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SOME EXPLICIT CONDITIONS FOR MAXIMAL LOCAL DIFFUSIONS IN ONE-DIMENSIONAL CASE

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The notions of maximal and strongly maximal matrix functions were defined in article [1] (see also Definition 1 in this article). The general criteria for maximality and strong maximality are given by Theorems 1 and 2 in [1] (Theorem 2 is also reformulated as Conclusion 1 in this article). These Theorems are valid if condition (A) given in [1] is fulfilled and in [1] there are also given more explicit assumptions under which condition (A) is fulfilled (see Lemma 4 and 5 in [1]). Due to recent results these assumptions can be simplified. Theorem 1 of this article concerns to this subject. The assumptions of Theorems 1 and 2 from [1] have no explicit form since we need to solve some parabolic equation and we must decide if the solution is convex in spatial variables. Only Theorem 3 from [1] (reformulated here as Theorem 2) has an explicit form, but the nonstochastic part $a(t, x)$ of Ito stochastic equation has to be linear in x. We generalize this result to include nonlinear $a(x)$ in one-dimensional case – see Theorems 3 and 4. Considering all this a question arises what conditions on $a(x)$ are needed at all. Theorem 5 shows that even to constant $b(x)$ there are $a(x)$ such that $b(x)$ is not maximal with respect to $a(x)$. In § 8 it is made clear that Theorem 5 expresses some necessary condition. Nevertheless there is some gap between sufficient conditions of Theorems 3,4 and the necessary condition of Theorem 5.

Example 1 in [1] shows that the unit matrix is not strongly maximal with respect to $a(t, x, y) \equiv 0$ and with respect to $Q = (0, L) \times D$ where D is a square. A far going generalization of this result is Theorem 6.

1. Definitions and notations. Let R_n denote the n-dimensional Euclidean space with a norm $|.|$. Let $Q = (0, L) \times D$ be a given region in R_{n+1} where L is a positive number and D is a region in R_n. $\bar{D}$ denotes the closure of D and $\dot{D}$ the boundary of D. Put $S = \langle 0, L\rangle \times \dot{D}$. Let Ω be a set, $\mathscr{F}$ a σ-field of subsets of Ω and P a probability measure on $\mathscr{F}$. Random variables and processes may be considered as $\mathscr{F}$-measurable functions on Ω. We assume that the structure of Ω, $\mathscr{F}$, P enables us to express every random process as an $\mathscr{F}$-measurable function on Ω.

Let $a(t, x)$ be a vector function and $B(t, x)$ an $n \times n$ matrix function defined on $\bar{Q}$. Denote by $w(t)$ a vector Wiener process ($Ew(t) = 0$, $E|w(t)|^2 = t$, where E is the mathematical expectation). We shall consider Ito stochastic equation

$$\mathrm{d}x = a(t, x)\,\mathrm{d}t + B(t, x)\,\mathrm{d}w(t)\,. \tag{1,1}$$

First we recall the concept of a solution $x(t)$ of (1,1) with an adhesive barrier S which was defined in [1]. We extend the domain of definition of $a(t, x)$, $B(t, x)$ onto the whole $\langle 0, L \rangle \times R_n$ and we find the solution $x^*(t)$ of the extended equation fulfilling $x^*(0) = x(0)$. We put $x(t) = x^*(t)$ for $t \leqq \tau$ and $x(t) = x^*(\tau)$ for $t > \tau$ where τ is the Markovian moment of the first exit of $x^*(t)$ from D. In [1] was shown that $x(t)$ is independent of the extension (in the sense of equivalent processes and under assumptions that $x^*(t)$ exists and is unique).

Let a function $f(x)$ be given which is defined on D, is Lebesgue measurable and $\int_D f(x)\,\mathrm{d}x = 1$. A solution $x(t)$ of (1,1) has the initial density f, if it is unique and $P\{x(0) \in A\} = \int_A f(x)\,\mathrm{d}x$ for any Borel subset A of D. We shall write $x_f(t)$ if $x(t)$ has the initial density $f(x)$. If $x(0) = x_0$ is a nonstochastic value in D then we shall write $\delta(x_0)$ instead of f. As in [1] we define

$$P(B, a, f, Q) = P\{\exists\, \{\xi : x_f(\xi) \notin D, \quad \xi \in \langle 0, L \rangle\}\}$$

i.e., $P(B, a, f, Q)$ is the probability that the solution $x_f(t)$ of (1,1) leaves the region D during the time interval $\langle 0, L \rangle$ at least once.

A region D is regular if it is bounded and if it has the outside strong sphere property, see [2].

2. Theorem 1. *Let a vector function $a(t, x)$ and an $n \times n$ matrix function $B(t, x)$ be defined on $\bar{Q}$ ($Q = (0, L) \times D$), Lipschitz continuous in x and Hölder continuous in t. Let the matrix function $A(t, x) = B(t, x)\,B^T(t, x)$ (B^T is the transposed matrix) be positive definite in $\bar{Q}$. If the region D is regular, then:*

i) *Ito equation (1,1) has a unique solution with the adhesive barrier S for every initial density in D and for every nonstochastic initial value from D.*

ii) *The parabolic equation*

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_{i,j} A_{ij}(L - t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum a_i(L - t, x) \frac{\partial u}{\partial x_i} \tag{2,1}$$

has the unique bounded solution fulfilling

$$\lim_{t \to 0+} u(t, x) = 0 \quad \textit{for} \quad x \in D\,, \tag{2,2}$$

$$\lim_{x \to \bar{x}} u(t, x) = 1 \quad \textit{for} \quad t > 0\,, \quad \bar{x} \in \dot{D}\,. \tag{2,3}$$

iii) *The bounded solution $u(t, x)$ fulfils*

$$P(B, a, f, Q) = \int_D f(x)\, u(L, x)\, \mathrm{d}x$$

for every density f in D and also for $f = \delta(x)$, $x \in D$: $P(B, a, \delta(x), Q) = u(L, x)$.

Since we shall use results from [1] the proof will be sketched only. First we shall deal with the item ii). The existence of such solution of (2,1) can be proved similarly as in the proof of Lemma 2 [1]. Let $u_m(t, x)$ be the solution of (2,1) fulfilling $u_m(0, x) = 0$ for $x \in D$, $u_m(t, \bar{x}) = 0$ for $0 \leqq t \leqq 2^{-m}$, $u_m(t, \bar{x}) = 2^m(t - 2^{-m})$ for $2^{-m} < t \leqq 2^{-m+1}$, $u_m(t, \bar{x}) = 1$ for $t > 2^{-m+1}$, $\bar{x} \in \dot{D}$. Let $u_{-m}(t, x)$ be solution of (2,1) fulfilling $u_{-m}(0, x) = 0$ outside of 2^{-m+1}-neighbourhood of $\dot{D}$, $u_{-m}(0, x) = 1$ in 2^{-m}-neighbourhood of $\dot{D}$ and such that $u_{-m}(0, x)$ is continuous and $0 \leqq u_{-m}(0, x) \leqq$ $\leqq 1$ in D, $u_{-m}(t, \bar{x}) = 1$ for $t > 0$, $\bar{x} \in \dot{D}$. As in [1] $u_m(t, x)$ and $u_{-m}(t, x)$ form convergent sequences, $u^{**}(t, x) = \lim_{m \to \infty} u_m(t, x)$, $u^*(t, x) = \lim_{m \to \infty} u_{-m}(t, x)$. The functions u^{**}, u^* fulfil (2,2) and (2,3). By Theorem 5 Chap III [2] the sequences u_m, u_{-m} converge locally uniformly which means that u^*, u^{**} are also solutions of (2,1). With respect to [3] we obtain $u^* = u^{**}$. The item ii) is proved.

We pass to the analysis of item iii). If the region D fulfils condition (B), i.e. if the frontier $\dot{D}$ can be locally expressed by means of functions h which have Hölder continuous second derivatives and if $\partial a_i/\partial x_i$, $\partial A_{ij}/\partial x_j$, $\partial^2 A_{ij}/\partial x_i\, \partial x_j$ are Hölder continuous, then with respect to Lemma 5 [1] item iii) is fulfilled.

First we shall weaken assumptions about a, B while retaining the assumptions about D, i.e. D fulfils condition (B).

Let a, B fulfil the conditions of Theorem 1 only. Let a, B be uniformly approximated by a^n, B^n which fulfil the additional conditions, i.e. $\partial a_i^n/\partial x_i$,, $\partial A_{ij}^n/\partial x_j$, $\partial^2 A_{ij}^n/\partial x_i\, \partial x_j$ are Hölder continuous. By Lemma 5 [1]

$$P(B^n, a^n, f, Q) = \int f(x)\, u^n(L, x)\, \mathrm{d}x \tag{2,4}$$

where $u^n(t, x)$ is the bounded solution of (2,1) fulfilling (2,2), (2,3) with a, A replaced by a^n, A^n. The solutions $u^n(t, x)$ can be approximated by $u_m^n(t, x)$ as in the previous part of this proof. By Theorem 15 Chap. III [2] $u_m^n(t, x)$ converge to some solution of (2,1) for $n \to \infty$. By Theorem 6 Chap. III [2] the derivatives are bounded uniformly with respect to n for m fixed. It implies

$$\lim_{n \to \infty} u_m^n(t, x) = u_m(t, x)\,.$$

As $u_m^n(t, x) \leqq u^n(t, x) \leqq u_{-m}^n(t, x)$ we obtain

$$\lim_{n \to \infty} u^n(t, x) = u(t, x)\,. \tag{2,5}$$

We can assume that a, B, a^n, B^n are defined on the whole $\langle 0, L\rangle \times R_n$ so that they are Lipschitz continuous in x, Hölder continuous in t and $a^n \to a$, $B^n \to B$ uniformly on the whole $\langle 0, L\rangle \times R_n$. Let $x_f^{(n)*}(t)$ denote the solution of (1,1) where a, B are replaced by a^n, B^n. As in the proof of Lemma 5 [1] or [4] we have $|||x_f^*(t) - x_f^{(n)*}(t)||| \to 0$ for $n \to \infty$ where $|||z||| = \sqrt{(E \sup_{\langle 0,L\rangle} |z(t)|^2)}$. Since $\tau^{(n)} \to \tau$ in probability where $\tau^{(n)}$ and τ are the first exit times of $x_f^{(n)*}$ and x_f^* from D, respectively (this follows from (17,9) [1]) we have

$$(2,6) \qquad P(B^n, a^n, f, Q) \to P(B, a, f, Q)\,.$$

Relations (2,4) to (2,6) imply that item iii) is valid if a, B fulfil conditions of Theorem 1 and if the region D fulfils condition (B).

Secondly we shall prove that iii) is valid also for regular regions. Let D be a regular region. We can construct a sequence D_s, $D_1 \subset \bar{D}_1 \subset D_2 \subset \bar{D}_2 \subset \ldots \subset D_s \subset \bar{D}_s \subset \ldots$, $\cup D_s = D$, all D_s fulfilling condition (B). Let $u_s(t, x)$ be the solution of (2,1) fulfilling (2,2), (2,3) with D replaced by D_s. We can prove again that $\lim_{s\to\infty} u_s(t, x) = u(t, x)$. Since $P(B, a, \delta, Q_s) = u_s(L, x)$ and $P(B, a, \delta, Q) \leqq P(B, a, \delta, Q_s)$ we obtain

$$(2,7) \qquad P(B, a, \delta, Q) \leqq u(L, x)$$

(where $Q_s = (0, L) \times D_s$). We assume that a, B are defined on the whole $\langle 0, L\rangle \times R_n$ (and fulfil there the conditions of the Theorem). Let $[\bar{t}, \bar{x}]$ be a point on the frontier $0 \leqq \bar{t} < L$, $\bar{x} \in \dot{D}$. With respect to the outside strong sphere property there exists a ball K such that $\bar{x} \in \dot{K}$ and D is outside of K. Let, generally, $P(t, x, U)$ (U being a region in R_n) be the probability that the solution of (1,1) (with the extended domain of definition of a, B) fulfilling $x^*(t) = x$ intersects U during the time interval $\langle t, L)$, i.e. $P(t, x, U) = P\{\exists \{\xi : x^*(\xi, t, x) \in U,\ t \leqq \xi < L\}\}$. By (17,9) [1] we obtain $P(\bar{t}, \bar{x}, K) = 1$. Since $P(t, x, K)$ is continuous in t, x (see Theorem 2 §6 Chap. VIII [5] – this Theorem can be generalized for multidimensional case without much difficulty if the considered continuity is the continuity in probability) we have $P(t_s, x_s, K) \to 1$ if $[t_s, x_s] \to [\bar{t}, \bar{x}]$, $0 \leqq \bar{t} < L$, $\bar{x} \in \dot{D}$, $[t_s, x_s] \in D$. Further

$$(2,8) \qquad P(t_s, x_s, R_n - \bar{D}) \to 1 \text{ if } x_s \in \dot{D}_s,\ 0 \leqq t_s \leqq L - \varepsilon$$

where ε is a fixed positive number. (The set $\dot{D}$ is compact.) Since $P(B, a, \delta(x_0), Q) = \int P(t, x, R_n - \bar{D})\, \mathrm{d}\nu_s$ where $x^*(t, x_0)$ is the solution of (1,1) fulfilling $x^*(0, x_0) = x_0 \in D_s$ and ν_s is the measure given on S_s by $\nu(M) = P\{\exists \{\xi : [\xi, x^*(\xi, x_0)] \in M, 0 \leqq \xi < L\}\}$, $M \subset S_s$, $S_s = \langle 0, L\rangle \times \dot{D}_s$, we conclude by (2,8)

$$P(B, a, \delta(x), Q) \geqq \lim_{s\to\infty} P(B, a, \delta(x), Q_s) = \lim_{s\to\infty} u_s(L, x)$$

which together with (2,7) proves iii) in general case.

Item i) is a clear consequence of considerations in [1] page 571.

Remark 1. From Theorem 1 it follows immediately that under conditions of this Theorem the vector function $a(t, x)$ and the matrix function $B(t, x)$ fulfil condition (A). (See Definition 4 [1].)

Remark 2. In the proof of ii) we needed only that $a(t, x)$, $B(t, x)$ are Hölder continuous with respect to all variables in Q and that $\Lambda(t, x)$ is positive definite in Q.

3. In this section we reformulate some definitions and results from [1].

Definition 1. A matrix function $B(t, x)$ is strongly maximal with respect to a vector function $a(t, x)$ and to a region Q ($Q = (0, L) \times D$) if a, B, D fulfil conditions of Theorem 1 and if

$$P(B, a, f, Q) = \max P(B', a, f, Q)$$

for all densities f in D where the maximum is taken over the set of matrix functions $B'(t, x)$ fulfilling conditions of Theorem 1 and $\Lambda(t, x) - \Lambda'(t, x)$ is a positive semi-definite matrix for every $[t, x] \in Q$ ($\Lambda(t, x) = B(t, x)\, B^T(t, x)$, $\Lambda'(t, x) = B'(t, x)\, .$ $.\, B'^T(t, x)$).

This definition is not identical to Definition 6 in [1] but the difference is not important for us now.

The definition of maximal matrix function could be presented in a similar manner but we shall not use it.

Using Theorem 1 we can reformulate Theorem 2 from [1].

Conclusion 1. *Let $a(t, x)$, $B(t, x)$ and D fulfil conditions of Theorem* 1. *The matrix function $B(t, x)$ is strongly maximal with respect to the vector function $a(t, x)$ and to the region $Q = (0, L) \times D$ if and only if the bounded solution of* (2,1) *fulfilling* (2,2) *and* (2,3) *is convex with respect to x in Q.*

Theorem 3 and Lemma 6 of [1] can be generalized.

Theorem 2. *Let a region Q be defined by $Q = (0, L) \times (x_1, x_2)$ where x_1, x_2 are real numbers, $x_1 < x_2$. Let a function $B(t, x)$ be defined on $\bar{Q}$ such that $B(t, x)$ is Lipschitz continuous in x, Hölder continuous in t and $B(t, x) \neq 0$ on $\bar{Q}$. Let $\alpha(t)$, $\beta(t)$ be Hölder continuous functions on $\langle 0, L \rangle$. Put $a(t, x) = \alpha(t) + \beta(t)\, x$. If $a(t, x_2) \leqq$ $\leqq 0$, $a(t, x_1) \geqq 0$, then the function $B(t, x)$ is (stongly) maximal with respect to the function $a(t, x)$ and to the region Q. It means that the bounded solution of*

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} B^2(L - t, x) \frac{\partial^2 u}{\partial x^2} + a(L - t, x) \frac{\partial u}{\partial x} \tag{3,1}$$

fulfilling

$$u(0, x) = 0 \quad \textit{for} \quad x_1 < x < x_2\,, \tag{3,2}$$

$$u(t, x_i) = 1 \quad \textit{for} \quad t > 0\,, \quad i = 1, 2 \tag{3,3}$$

is convex in x.

Remark 3. For the statement on convexity we need only that $a(t, x)$, $B(t, x)$ are Hölder continuous in $\bar{Q}$ and $B(t, x) \neq 0$ on $\bar{Q}$.

Remark 4. If $\varphi(x)$ is a convex function fulfilling $\varphi(x_i) = 1$, $i = 1, 2$ and condition (3,2) is replaced by

$$u(0, x) = \varphi(x) \quad \text{for} \quad x_1 < x < x_2 \tag{3,4}$$

then $u(t, x)$ is convex in x again. This follows immediately from the proof of Theorem 3 in [1].

4. An auxiliary Lemma will be proved using Theorem 2. Hovewer, this Lemma is important by itself.

Lemma 1. *Let Q be a region $Q = (0, L) \times (x_1, x_2)$ where x_1, x_2 are real numbers $x_1 < x_2$. Let $a(x)$, $b(x)$ be Hölder continuous functions defined on $\bar{Q}$, $b(x) > 0$ on Q. Let $u(t, x)$ be the bounded solution of*

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} b(x) \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial x} \tag{4,1}$$

fulfilling

$$u(0, x) = 0 \quad \textit{for} \quad x_1 < x < x_2\,, \tag{4,2}$$

$$u(t, x_i) = 1 \quad \textit{for} \quad t > 0\,, \quad i = 1, 2\,. \tag{4,3}$$

We have $(\partial^2 u / \partial x^2)(t, x) \geqq 0$ at all points $[t, x] \in Q$ for which $(\partial u / \partial x)(t, x)\, a(x) \leqq 0$.

Proof. Let $\psi(y)$ be the solution of

$$\psi' = \exp\left\{2 \int_{x_1}^{\psi} \frac{a(\xi)}{b(\xi)}\, \mathrm{d}\xi\right\}$$

fulfilling $\psi(0) = x_2$. There exists $y_{-1} < 0$ such that $\psi(y_{-1}) = x_1$. The derivative $\psi'(y)$ is positive on $\langle y_{-1}, 0\rangle$ and

$$\psi''(y) = 2\, \frac{a(\psi(y))}{b(\psi(y))}\, \psi'^2(y)\,. \tag{4,4}$$

Using the transformation $x = \psi(y)$, $u(t, x) = v(t, y)$, equation (4,1) is transformed onto

$$\frac{\partial v}{\partial t} = \frac{1}{2}\, \frac{b(\psi(y))}{\psi'^2(y)}\, \frac{\partial^2 v}{\partial y^2}$$

and conditions (4,2), (4,3) onto $v(0, y) = 0$ for $y_{-1} < y < 0$, $v(t, y_{-1}) = v(t, 0) = 1$ for $t > 0$.

Theorem 2 implies $(\partial^2 v/\partial y^2)(t, y) \geqq 0$. With respect to (4,4) $\psi''(y)$ has the same sign as $a(x)$ and since

$$\frac{\partial u^2}{\partial x^2}(t, x) = \frac{1}{\psi'^2(y)} \frac{\partial^2 v}{\partial y^2}(t, y) - \frac{\psi''(y)}{\psi'^2(y)} \frac{\partial u}{\partial x}(t, x),$$

Lemma 1 is proved.

5. With respect to Conclusion 1, it is sufficient for our purpose to give criteria for the convexity of $u(t, x)$. In this section we shall give such criteria under the assumption that $a(t, x)$ does not change its sign.

Theorem 3. *Let Q be the same as in Lemma 1 and functions $a(x)$, $b(x)$ be defined and Hölder continuous on $\bar{Q}$, $b(x) > 0$ on $\bar{Q}$, $K_1 \leqq \min b(x)$, $K_2 \geqq \max b(x)$. Let $u(t, x)$ be the bounded solution of (4,1) fulfilling (4,2) and (4,3). If $a(x) \geqq 0$ and if*

$$(5,1) \qquad a(x) \leqq \frac{x_2 - x}{2} \frac{K_2}{(x_2 - x_1)^2} \arcsin^2 \sqrt{\frac{K_1}{K_2}}$$

for all x for which there exists $t : 0 < t < L$ such that $(\partial u/\partial x)(t, x) > 0$, then $\partial^2 u/\partial x^2 \geqq 0$ in Q.

Before we pass to the proof of Theorem 3 we have to deal with some auxiliary Lemmas and we shall for the moment assume that $a(x)$ is not identically equal to zero in any neighbourhood of x_1.

Lemma 2. *Let all assumptions of Theorem 3 be satisfied and let there not exist any neighbourhood U of x_1 such that $a(x) \equiv 0$ on U. Let α be a number $0 \leqq \alpha \leqq \leqq K_2(x_2 - x_1)^{-2} \arcsin^2 \sqrt{(K_1/K_2)}$. If $\psi(y)$ is a solution of*

$$(5,2) \qquad \psi''(y) = 2 \frac{a(\psi(y))}{b(\psi(y))} \psi'^2 + \alpha \frac{y}{b(\psi(y))} \psi'^3(y)$$

fulfilling $\psi(0) = x_2$, $\psi'(0) < 0$, then there exists $y_1 > 0$ such that $\psi(y_1) = x_1$, $\psi'(y) < 0$ for $y \in \langle 0, y_1 \rangle$.

Proof. The solution $\psi(y)$ is defined on some maximal interval $\langle 0, y_0)$ (it can be $y_0 = \infty$). Multiplying (5,2) by ψ'^{-1} and integrating we obtain

(5,3)

$$\psi'(y) = \psi'(0) \exp \left\{ 2 \int_{x_2}^{\psi(y)} \frac{a(\xi)}{b(\xi)} \, d\xi + \alpha \int_0^y \frac{\xi}{b(\psi(\xi))} \psi'^2(\xi) \, d\xi \right\} \quad \text{for} \quad 0 \leqq y < y_0 .$$

Multiplying (5,2) by ψ'^{-3} and integrating we obtain

$$(5,4) \quad \psi'(y) = \psi'(0) \left[1 + 4\psi'^2(0) \int_0^y \frac{a(\psi)}{b(\psi)(-\psi')} \, d\xi - 2\alpha\psi'^2(0) \int_0^y \frac{\xi}{b(\psi)} \, d\xi \right]^{-1/2}$$

for $0 \leqq y < y_0$. Let G be the region of R_2 defined by $G = \{[x, y];\ x_1 < x < x_2,\ y < 0\}$. The frontier $\dot{G}$ consists of three parts:
$\dot{G}_1 = \{[x_2, y];\ y < 0\}$, $\dot{G}_2 = \{[x, 0];\ x_1 \leqq x \leqq x_2\}$, $\dot{G}_3 = \{[x_1, y];\ y < 0\}$. From $\psi'(0) < 0$ it follows that $[\psi(y), \psi'(y)]$ enters G for $y > 0$. The curve $[\psi(y), \psi'(y)]$ cannot intersect $\dot{G}_1$ since there is $\psi' < 0$ and cannot intersect $\dot{G}_2$ since every point of $\dot{G}_2$ is a constant solution of (5,2) whose unicity is guaranteed.

We shall assume that $[\psi(y), \psi'(y)]$ does not intersect $\dot{G}_3$. In such case $y_0 < \infty$ since with respect to (5,3) $\psi'(y) \leqq \psi'(0)\, e^{-c}$ where c is a positive constant and $\psi'(0) < 0$. Put

$$f(y) = 1 + 4\psi'^2(0) \int_0^y \frac{a(\psi(\xi))}{b(\psi(\xi))\,(-\psi'(\xi))}\, d\xi - 2\alpha\, \psi'^2(0) \int_0^y \frac{\xi}{b(\psi(\xi))}\, d\xi$$

for $0 \leqq y < y_0$. Since $\psi'(y) < 0$ for $0 \leqq y < y_0$ and $y_0 < \infty$ there exist $\lim_{y \to y_0} f(y) \geqq 0$ and $\lim_{y \to y_0 -} \psi(y) \geqq x_1$. If $\lim_{y \to y_0 -} f(y) > 0$ then by (5,4) $\lim_{y \to y_0 -} \psi'(y)$ exists and is finite. It would mean that $[\psi(y), \psi'(y)]$ converges to $[\lim_{y \to y_0 -} \psi(y), \lim_{y \to y_0 -} \psi'(y)]$ where $y_0 < \infty$. It would be a contradiction with the assumption that $\langle 0, y_0)$ is the maximal interval of definition of the solution $\psi(y)$. Thus we have proved

$$\lim_{y \to y_0 -} f(y) = 0\,. \tag{5,5}$$

With respect to (5,4) and to the inequality $a(x) \geqq 0$,

$$\psi'(y) \leqq \psi'(0) \left[1 + 4\psi'^2(0) \int_0^{y_0} \frac{a(\psi)}{b(\psi)\,(-\psi')}\, d\xi - \frac{\alpha}{K_2}\, \psi'^2(0)\, y^2\right]^{-1/2} \tag{5,6}$$

and by simple calculation

$$\psi(y) = x_2 + \int_0^y \psi'(z)\, dz \leqq \tag{5,7}$$

$$\leqq x_2 - \sqrt{\left[\frac{K_2}{\alpha}\right]} \arcsin \left\{ y |\psi'(0)|\, \alpha^{1/2} \left[K_2 \left(\left(1 + 4\psi'^2(0) \int_0^{y_0} \frac{a(\psi)}{b(\psi)\,(-\psi')}\, d\xi\right)\right]^{-1/2} \right\}.$$

By (5,5) we obtain $\lim_{y \to y_0 -} \psi(y) \leqq x_2 - \sqrt{[K_2/\alpha]} \arcsin \sqrt{[K_1/K_2]}$ and by $\alpha \leqq K_2(x_2 - x_1)^{-2} \arcsin^2 \sqrt{[K_1/K_2]}$ we have $\lim_{y \to y_0 -} \psi(y) \leqq x_1$. Due to the additional assumption about the behaviour of $a(x)$ near to x_1 the inequalities (5,6), (5,7) and the last one are sharp. It means that there exist a first $y_1 > 0$, $y_1 < y_0$ such that $\psi(y_1) = x_1$ and $\psi'(y) < 0$ for $y \in \langle 0, y_1 \rangle$.

Lemma 3. *Let the assumptions of Theorem 3 be fulfilled. If there exists $x_0 : x_1 \leqq x_0 < x_2$ such that $a(x) \leqq \alpha(x_2 - x)/2$ for $x \in \langle x_0, x_2 \rangle$, then $\psi''(y) \leqq 0$ for those y for which $x_0 \leqq \psi(y) \leqq x_2$.*

Proof. With respect to (5,2) $\psi''(y) \leqq 0$ is equivalent to the inequality $2a(\psi(y)) + \alpha \psi'(y) y \leqq 0$ and it is sufficient to prove that $u(y) = 2a(\psi(y)) + \alpha y \psi'(y) \leqq 0$. The function $u(y)$ is the solution of

$$u'(y) = \frac{\alpha y}{b(\psi(y))} \psi'^2(y) u(y) + [2a'(\psi(y)) + \alpha] \psi'(y) \tag{5,8}$$

fulfilling $u(0) = 0$. Let $u_\varepsilon(y)$ be the solution of (5,8) fulfilling $u_\varepsilon(0) = -\varepsilon < 0$. Denote by y^* the first point for which $u_\varepsilon(y^*) = 0$. For $y \in \langle 0, y^*)$ we have $u_\varepsilon'(y) \leqq \leqq 2a'(\psi(y)) \psi'(y) + \alpha \psi'(y)$ so that $u_\varepsilon(y^*) = -\varepsilon + 2a(\psi(y^*)) + \alpha(\psi(y^*) - x_2)$. With respect to the assumptions of Lemma 3 we have $\psi(y^*) < x_0$. For $\varepsilon \to 0$ the solution $u_\varepsilon(y)$ converges to $u(y)$, hence $u(y) \leqq 0$ for $\psi(y) \geqq x_0$.

Now we pass to the proof of Theorem 3 retaining the assumption about the behaviour of $a(x)$ near to x_1.

Using Lemma 1 we find that $(\partial^2 u/\partial x^2)(t, x) \geqq 0$ at all points at which $(\partial u/\partial x)(t, x) \leqq 0$.

If $(\partial u/\partial x)(t, x) > 0$, then with respect to the assumptions of Theorem 3 the function $a(x)$ fulfils inequality (5,1). We perform the transformation $x = \psi(y)$, $u(t, x) = = v(t, y)$. The function $v(t, y)$ is the bounded solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2} \frac{b(\psi(y))}{\psi'^2(y)} \frac{\partial^2 v}{\partial y^2} - \frac{\alpha}{2} y \frac{\partial v}{\partial y} \tag{5,9}$$

fulfilling

$$v(0, y) = 0 \quad \text{for} \quad 0 < y < y_1, \quad v(t, 0) = v(t, y_1) = 1 \quad \text{for} \quad t > 0. \tag{5,10}$$

With regard to Theorem 2 the function $v(t, y)$ fulfils $(\partial^2 v/\partial y)(t, y) \geqq 0$. Since

$$\frac{\partial^2 u}{\partial x^2}(t, x) = \psi'^{-2}(y) \frac{\partial^2 v}{\partial y^2}(t, y) - \frac{\psi''(y)}{\psi'^2(y)} \frac{\partial u}{\partial x}(t, x) \tag{5,11}$$

and due to the statement of Lemma 3 we have $(\partial^2 u/\partial x^2)(t, x) \geqq 0$ at all points $[t, x]$ for which $a(x) \leqq \alpha(x_2 - x)/2$. By the assumption of Theorem 3 this holds at all points at which $(\partial u/\partial x)(t, x) > 0$ and with respect to the convexity of $v(t, x)$ we obtain that $(\partial u/\partial x)(t, \xi) > 0$ for $\xi \geqq x$ if $(\partial u/\partial x)(t, x) > 0$.

If $a(x)$ is identically equal to zero in a neighbourhood of x_1 then $a(x)$ can be approximated by functions which fulfil the additional assumption and which fulfil all conditions of Theorem 3. This proves completely Theorem 3.

6. In the previous section the coefficient $a(x)$ was supposed not to change the sign. Now we shall deal with the case when $a(x)$ may change the sign. However, conditions obtained will be stronger then (5,1).

Theorem 4. *Let Q, K_1, K_2 have the same meaning as in Theorem 3. Let $a(x), b(x)$ be defined and Hölder continuous in $\bar{Q}$, $b(x) > 0$ on $\bar{Q}$. Let $a(x_2) \leqq 0 \leqq a(x_1)$,*

$$\beta \leqq \min \int_{x_2}^{x} \frac{a(\xi)}{b(\xi)} \mathrm{d}\xi \,, \quad \gamma \geqq \max \int_{x_2}^{x} \frac{a(\xi)}{b(\xi)} \mathrm{d}\xi$$

and let $u(t, x)$ be the bounded solution of (4,1) *fulfilling* (4,2) *and* (4,3). *If*

$$(6,1) \qquad a(x) - a(x_2) \leqq \frac{x_2 - x}{2(x_2 - x_1)^2} K_2 \arcsin^2 \left[e^{2(\beta - \gamma)} \sqrt{\frac{K_1}{K_2}} \right]$$

for all x for which there exists $t : 0 < t < L$ such that $(\partial u/\partial x)(t, x) > 0$ and

$$(6,2) \qquad a(x) - a(x_1) \geqq - \frac{x - x_1}{2(x_2 - x_1)^2} K_2 \arcsin^2 \left[e^{2(\beta - \gamma)} \sqrt{\frac{K_1}{K_2}} \right]$$

for all x for which there exists $t : 0 < t < L$ such that $(\partial u/\partial x)(t, x) < 0$, then $(\partial^2 u/\partial x^2)(t, x) \geqq 0$ on Q.

The proof is based again on some auxiliary lemmas.

Lemma 4. *Let all conditions of Theorem 4 be fulfilled and*

$$(6,3) \qquad 0 \leqq \alpha \leqq \frac{K_2}{(x_2 - x_1)^2} \arcsin^2 \left[e^{2(\beta - \gamma)} \sqrt{\frac{K_1}{K_2}} \right].$$

Then either there exist $y_1 > 0$ and a solution $\psi(y)$ of (5,2) *fulfilling $\psi(0) = x_2$, $\psi'(0) < 0$, $\psi(y_1) = x_1$, $\psi'(y) < 0$ for $y \in \langle 0, y_1 \rangle$ or the statement of Theorem 4 is valid.*

The proof of Lemma 4 is similar to that of Lemma 2 but not quite the same. Let G, $\dot{G}_i$, $i = 1, 2, 3$ be the same as in the proof of Lemma 2. The solution $\psi(y)$ is defined on some maximal interval $\langle 0, y_0)$ and the curve $[\psi(y), \psi'(y)]$ must enter G again. $[\psi(y), \psi'(y)]$ cannot intersect $\dot{G}_i$, $i = 1, 2$ for the same reasons as before. We assume that $[\psi(y), \psi'(y)]$ does not intersect $\dot{G}_3$. Since (5,3) is valid again we obtain $y_0 < \infty$.

We put

$$\zeta(z) = \alpha \int_0^z \frac{\xi}{b(\psi(\xi))} \psi'^2(\xi) \, \mathrm{d}\xi \,.$$

If we square both sides of (5,3) and multiply by $y(b(\psi(y)))^{-1}$ we obtain a differential equation for $\zeta(z)$. Consequently

$$\zeta(z) = \lg \left[1 - 2\alpha\psi'^2(0) \int_0^z \frac{y}{b(\psi(y))} \exp \left\{ 4 \int_{x_2}^{\psi(y)} \frac{a(\xi)}{b(\xi)} \mathrm{d}\xi \right\} \mathrm{d}y \right]^{-1/2}$$

and

$$\psi'(y) = \psi'(0) \exp \left\{ 2 \int_{x_2}^{\psi(y)} \frac{a(\xi)}{b(\xi)} \, d\xi \right\} \times \tag{6,4}$$

$$\times \left[1 - 2\alpha\psi'^2(0) \int_0^y \frac{\eta}{b(\psi(\eta))} \exp \left\{ 4 \int_{x_2}^{\psi(\eta)} \frac{a(\xi)}{b(\xi)} \, d\xi \right\} d\eta \right]^{-1/2}.$$

Equation (6,4) implies as in the previous case

$$1 - 2\alpha \, \psi'^2(0) \int_0^{y_0} \frac{\eta}{b(\psi(\eta))} \exp \left\{ 4 \int_{x_2}^{\psi(\eta)} \frac{a(\xi)}{b(\xi)} \, d\xi \right\} d\eta = 0 \, . \tag{6,5}$$

With respect to definitions of K_i, β, γ, (6,4) implies

$$\psi(y) = x_2 + \int_0^y \psi'(y) \, dy \leqq x_2 - \sqrt{\left[\frac{K_2}{\alpha}\right]} \arcsin \left\{ y e^{2\beta} |\psi'(0)| \sqrt{\frac{\alpha}{K_2}} \right\}. \tag{6,6}$$

Equation (6,5) gives $y_0 \geqq e^{-2\gamma} |\psi'(0)|^{-1} \sqrt{(K_1/\alpha)}$ and together with (6,6) we obtain

$$\lim_{y \to y_0 -} \psi(y) \leqq x_2 - \sqrt{\left[\frac{K_2}{\alpha}\right]} \arcsin \left\{ e^{2(\beta - \gamma)} \sqrt{\frac{K_1}{K_2}} \right\}. \tag{6,7}$$

By (6,3) we have $\lim_{y \to y_0 -} \psi(y) \leqq x_1$.

Inequalities (6,6), (6,7) and the last one need not be sharp only in the case $a(x) \equiv 0$. But in this case we can apply Theorem 2 immediately. If inequalities just mentioned are sharp, then there exists $y_1 > 0$, $y_1 < y_0$ which has desired properties.

Lemma 5. *Let $\psi(y)$ be the function determined in Lemma* 4 *and let all conditions of Theorem* 4 *be satisfied. The second derivative fulfils: $\psi''(z) \leqq 0$ at all points z for which*

$$2a(\psi(y)) - 2a(x_2) + \alpha(\psi(y) - x_2) \leqq 0 \quad \textit{for all} \quad 0 \leqq y \leqq z \, . \tag{6,8}$$

The proof of Lemma 5 is almost the same as that of Lemma 3.

Proof of Theorem 4. Let $a(x_0) \geqq 0$, we shall prove $(\partial^2 u/\partial x^2)(t, x_0) \geqq 0$ for all t. If $(\partial u/\partial x)(t, x_0) \leqq 0$ then the assertion follows from Lemma 1. If $(\partial u/\partial x)(t, x_0) > 0$ then we perform the transformation $x = \psi(y)$, $u(t, x) = v(t, y)$ where ψ is determined in Lemma 4. The function $v(t, y)$ is the bounded solution of (5,9) fulfilling (5,10). Using Theorem 2 we obtain $(\partial^2 v/\partial y^2)(t, y) \geqq 0$. With respect to (5,11), $(\partial u/\partial x)(t, x_0) > 0$ and (6,8) (which is fulfilled owing to the assumptions of Theorem 4 since $(\partial u/\partial x)(t, x_0) > 0$) we obtain $(\partial^2 u/\partial x^2)(t, x_0) \geqq 0$.

The case $a(x_0) < 0$ may be reduced by the transformation $x = -\xi$ onto the previous one ($a(x_0) \geqq 0$).

7. Conditions of Theorems 3 and 4 contain the expression

$$g(K_1, K_2) = K_2 \arcsin^2 \alpha \sqrt{\frac{K_1}{K_2}}$$

which is defined for $K_2 \geqq \alpha^2 K_1$. The dependence of the expression on K_1, α is obvious.

Remark 5. The function $g(K_1, K_2)$ is decreasing with respect to K_2 for $K_2 \geqq \alpha^2 K_1$ and

$$\lim_{K_2 \to \infty} g(K_1, K_2) = \alpha^2 K_1 .$$

Another type od criteria can be derived for symmetric equations using Lemma 1.

Remark 6. Let $Q = (0, L) \times (-x_0, x_0)$ where $x_0 > 0$. Let functions $a(x), b(x)$ be defined and Hölder continuous on $\bar{Q}$, $b(x) > 0$ on $\bar{Q}$. Let $u(t, x)$ be the bounded solution of (4,1) fulfilling (4,2) and (4,3) where x_1, x_2 are replaced by $-x_0, x_0$, respectively. If $a(x) = -a(-x)$, $b(x) = b(-x)$ and $a(x) \leqq 0$ for $x > 0$, then $\partial^2 u / \partial x^2 \geqq 0$ on Q.

8. The function $\frac{1}{2}(x_2 - x)(x_2 - x_1)^{-2} K_2 \arcsin^2 \sqrt{(K_1/K_2)}$ appearing in Theorem 3 guarantees the convexity of $u(t, x)$ provided that $a(x)$ is nonnegative and smaller than the given function. The function occurring in Theorem 4 has a similar meaning.

With respect to Definition 2 below these functions will be briefly referred to as functions havig property (C). It is worth knowing something more about functions with property (C) since they enable us to decide whether $u(t, x)$ is convex in x even if the function $a(x)$ is known only approximately. Now the problem arises to find the best function which has property (C). We shall not solve the problem entirely but we shall give a necessary condition (see (8,1)) for a function to have property (C).

On the other hand, Theorem 2 implies that for linear $a(x)$ no condition on absolute values is needed and it means that the condition $a(x) \leqq \varphi(x, x_1, x_2)$ where φ is a function with property (C) is not necessary.

Definition 2. Let $Q = (0, L) \times (0, 1)$. A function $\varphi(x, K_1, K_2)$ has property (C) if it is defined, continuous and positive for $x \in \langle 0, 1 \rangle$, $0 < K_1 \leqq K_2$ and if the bounded solution of (4,1) fulfilling $u(0, x) = 0$ for $x \in (0, 1)$, $u(t, 0) = u(t, 1) = 1$ for $t > 0$ is a convex function with respect to x on Q under the condition that $0 \leqq \leqq a(x) \leqq \varphi(x, K_1, K_2)$ for $x \in \langle 0, 1 \rangle$, $K_1 \leqq b(x) \leqq K_2$ for $x \in \langle 0, 1 \rangle$.

Theorem 5. *Let a function $\varphi(x, K_1, K_2)$ have property* (C), *then*

$$\varphi(x, 1, 1) \leqq \frac{1}{2} \min_t \left[\frac{\partial^2 v}{\partial x^2}(t, x) \Big/ \frac{\partial v}{\partial x}(t, x) \right] \quad \textit{for} \quad x > \tfrac{1}{2} \tag{8,1}$$

where $v(t, x)$ is the bounded solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \tag{8,2}$$

fulfilling $v(0, x) = 0$ for $x \in (0, 1)$ and $v(t, 0) = v(t, 1) = 1$ for $t > 0$.

Proof. On the contrary we shall suppose

$$\varphi(x_0, 1, 1) > \frac{1}{2} \min_t \left[\frac{\partial^2 v}{\partial x^2}(t, x_0) \Big/ \frac{\partial v}{\partial x}(t, x_0) \right] \quad \text{for} \quad x_0 \in (1/2, 1\rangle ;$$

then there exist numbers β and $\delta > 0$ such that

$$\varphi(x, K_1, K_2) > \beta + \delta \quad \text{for} \quad |x - x_0| \leqq \delta , \quad 1 - \delta \leqq K_1 \leqq K_2 \leqq 1 + \delta , \tag{8,3}$$

$$\frac{1}{2} \min_t \left[\frac{\partial^2 v}{\partial x^2}(t, x) \frac{\partial v}{\partial x}(t, x) \right] < \beta - \delta \quad \text{for} \quad |x - x_0| \leqq \delta . \tag{8,4}$$

Further we choose positive numbers $\alpha > 0$, $\varepsilon > 0$ such that

$$\alpha\varepsilon^3 < 6 , \quad \varepsilon < \delta , \quad \alpha\varepsilon^3(\varepsilon + 2x_0) < 12 , \tag{8,5}$$

$$\frac{\alpha\varepsilon^2}{2} \left[\frac{6 - \alpha\varepsilon^3}{12 - \alpha\varepsilon^3(\varepsilon + 2x_0)} \right]^2 > \beta - \delta , \tag{8,6}$$

$$\frac{54\alpha\varepsilon^2}{(6 - \alpha\varepsilon^3)^2 (12 - \alpha\varepsilon^3(\varepsilon + 2x_0))} < \beta + \delta , \tag{8,7}$$

$$\frac{144}{1 + \delta} < [12 - \alpha\varepsilon^3(\varepsilon + 2x_0)]^2 , \tag{8,8}$$

$$\left[\frac{12 - \alpha\varepsilon^3(\varepsilon + 2x_0)}{6 - \alpha\varepsilon^3} \right]^2 < \frac{4}{1 - \delta} , \tag{8,9}$$

$$x_0 - \delta < 2x_0 \frac{6 - \alpha\varepsilon^3}{12 - \alpha\varepsilon^3(\varepsilon + 2x_0)} , \tag{8,10}$$

$$2(x_0 + \varepsilon) \frac{6 - \alpha\varepsilon^3}{12 - \alpha\varepsilon^3(\varepsilon + 2x_0)} < x_0 + \delta . \tag{8,11}$$

It is always possible since, if α diverges to ∞ while ε converges to 0 so that $\varepsilon^2\alpha$ converges to a number $d > 0$, then $\alpha\varepsilon^n$ converges to zero for $n > 2$ and inequalities (8,6) to (8,11) convert successively into inequalities $d/8 > \beta - \delta$, $d/8 < \beta + \delta$, $1/(1 + \delta) < 1$, $1 < 1/(1 - \delta)$, $x_0 - \delta < x_0$, $x_0 < x_0 + \delta$. We shall consider further a transformation $x = \psi(z) = \psi_1(\psi_2(z))$ where ψ_i, $i = 1, 2$ are defined as follows:

$$x = \psi_1(y) = y \quad \text{for} \quad 0 \leqq y \leqq x_0 ,$$

$$x = \psi_1(y) = y - \alpha(-\tfrac{1}{12}y^4 + \tfrac{1}{6}(2x_0 + \varepsilon)\, y^3 - \tfrac{1}{2}x_0(x_0 + \varepsilon)\, y^2 + $$
$$+ x_0^2(\tfrac{1}{3}x_0 + \tfrac{1}{2}\varepsilon)\, y - \tfrac{1}{12}x_0^3(x_0 + 2\varepsilon)) \quad \text{for} \quad x_0 < y \leqq x_0 + \varepsilon ,$$

$$x = \psi_1(y) = \tfrac{1}{6}(6 - \alpha\varepsilon^3)\, y + \frac{\alpha}{12}\varepsilon^4 + \frac{\alpha}{6}\varepsilon^3 x_0 \quad \text{for} \quad x_0 + \varepsilon < y \leqq y_1$$

where

$$y_1 = \frac{1}{2}\,\frac{12 - \alpha\varepsilon^3(\varepsilon + 2x_0)}{6 - \alpha\varepsilon^3}$$

and

$$y = \psi_2(z) = y_1 z .$$

(8,5) implies that $\psi(z)$ maps the interval $\langle 0, 1\rangle$ one-to-one onto itself and equation (8,2) is transformed onto

$$\frac{\partial w}{\partial t} = \tfrac{1}{2}b^*(z)\,\frac{\partial^2 w}{\partial z^2} + a^*(z)\,\frac{\partial w}{\partial z} \tag{8,12}$$

where $v(t, x) = w(t, z)$ and $w(t, z)$ fulfils the same initial and boundary conditions as $v(t, x)$. For coefficients a^*, b^* we obtain

$$a^*(z) = -\frac{\psi''(z)}{2\psi'^3(z)} = -\frac{\psi_1''(\psi_2(z))}{2\psi_1'^3(\psi_2(z))\,\psi_2'(z)} , \tag{8,13}$$

$$b^*(z) = \frac{1}{\psi'^2(z)} = \frac{1}{\psi_1'^2(\psi_2(z))\,\psi_2'^2(z)} .$$

Since $\psi_1''(y) \neq 0$ only on the interval $(x_0, x_0 + \varepsilon)$ and this interval is transformed by $z = \psi_2^{-1}(y)$ into the interval $(x_0 - \delta, x_0 + \delta)$ owing to (8,10), (8,11), the function $a^*(z)$ is nonzero only in $(x_0 - \delta, x_0 + \delta)$. With respect to (8,7) and (8,13) we have $a^*(z) < \beta + \delta$ and with regard to (8,3) we obtain

$$a^*(z) \leqq \varphi(z, 1 - \delta, 1 + \delta) . \tag{8,14}$$

With respect to (8,13) inequalities (8,8) and (8,9) imply

$$1 - \delta \leqq b^*(z) \leqq 1 + \delta .$$

Since the function $\varphi(x, K_1, K_2)$ has property (C) this inequality together with (8,14) should mean that the solution $w(t, z)$ of (8,12) is convex in z. But by simple calculation we obtain $\psi_1''(x_0 + \frac{1}{2}\varepsilon) = -(\alpha\varepsilon^2/4)$ and using (8,6) and (8,13) we derive

$$(8,15) \qquad a^*(z^*)\,\psi'(z^*) > \beta - \delta \quad \text{where} \quad z^* = \frac{1}{y_1}\left(x_0 + \frac{\varepsilon}{2}\right).$$

Since $\psi(z^*) = \psi_1(x_0 + \varepsilon/2) \leqq \psi_1(x_0 + \varepsilon) = x_0 + \varepsilon - (\alpha/12)\,\varepsilon^4 < x_0 + \delta$ and

$$(8,16) \qquad \psi(z^*) = \psi_1\left(x_0 + \frac{\varepsilon}{2}\right) \geqq \psi_1(x_0) = x_0\,,$$

inequalities (8,4) and (8,15) imply

$$(8,17) \qquad a^*(z^*)\,\psi'(z^*) > \frac{1}{2}\min_t\left[\frac{\partial^2 v}{\partial x^2}(t, \psi(z^*)) \Big/ \frac{\partial v}{\partial x}(t, \psi(z^*))\right].$$

From $w(t, z) = v(t, x) = v(t, \psi(z))$ we obtain

$$\frac{\partial^2 w}{\partial z^2}(t, z^*) = \left[\frac{\partial^2 v}{\partial x^2}(t, \psi(z^*)) - 2\frac{\partial v}{\partial x}(t, \psi(z^*))\,a^*(z^*)\,\psi'(z^*)\right]\psi'^2(z^*)\,.$$

Since $\psi(z^*) \geqq x_0 > \frac{1}{2}$ (see (8,16)), we have $(\partial v/\partial x)(t, \psi(z^*)) > 0$ and (8,17) implies that there exists $t^* \in (0, L)$ such that $(\partial^2 w/\partial z^2)(t^*, z^*) < 0$. But this means that w cannot be convex with respect to z and Theorem is proved.

9. An example was introduced in [1] showing that the unit matrix is not strongly maximal with respect to the square and to the vector function $a_1(t, x, y) \equiv \equiv a_2(t, x, y) \equiv 0$. It will be shown in this section that if a matrix function defined in a cylindric region $Q = (0, L) \times D$ is strongly maximal, then D is convex and has a tangent at every point of $\dot{D}$. The proof is carried out for two-dimensional space only since for multidimensional spaces it would be much more complicated.

A function $u(t, x)$ is a solution of (9,1) if there exist continuous derivative $\partial u/\partial t$ and continuous second derivatives $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\,\partial y$, $\partial^2 u/\partial y^2$ which fulfil (9,1).

A solution is locally convex if $\partial^2 u/\partial x^2 \geqq 0$, $\partial^2 u/\partial y^2 \geqq 0$, $(\partial^2 u/\partial x\,\partial y)^2 \leqq \leqq (\partial^2 u/\partial x^2)(\partial^2 u/\partial y^2)$ at any point of Q and is convex if

$$u(t, \lambda x_1 + (1 - \lambda)\,x_2\,, \quad \lambda y_1 + (1 - \lambda)\,y_2) \leqq \lambda u(t, x_1, y_1) + (1 - \lambda)\,u(t, x_2, y_2)$$

provided that $0 \leqq \lambda \leqq 1$ and all mentioned points belong to Q. If D is convex these both definitions are equivalent.

A matrix function $B(t, x, y)$ is uniformly positive definite in Q if there exists a constant $m > 0$ such that $(Bz, z) \geqq m|z|^2$ for every $z \neq 0$.

Theorem 6. *Let functions $a_1(t, x, y)$, $a_2(t, x, y)$ be defined, measurable and bounded on Q. Let the elements $B_{ij}(t, x, y)$ of a symmetric matrix function $B(t, x, y)$*

be defined and locally integrable on Q. Assume that the matrix function $B(t, x, y)$ is uniformly positive definite in Q. Let $u(t, x, y)$ be a solution of

$$(9,1)\qquad \frac{\partial u}{\partial t} = \frac{1}{2}\left[B_{11}(t, x, y)\frac{\partial^2 u}{\partial x^2} + 2B_{12}(t, x, y)\frac{\partial^2 u}{\partial x\,\partial y} + B_{22}(t, x, y)\frac{\partial^2 u}{\partial y^2}\right] + $$
$$+ a_1(t, x, y)\frac{\partial u}{\partial x} + a_2(t, x, y)\frac{\partial u}{\partial y}$$

fulfilling

$$(9,2)\qquad \lim_{t\to 0+} u(t, x, y) = 0 \quad in \quad D\,,$$

$$(9,3)\qquad \lim_{[x,y]\to[\bar{x},\bar{y}]} u(t, x, y) = 1 \quad for \quad t > 0\,,\quad [\bar{x}, \bar{y}] \in \dot{D}\,.$$

If $u(t, x, y)$ is bounded for every $t > 0$ and locally convex with respect to x, y, then the region D is convex and there exists a tangent at every point of $\dot{D}$.

Proof. First of all two auxiliary lemmas will be proved. We can obviously assume that $\dot{D}$ is nonempty.

Lemma 6. *The solution u fulfils $u(t, x, y) \leqq 1$ on Q.*

There exists a point $[\bar{x}, \bar{y}]$ on $\dot{D}$. Let P be the straight line passing through $[\bar{x}, \bar{y}]$ and $[x, y]$. Denote by J the component of $P \cap D$ containing the point $[x, y]$. There are two possibilities:

i) J is a bounded interval. Since the boundary points of J belong to $\dot{D}$ we have $u = 1$ at these points owing to (9,3). Since u is locally convex on J and every inner point of J belong to D the function u is convex on J and inequality $u \leqq 1$ is proved.

ii) J is an unbounded interval. In this case J has only one boundary point and at this point is $u = 1$. Since u is convex on J again and u is bounded (the variable t is fixed) we obtain $u \leqq 1$ again. The lemma is proved.

Lemma 7. *Let $[t_0, x_0, y_0] \in Q$. If $u(t_0, x_0, y_0) = 1$, then $u(t_0, x, y) = 1$ on D.*

Let P be an arbitrary straight line passing through $[x_0, y_0]$. Denote by J the component of $P \cap D$ containing the point $[x_0, y_0]$. Now there are three possibilities.

i) J is a bounded interval. ii) J is an unbounded interval with one boundary point, iii) J is the whole P. Since u is bounded by 1 and convex on J we easily obtain that u is identically equal to one on J. It means that u equals one on every polygonal line which lies in D. Since the region D is connected the lemma is proved.

Now we shall prove that D is convex. Owing to (9,2) and Lemma 7 there exists a number $\delta > 0$ such that $u < 1$ on $(0, \delta) \times D$. We can suppose $L = \delta$ so that

$$(9,4)\qquad u(t, x, y) < 1 \quad \text{on} \quad Q\,.$$

Let $[u, v]$ and $[x, y]$ be two points belonging to D. Denote by $J(u, v; x, y)$ the segment with the end points $[u, v]$ and $[x, y]$. Assume $J(u, v; x, y) \not\subset D$. There exists a polygonal line S, $S \subset D$ with the end points $[u, v]$ and $[x, y]$. Obviously there is a line segment J^* which is parallel to $J(u, v; x, y)$, $J^* \subset \bar{D}$, $J^* \cap \dot{D} \neq \emptyset$, the end points of J^* being on S. Further there exist line segments J_n which are parallel to J^* with the end points on S and $J_n \subset D$. The solution u is continuous on J^* (for fixed $t > 0$) and since u is convex on J_n it is convex also on J^* but this is a contradiction with (9,4) and with the fact that $u = 1$ on $\dot{D}$ and $J^* \cap \dot{D} \neq \emptyset$. We have proved that D is convex.

The next part of the proof concerns the assertion that D has a tangent at every point of $\dot{D}$. Without loss of generality we can assume that the origin belongs to $\dot{D}$ and that D lies in the half plane $y > 0$, and that there exists a $\delta > 0$ such that the frontier of D can be locally expressed by a function $y = d(x)$, $|x| < \delta$. Obviously $d(x)$ is a convex function which has derivatives from the right and from the left and we assume $d'_+(0) = k > 0$, $d'_-(0) = -k$. Since u is convex in D there exists

$$\alpha(t, x) = \frac{\partial u}{\partial y}(t, x, d(x)) = \lim_{y \to d(x)+} \frac{\partial u}{\partial y}(t, x, y).$$

We shall prove

$$(9,5) \qquad \alpha(t, x) < 0 \quad (\text{it could be } -\infty) \quad \text{for} \quad |x| < \delta, \quad t > 0.$$

If $\alpha(t, x_0) \geqq 0$, then $(\partial u/\partial y)(t, x_0, y) \geqq 0$ for $y \geqq d(x)$ since u is convex, and with respect to (9,3) it is $u(t, x_0, y) \geqq 1$ which is a contradiction with (9,4), i.e. (9,5) is proved.

In the same way the following inequalities can be proved

$$(9,6) \qquad \lim_{y \to d(x)-} \frac{\mathrm{d}u}{\mathrm{d}y}\left(t, x + \frac{2}{k}(y - d(x)), y\right) > 0 \quad \text{for} \quad 0 < x < \delta, \quad t > 0,$$

and

$$\lim_{y \to d(x)-} \frac{\mathrm{d}u}{\mathrm{d}y}\left(t, x - \frac{2}{k}(y - d(x)), y\right) > 0 \quad \text{for} \quad -\delta < x < 0, \quad t > 0.$$

Further we shall prove that to any numbers $\delta > 0$, t_1, t_2, $0 < t_1 < t_2 < L$ there exists a number $\gamma > 0$ such that

$$(9,7) \qquad \alpha(t, x) < -\gamma \quad \text{for} \quad t \in \langle t_1, t_2 \rangle, \quad |x| \leqq \delta.$$

Let a point $[t_0, x_0]$, $t_1 \leqq t_0 \leqq t_2$, $|x_0| \leqq \delta$ be given. Since

$$\alpha(t_0, x_0) = \lim_{y \to d(x_0)+} \frac{\partial u}{\partial y}(t_0, x_0, y) < 0,$$

there exists such y_0 that

$$\frac{\partial u}{\partial y}(t_0, x_0, y_0) < \max\left[\alpha(t_0, x_0) + \varepsilon, -\frac{1}{\varepsilon}\right]$$

for every positive ε. Since $[t_0, x_0, y_0] \in Q$ and $\partial u/\partial y$ is continuous in Q there exists $\eta > 0$ such that

$$\frac{\partial u}{\partial y}(t, x, y_0) < \varepsilon + \max\left[\alpha(t_0, x_0) + \varepsilon, -\frac{1}{\varepsilon}\right]$$

for $|t - t_0| < \eta$, $|x - x_0| < \eta$. Since $\partial u/\partial y$ is nondecreasing in y we obtain $\alpha(t, x) < < \varepsilon + \max[\alpha(t_0, x_0) + \varepsilon, -1/\varepsilon]$. This means that the function $\alpha(t, x)$ is semi-continuous from above in $\langle t_1, t_2 \rangle \times \langle -\delta, \delta \rangle$ and with respect to (9,5) inequality (9,7) is proved.

Since u is sufficiently smooth in Q and (9,7), (9,6) are valid we find functions $g_1(y)$, $g_2(y)$ which are defined for $0 \leqq y \leqq \delta_1$ where δ_1 is a sufficiently small positive number, such that $g_1(y) < 0 < g_2(y)$ for $\delta_1 \geqq y > 0$, $g_i(y)$ are continuous, $g_i(0) = = 0$, $g_1(d(x)) \geqq x$ for $x \leqq 0$, $g_2(d(x)) \leqq x$ for $x \geqq 0$ and

(9,8) $$\frac{\partial u}{\partial y}(t, g_i(y), y) \leqq -\gamma/2 \quad \text{for} \quad t \in \langle t_1, t_2 \rangle,$$

(9,9)

$$\frac{\mathrm{d}u}{\mathrm{d}y}\left(t, \frac{2}{k}(y - y_0) + g_2(y_0), y\right) \geqq 0, \quad \frac{\mathrm{d}u}{\mathrm{d}y}\left(t, -\frac{2}{k}(y - y_0) + g_1(y_0), y\right) \geqq 0,$$

for $y = y_0$, $t \in \langle t_1, t_2 \rangle$. As it will be proved these inequalities imply

(9,10)

$$\frac{\partial u}{\partial x}(t, g_1(y), y) \leqq -k\gamma/4, \quad \frac{\partial u}{\partial x}(t, g_2(y), y) \geqq k\gamma/4 \quad \text{for} \quad 0 < y < \delta_1 \quad t \in \langle t_1, t_2 \rangle.$$

We shall prove the latter inequality only since the former can be reduced to the latter by the transformation $x = -\xi$. We choose points

$$L \equiv [g_2(y_0) - 2\Theta, \quad y_0 - k\Theta], \quad S \equiv [g_2(y_0) - \Theta, y_0],$$
$$P \equiv [g_2(y_0), y_0 + k\Theta]$$

where $\Theta > 0$. These points lie on the straight line $y = k(x - g_2(y_0) + \Theta) + y_0$. Since u is convex we have

$$2u(t, S) \leqq u(t, L) + u(t, P).$$

This inequality can be rewritten

$$2\,\frac{u(t,g_2(y_0)-\Theta,y_0)-u(t,g_2(y_0),y_0)}{\Theta}\leqq$$

$$\leqq\frac{u(t,g_2(y_0)-2\Theta,y_0-k\Theta)-u(t,g_2(y_0),y_0)}{\Theta}+$$

$$+\frac{u(t,g_2(y_0),y_0+k\Theta)-u(t,g_2(y_0),y_0)}{\Theta}$$

and with respect to (9,9) for $\Theta\to 0+$

$$(9{,}11)\qquad -2\frac{\partial u}{\partial x}(t,g_2(y_0),y_0)\leqq k\frac{\partial u}{\partial y}(t,g_2(y_0),y_0)\quad\text{for}\quad t\in\langle t_1,t_2\rangle\,.$$

We obtain (9,10) using (9,8).

Choose $h>0$ such that $B_{ij}(t,x,h)$ is integrable with respect to x over the interval $(g_1(h),g_2(h))$ for almost all t from $\langle t_1,t_2\rangle$. Put

$$(9{,}12)\qquad \varphi(t,h)=\int_{g_1(h)}^{g_2(h)}u(t,x,h)\,\mathrm{d}x\,.$$

Since u is sufficiently smooth we derive

$$(9{,}13)\qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t}=\int_{g_1(h)}^{g_2(h)}\frac{\partial u}{\partial t}\,\mathrm{d}x=\int_{g_1(h)}^{g_2(h)}\left[a_1(t,x,h)\frac{\partial u}{\partial x}+a_2(t,x,h)\frac{\partial u}{\partial y}+\right.$$

$$\left.+\tfrac12B_{11}(t,x,h)\frac{\partial^2u}{\partial x^2}+B_{12}(t,x,h)\frac{\partial^2u}{\partial x\,\partial y}+\tfrac12B_{22}(t,x,h)\frac{\partial^2u}{\partial y^2}\right]\mathrm{d}x\,.$$

Let M be the l.u.b. for a_i, i.e. $|a_i(t,x,h)|\leqq M$, $i=1,2$. Using the method of integration by parts we obtain

$$(9{,}14)\qquad \int_{g_1(h)}^{g_2(h)}a_1(t,x,h)\frac{\partial u}{\partial x}(t,x,h)\,\mathrm{d}x=\frac{\partial u}{\partial x}(t,g_2(h),h)\int_{g_1(h)}^{g_2(h)}a_1(t,x,h)\,\mathrm{d}x-$$

$$-\int_{g_1(h)}^{g_2(h)}\frac{\partial^2u}{\partial x^2}(t,x,h)\int_{g_1(h)}^{x}a_1(t,\xi,h)\,\mathrm{d}\xi\,\mathrm{d}x\geqq-\frac{\partial u}{\partial x}(t,g_2(h),h)\int_{g_1(h)}^{g_2(h)}|a_1(t,x,h)|\,\mathrm{d}x-$$

$$-\int_{g_1(h)}^{g_2(h)}|a_1(t,x,h)|\,\mathrm{d}x\int_{g_1(h)}^{g_2(h)}\frac{\partial^2u}{\partial x^2}(t,x,h)\,\mathrm{d}x=\left[\frac{\partial u}{\partial x}(t,g_1(h),h)-2\frac{\partial u}{\partial x}(t,g_2(h),h)\right]\cdot$$

$$\cdot\int_{g_1(h)}^{g_2(h)}|a_1(t,x,h)|\,\mathrm{d}x\geqq\left[\frac{\partial u}{\partial x}(t,g_1(h),h)-2\frac{\partial u}{\partial x}(t,g_2(h),h)\right]M(g_2(h)-g_1(h))\geqq$$

$$\geqq\left[\frac{\partial u}{\partial x}(t,g_1(h),h)-2\frac{\partial u}{\partial x}(t,g_2(h),h)\right]2M\,\frac{h}{k}$$

and similarly

$$(9{,}15)\quad \int_{g_1(h)}^{g_2(h)} a_2(t, x, h) \frac{\partial u}{\partial y}(t, x, h)\,\mathrm{d}x = \frac{\partial u}{\partial y}(t, g_2(h), h) \int_{g_1(h)}^{g_2(h)} a_2(t, x, h)\,\mathrm{d}x -$$

$$- \int_{g_1(h)}^{g_2(h)} \frac{\partial^2 u}{\partial x\,\partial y}(t, x, h) \int_{g_1(h)}^{x} a_2(t, \xi, h)\,\mathrm{d}\xi\,\mathrm{d}x \geqq \frac{\partial u}{\partial y}(t, g_2(h), h)\, 2M \frac{h}{k} -$$

$$- \int_{g_1(h)}^{g_2(h)} \frac{\partial^2 u}{\partial x\,\partial y}(t, x, h) \int_{g_1(h)}^{x} a_2(t, \xi, h)\,\mathrm{d}\xi\,\mathrm{d}x\,.$$

With respect to (9,13) to (9,15) we obtain

(9,16)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \geqq \left[\frac{\partial u}{\partial x}(t, g_1(h), h) - 2\frac{\partial u}{\partial x}(t, g_2(h), h)\right] 2M\frac{h}{k} + \frac{\partial u}{\partial y}(t, g_2(h), h)\, 2M\frac{h}{k} +$$

$$+ \frac{1}{2}\int_{g_1(h)}^{g_2(h)} \left[B_{11}(t, x, h)\frac{\partial^2 u}{\partial x^2} + 2\left(B_{12}(t, x, h) - \int_{g_1(h)}^{x} a_2(t, \xi, h)\,\mathrm{d}\xi\right)\frac{\partial^2 u}{\partial x\,\partial y} +\right.$$

$$\left. + B_{22}(t, x, h)\frac{\partial^2 u}{\partial y^2}\right]\mathrm{d}x\,.$$

Let a symmetric matrix B^* be defined by $B^*_{11} = B_{11}$,

$$B^*_{12} = -\left(|B_{12}| + \left|\int_{g_1(h)}^{x} a_2(t, \xi, h)\,\mathrm{d}\xi\right|\right)\operatorname{sgn}\frac{\partial^2 u}{\partial x\,\partial y}\,, \quad B^*_{22} = B_{22}\,.$$

Since B is uniformly positive definite with a constant m, then $(B^*z, z) \geqq$ $\geqq (m - |\int_{g_1(h)}^{x} a_2(t, \xi, h)\,\mathrm{d}\xi|)\,|z|^2$. If h is sufficiently small then $\int_{g_1(h)}^{g_2(h)} |a_2(t, \xi, h)|\,\mathrm{d}\xi <$ $< m/2$ and B^* is also uniformly positive definite. Since $(\partial^2 u/\partial x\,\partial y)^2 \leqq (\partial^2 u/\partial x^2)\,.$ $.\,(\partial^2 u/\partial y^2)$ (the solution u is locally convex) and since the matrix B^* is positive definite with the same constant $m/2$ even if we change the sign of the element B^*_{12} we obtain

$$B_{11}\frac{\partial^2 u}{\partial x^2} + 2\left(B_{12} - \int_{g_1(h)}^{x} a_2\,\mathrm{d}\xi\right)\frac{\partial^2 u}{\partial x\,\partial y} + B_{22}\frac{\partial^2 u}{\partial y^2} \geqq B^*_{11}\frac{\partial^2 u}{\partial x^2} + 2B^*_{12}\frac{\partial^2 u}{\partial x\,\partial y} +$$

$$+ B^*_{22}\frac{\partial^2 u}{\partial y^2} \geqq B^*_{11}\frac{\partial^2 u}{\partial x^2} - 2|B^*_{12}|\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}\right]^{1/2} + B^*_{22}\frac{\partial^2 u}{\partial y^2} \geqq$$

$$\geqq \frac{m}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \geqq \frac{m}{2}\frac{\partial^2 u}{\partial x^2}\,.$$

If h is small enough, then $M(1 + 1/k)\,h/k < m/16$ and with respect to (9,11), (9,16) we have

$$\frac{d\varphi}{dt} \geqq \left(\frac{m}{4} - 4M\frac{h}{k}\right)\frac{\partial u}{\partial x}(t, g_2(h), h) - \left(\frac{m}{4} - 2M\frac{h}{k}\right)\frac{\partial u}{\partial x}(t, g_1(h), h) +$$

$$+ \frac{\partial u}{\partial y}(t, g_2(h), h)\, 2M\frac{h}{k} \geqq \left(\frac{m}{4} - 4M\left(1 + \frac{1}{k}\right)\frac{h}{k}\right)\frac{\partial u}{\partial x}(t, g_2(h), h) -$$

$$- \left(\frac{m}{4} - 2M\frac{h}{k}\right)\frac{\partial u}{\partial x}(t, g_1(h), h)$$

and by (9,10)

$$\frac{d\varphi}{dt} \geqq \frac{k\gamma m}{16}\,. \tag{9,17}$$

The solution u was defined as continuous on $\langle t_1, t_2\rangle \times D$ so that $\psi(h) = \min u(t, x, y)$ for $0 \leqq y \leqq h$, $[x, y] \in D$, $t \in \langle t_1, t_2\rangle$ exists and $\psi(h) \to 1$ for $h \to 0$. We choose h so small that $u(t, x, h)$ is nonnegative for $t \in \langle t_1, t_2\rangle$, $g_1(h) \leqq x \leqq g_2(h)$. By Lemma 6 we have $0 \leqq u(t, x, y) \leqq 1$ for $0 \leqq y \leqq h$, $g_1(h) \leqq x \leqq g_2(h)$, $t \in \langle t_1, t_2\rangle$. By (9,12) we have

$$\varphi(\tau_2, h) - \varphi(\tau_1, h) \leqq g_2(h) - g_1(h) \leqq 2h/k$$

for arbitrary $\tau_1, \tau_2 : t_1 \leqq \tau_1 < \tau_2 \leqq t_2$.

With respect to (9,17) we obtain

$$m\frac{k\gamma}{16}(\tau_2 - \tau_1) \leqq \varphi(\tau_2, h) - \varphi(\tau_1, h) \leqq \frac{2h}{k}\,.$$

However, the last inequality is a contradiction with the fact that the numbers τ_1, τ_2 are arbitrary numbers from the interval $\langle t_1, t_2\rangle$ and h is an arbitrarily small number. Theorem 6 is proved.

References

[1] *I. Vrkoč:* Some maximum principles for stochastic equations. Czech. Math. J. V. *19* (*94*), 1969, 569–604.

[2] *A. Friedman:* Partial differential equations of parabolic type. Prentice-Hall, Inc. 1964.

[3] *G. Schleinkofer:* Die erste Randwertaufgabe und das Cauchy — Problem für parabolische Differentialgleichungen mit unstetigen Anfangswerten. Mathematische Zeitschrift 1969, B 111, 87–97.

[4] *И. И. Гихман, А. В. Скороход:* Стохастические дифференциальные уравнения. Изд. Наукова Думка, Киев 1968.

[5] *И. И. Гихман, А. В. Скороход:* Введение в теорию случайных процессов. Изд. Наука Москва 1965.

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