edges incoming into u. We have proved that G is homogeneous.

Theorem 3. *Let H be a homogeneous bipartite undirected graph, let k be an even positive integer. Then there exists a connected homogeneous digraph G which is not alternatingly connected such that any graph of $\mathfrak{G}$ (considered as an undirected one) is isomorphic to H, any source set of a graph of $\mathfrak{G}$ is a sink set of another one and vice versa and $\tilde{G}$ is a cycle with k vertices.*

Proof. Let A, B be the two dominating sets of H, $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_r\}$. (As H is homogeneous, the sets A and B must have the same cardinality r.) Denote $l = \frac{1}{2}k$, which is an integer. The vertex set of the graph G is the union of k pairwise disjoint sets $A^{(1)}, \ldots, A^{(l)}$, $B^{(1)}, \ldots, B^{(l)}$ such that $A^{(i)} = \{a_1^{(i)}, \ldots, a_r^{(i)}\}$, $B^{(i)} = \{b_1^{(i)}, \ldots, b_r^{(i)}\}$ for $i = 1, \ldots, l$. There exist edges $\overrightarrow{a_m^{(i)}b_n^{(i)}}$, $\overrightarrow{b_n^{(i-1)}a_m^{(i)}}$ in G ($1 \leqq m \leqq r$, $1 \leqq n \leqq r$), if and only if there exists an edge $a_m b_n$ in H (for $i = 1, \ldots, l$). The difference $i - 1$ is taken modulo l. No other edges exist in G. Let $\alpha_{mn}, \beta_{mn}, \gamma_{mn}$ for $m = 1, \ldots, r$, $n = 1, \ldots, r$ be automorphisms of H such that $\alpha_{mn}(a_m) = a_n$, $\beta_{mn}(b_m) = b_n$, $\gamma_{mn}(a_m) = b_n$. To these automorphisms of H the automorphisms $\bar{\alpha}_{mn}, \bar{\beta}_{mn}, \bar{\gamma}_{mn}$ of G are assigned so that $\alpha_{mn}(u) = v$ implies $\bar{\alpha}_{mn}(u^{(i)}) = v^{(i)}$, $\beta_{mn}(u) = v$ implies $\bar{\beta}_{mn}(u^{(i)}) = v^{(i)}$, $\gamma_{mn}(u) = v$ implies $\bar{\gamma}_{mn}(u^{(i)}) = v^{(i)}$ for $i = 1, \ldots, l$. (If u is a vertex of H, we shall speak about the corresponding vertex $u^{(i)}$ in G for $i = 1, \ldots, l$ so that if $u = a_m$, then $u^{(i)} = a_m^{(i)}$, if $u = b_m$, then $u^{(i)} = b_m^{(i)}$ for any m.) Further let η be an automorphism of G such that $\eta(u^{(i)}) = u^{(i+1)}$ for $i = 1, \ldots, l$ (the sum $i + 1$ being taken modulo l). If we have two vertices $a_m^{(i)}$, $a_n^{(j)}$ of G, then $a_n^{(j)} = \eta^{j-i}\alpha_{mn}(a_m^{(i)})$.

If we have two vertices $b_m^{(i)}$, $b_n^{(j)}$ of G, then $b_n^{(j)} = \eta^{j-i}\beta_{mn}(b_m^{(i)})$. Finally, if we have two vertices $a_m^{(i)}$, $b_n^{(j)}$ of G, then $b_n^{(j)} = \eta^{j-i}\gamma_{mn}(a_m^{(i)})$. Let $a_m \in A$, let $h_1, \ldots, h_s$ be the edges incident with a_m in H. If $h_t = a_m b_n$ for some t, n, then $h_t^{(i)} = \overrightarrow{a_m^{(i)} b_n^{(i)}}$, $e_t = \overrightarrow{b_n^{(i-1)} a_m^{(i)}}$ in G. If p is some permutation of the number set $\{1, \ldots, s\}$, let ψ_p be an automorphism of H such that $\psi_p(h_i) = h_{p(i)}$ for $i = 1, \ldots, s$. Let us define $\bar{\psi}_p$ analogously to $\bar{\alpha}_{mn}$, $\bar{\beta}_{mn}$, $\bar{\gamma}_{mn}$. Then $\bar{\psi}_p$ is an automorphism of G and $\bar{\psi}_p(h_i) = h_{p(i)}$, $\bar{\psi}_p(e_i) = e_{p(i)}$ for $i = 1, \ldots, s$. We proceed analogously for a vertex $b_m \in B$. The digraph G is homogeneous, q.e.d.

Theorem 4. *Let k, r be positive integers. Then there exists a connected strongly homogeneous digraph G which is not alternatingly connected and such that any graph of $\mathfrak{G}$ is (considered as an undirected one) isomorphic to a complete bipartite graph K_{rr}, the source set of any graph of $\mathfrak{G}$ is the sink set of another one and vice versa and $\tilde{G}$ is a cycle with k vertices.*

Remark. The vertex set of the graph K_{rr} is the union of two disjoint sets A and B, both having the cardinality r, and any vertex of A is joined by an edge with any vertex of B.

Proof. The vertex set of G is the union of k pairwise disjoint sets $A^{(1)}, \ldots, A^{(k)}$. We have $A^{(i)} = \{a_1^{(i)}, \ldots, a_r^{(i)}\}$. The edges of G are $\overrightarrow{a_m^{(i)} a_n^{(i+1)}}$ for $1 \leqq m \leqq r$, $1 \leqq \leqq n \leqq r$ and $i = 1, \ldots, k$. (The sum $i + 1$ is taken modulo k.) No other edge is contained in G. If p is some permutation of the number set $\{1, \ldots, r\}$ and $1 \leqq i \leqq k$, let $\varphi_p^{(i)}$ be an automorphism of G such that $\varphi_p^{(i)}(a_m^{(i)}) = a_{p(m)}^{(i)}$ and $\varphi_p^{(i)}(a_m^{(j)}) = a_m^{(j)}$ for $j \neq i$ and $m = 1, \ldots, r$. Further let η be an automorphism of G such that $\eta(a_m^{(i)}) = = a_m^{(i+1)}$ for $m = 1, \ldots, r$, $i = 1, \ldots, k$. (The sum $i + 1$ is again taken modulo k.) Thus for two vertices $a_m^{(i)}$, $a_n^{(j)}$ of G we have $a_n^{(j)} = \eta^{j-i}\varphi_p^{(i)}(a_m^{(i)})$ where p is a permutation of the number set $\{1, \ldots, r\}$ such that $p(m) = n$. Now let us have a vertex $a_m^{(i)}$. The edges outgoing from it are $h_n = \overrightarrow{a_m^{(i)} a_n^{(i+1)}}$ for $n = 1, \ldots, r$, the edges incoming into it are $e_n = \overrightarrow{a_n^{(i-1)} a_m^{(i)}}$ for $n = 1, \ldots, r$. If p, q are two permutations of the number set $\{1, \ldots, r\}$, then let ψ_{pq} be the automorphism of G such that $\psi_{pq}(a_n^{(i+1)}) = a_{p(n)}^{(i+1)}$, $\psi_{pq}(a_n^{(i-1)}) = a_{q(n)}^{(i-1)}$ for $n = 1, \ldots, r$ and all other vertices of G are fixed in ψ_{pq}. The mapping ψ_{pq} evidently is an automorphism of G, thus G is strongly homogeneous.

The following theorems are evidently true.

Theorem 5. *Let H be a connected homogeneous undirected graph. By substituting each undirected edge of H by two directed edges joining the same vertices and having opposite directions we obtain an alternatingly connected homogeneous digraph.*

Theorem 6. *Let G be a homogeneous digraph which is not connected. Then any of its connected components is a homogeneous digraph and any two of them are isomorphic.*

In the end we shall mention some examples of homogeneous undirected graphs.

1. The complete graph with n vertices for any positive integer n.

2. The complete bipartite graph K_{nn} for any positive integer n.

3. The circuit with n vertices for any positive integer $n \geqq 2$.

4. The graph Q_n of the n-dimensional cube for any positive integer n.

5. The graph of the regular dodecahedron.

6. The graph of the simplest finite projective plane (with seven points and seven straight lines). The vertices of the graph are points and straight lines of the plane, a point is joined by an edge with a line if and only if they are incident in the plane. This graph is on Fig. 1.

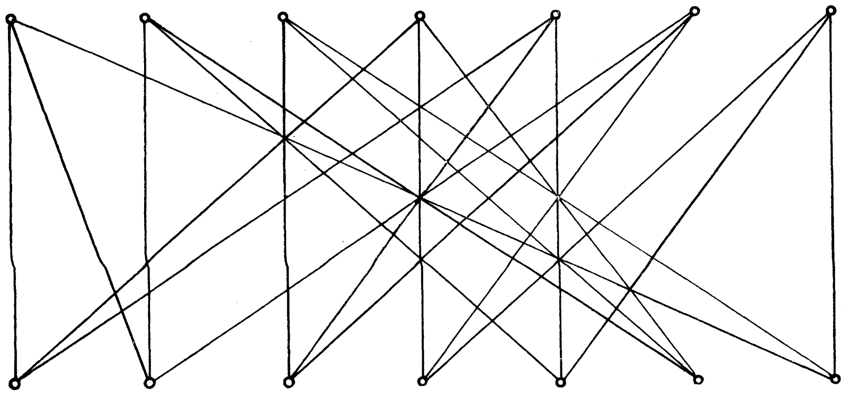

Fig. 1.

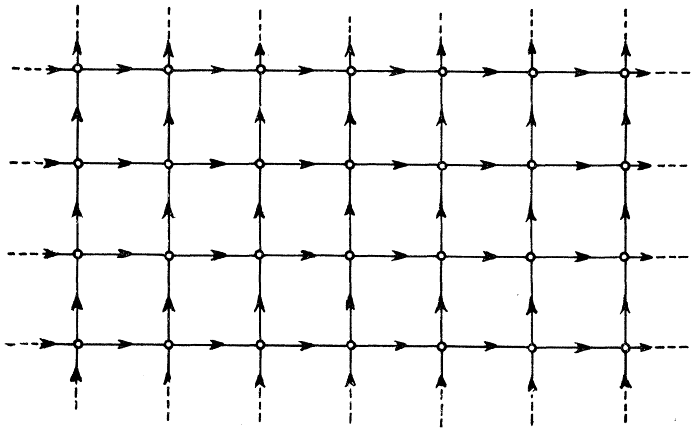

Fig. 2.

The graphs sub 2), sub 3) at even n, sub 4) and sub 6) are bipartite.

Problem 1. Does there exist a semihomogeneous bipartite graph both dominating sets of which have equal cardinalities and which is not homogeneous?

Problem 2. Does there exist a strongly homogeneous alternatingly connected digraph?

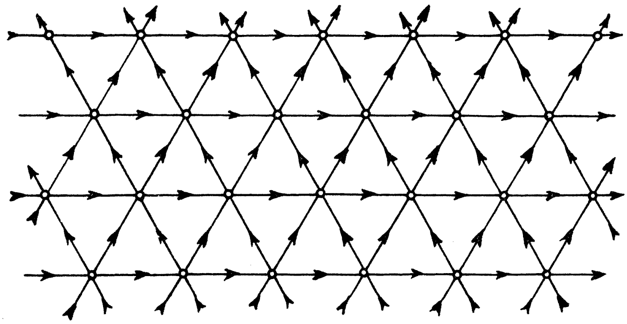

Fig. 3.

On Figs. 2 and 3 two examples of infinite homogeneous digraphs are shown.

References

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