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A DUALITY THEOREM FOR STANDARD THREADS*)

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I. INTRODUCTION

The characters of a topological groups are continuous complex-valued homomorphisms for the simple reason that no continuous non-trivial real-valued characters exist. In the case of a topological semigroup the situation is very different since there are semigroups for which no continuous non-trivial semicharacters exist. One approach is to consider equivalence classes of measurable complex-valued semicharacters as was done in [1]. In this paper we show that by considering such equivalence classes for real-valued semicharacters a Pontrjagin type of duality theorem can be obtained for a certain class of semigroups.

A *standard thread* [2] is a compact semigroup S with a total order such that:

i) the order topology is the given topology;

ii) S is connected in the order topology;

iii) S has a maximal element and it is an identity;

iv) S has a minimal element and it is a zero.

A *nil thread* [2] is a standard thread having no interior idempotent but at least one non-zero nilpotent element. A *unit thread* [2] is a standard thread with no interior idempotent element and no nilpotent element. With any thread S we associate its order dual S^0 obtained from S by inverting the order relation.

Suppose E is a compact totally ordered set. We define two distinct points of E to be adjacent if there is no element of E between them. The lesser of two adjacent elements will be called an initial element; the greater a terminal element. An

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element which is not an initial element will be called a limit element. With each initial element e of E we associate a nil or unit thread S_e. Identify e with the zero element of S_e and let S'_e denote S_e with its identity deleted. If e is a limit point, define $S_e = S'_e =$ one point semigroup $\{e\}$. The *ordinal sum* of the semigroups $\{S'_e\}_{e \in E}$ is denoted by $\sum_e S'_e$ and is the semigroup $S = \bigcup_{e \in E} S'_e$ where the product $a \circ b$ of two elements in S is given by

$$a \circ b = \text{usual product in } S'_e \text{ if } a, b \in S'_e =$$
$$= ab = ba = a \text{ if } a \in S'_e,\ b \in S'_f \text{ and } e < f\,.$$

The canonical representations in the following theorems are due to CLIFFORD [2] and form the basis for our work.

Theorem. *Every standard thread S is the ordinal sum of a compact totally ordered set of half open nil threads, half open unit threads, and one element semigroups.*

Theorem. *Every unit thread is isomorphic to $[0, 1]$ with ordinary multiplication, and every nil thread N is isomorphic to $[\frac{1}{2}, 1]$ with $x \circ y = \max\{\frac{1}{2}, xy\}$ for $x, y \in \in [\frac{1}{2}, 1]$.*

II. THE REAL DUAL OF A STANDARD THREAD

In order to investigate the nature of the real-valued semicharacters of a standard thread it suffices to examine the duals of the canonical elemenets.

Proposition 1. *Let S be the semigroup $(0, 1)$ under ordinary multiplication and let S^* denote the semigroup of bounded real-valued non-constant semicharacters on S.*

Then S^* is isomorphic to the order dual of the semi-group of positive real numbers under addition.

Proof. The proof of this proposition, although very simple, is rather long and is broken down into six steps.

i) Consider the possible zeros for an element $\tau \in S^*$:

If $\tau(x) = 0$, then clearly $\tau(y) = 0$ for all $y < x$. Moreover, if $\tau \not\equiv 0$, then there exists a z such that $\tau(z) \neq 0$ but $z < 1$ implies that for some n we have $z^n < x$ so $\tau(z^n) = 0 = \tau(z)^n$. Hence $\tau \equiv 0$ if it is zero at any point.

ii) Every element of S^* is order preserving:

Suppose $x < y$. Then $x = sy$ for some $s \in S$ hence $\tau(x) = \tau(sy) = \tau(s) \,.\, \tau(y)$ but $\tau(s) \leqq 1$ and thus $\tau(x) \leqq \tau(y)$.

iii) $\tau \in S^*$ implies $\tau(y^r) = (\tau(y))^r$ for any rational $r > 0$:

iv) $\tau \in S^*$ implies τ is continuous in the order topology:

As result of (ii), τ is monotone and hence has only jump discontinuities. In particular, suppose

$$\tau^-(y) = \sup\{\tau(x) : x < y\} < \tau(y)$$

for some y. Pick an $x_0 < y$ and choose a rational r such that

$$\tau^-(y) < [\tau(x_0)^r] < \tau(y).$$

Then, from (iii), we have $\tau^-(y) < \tau(x_0^r) < \tau(y)$ which is a contradiction. A similar argument is used for $\tau^+(y) = \inf\{\tau(x) : x > y\}$.

v) If $\tau \in S^*$, then $\tau(x) = x^{\alpha_0}$ for some $\alpha_0 \in (0, \infty)$:

It is clear that $\tau(x) = 1$ for some x implies $\tau \equiv 1$. If $\tau \not\equiv 1$, then $\tau(\frac{1}{2}) = (\frac{1}{2})^{\alpha_0}$ for some real number α_0 and it follows from (iii) and (iv) that $\tau(x) = x^{\alpha_0}$ for all $x \in (0, 1)$.

vi) S^* is isomorphic to $\{R^+, +\}^0$:

Use the obvious mapping $\Phi : S^* \to R^+$ by $\Phi : x^\alpha \to \alpha$.

Corollary 2. *If S is the semigroup $(0, 1)$ under ordinary multiplication and S^* is the semigroup of bounded real-valued non-constant semicharacters on S, then S^{**} is isomorphic to S.*

Proof. Define the mapping $\psi : (R^+)^0 \to (0, 1)$ by $\psi : \alpha \to 1/e^\alpha$.

Corollary 3. *If T is a unit thread, m is Lebesgue measure on T, and T^* is the semigroup of equivalence classes of bounded, measurable, real-valued semicharacters on T, then T^* is isomorphic to the order dual of $[0, \infty]$ under addition and T^{**} is isomorphic to T.*

Proof. We let $[\theta]$ denote the equivalence class containing the two semicharacters τ_0 and τ_0' where

$$\tau_0(x) \equiv 0 \quad \tau_0'(x) = 0 \quad \text{if} \quad 0 \leqq x < 1$$
$$= 1 \quad \text{if} \quad x = 1$$

and similarly $[1]$ denotes the equivalence class containing τ_1 and τ_1' where

$$\tau_1(x) \equiv 1 \quad \tau_1'(x) = 0 \quad \text{if} \quad x = 0$$
$$= 1 \quad \text{if} \quad 0 < x \leqq 1.$$

Since any semicharacter has the value 0 or 1 at an idempotent and any non-constant semicharacter must have values 0 at zero and 1 at the identity it follows that $T^* = S^* \cup \{[\theta]\} \cup \{[1]\}$. We can now map T^* directly onto $[0, 1]$ by means of the function λ defined by

$$\begin{aligned} \lambda(\tau) &= 0 \quad &\text{if} \quad &\tau \in [\theta] \\ &= 1/e^{\alpha} \quad &\text{if} \quad &\tau = f_{\alpha} = x^{\alpha} \\ &= 1 \quad &\text{if} \quad &\tau \in [1]. \end{aligned}$$

From this it follows that T^{**} is isomorphic to T.

Theorem 4. *Let S be a standard thread with m the induced Lebesgue measure on S, and let S^* be the semigroup of equivalence classes of bounded, measurable, real-valued semicharacters.*

Then S^{**} is isomorphic to S if and only if S has no non-idempotent nilpotent elements.

Proof. As an immediate consequence of Clifford's theorems, we infer that a standard thread with no non-idempotent nilpotents is the ordinal sum $S = \sum_{e \in E} S'_e$ where $\{S'_e\}_{e \in E}$ are either unit threads or one element semigroups. We shall prove that the dual S^* of such an ordinal sum is the ordinal sum over the order dual, E^0, of the duals $(S'_e)^*$; that is, $S^* = \sum_{e \in E^0} (S'_e)^*$. The theorem then follows immediately from the preceding corollary and the trivial fact that $(E^0)^0 = E$.

We begin by proving that any element of S^* either belongs to the equivalence class of the characteristic function $X_{[e_\alpha e_m]}$ for some $e_\alpha \in E$ (where e_m denotes the maximal element of S); or is a natural extension of an element of $(S'_e)^*$ for some $S'_e \neq \{e\}$. This is done in the following lemmas.

Lemma 5. If we fix a semicharacter ψ_0 in the dual of S and define $U\psi_0 = \{e : e \in E, \psi_0 \mid S'_e \not\equiv 0\}$, $L\psi_0 = \{e : e \in E, \psi_0 \mid S'_e \equiv 0\}$, then $\{L\psi_0, U\psi_0\}$ is a cut for E.

Proof. i) Suppose $e < f$ with $f \in L\psi_0$. For any element $a \in S'_e$ and $b \in S'_f$, since $ab = ba = a$, we have $\psi_0(a) = 0$ which implies $e \in L\psi_0$.

ii) Similarly if $e \in U\psi_0$ with $f > e$ one can show $f \in U\psi_0$.

Remark. It also follows from (ii) that if $e, f \in U\psi_0$ with $f > e$, then $\psi_0 \mid S'_f \equiv 1$.

Lemma 6. *If $\{L\psi_0, U\psi_0\}$ is the cut defined in Lemma 5, then $(L\psi_0, U\psi_0\}$ determines a point $e_{\alpha 0}$ of E.*

Proof. Case 1. Suppose $U\psi_0$ has no least element. Then, from the remark above, it follows that $\psi_0 \equiv 1$ on $U\psi_0$. If we let e_α denote the greatest element of $L\psi_0$ it follows that $S'_{e_\alpha} = \{e_\alpha\}$ and hence $\psi_0 = \chi(e_\alpha, e_m)$.

Case 2. Suppose $L\psi_0$ has no greatest element whence $U\psi_0$ has a least element e_β. This least element must then be a one element semigroup and thus $\psi_0 = \chi[e_\beta, e_m]$.

Case 3. Suppose $L\psi_0$ and $U\psi_0$ have greatest and least elements e_α and e_β respectively. Then, since $\psi_0(x) = 0$ for $x \leqq e_\alpha$ and $\psi_0(x) = 1$ for $x \geqq e_\beta$ we conclude that ψ_0 is the natural extension of a semicharacter τ on S'_{e_α}. By natural extension we mean the semicharacter $\hat{\tau}$ on S obtained from τ in the dual of S'_{e_α} by

$$\begin{aligned} \hat{\tau}(x) &= 0 \quad \text{if} \quad x \in S'_e \quad e < e_\alpha \\ &= (x) \quad \text{if} \quad x \in S'_{e_\alpha} \\ &= 1 \quad \text{if} \quad x \in S'_e \quad e > e_\alpha \,. \end{aligned}$$

Combining the three cases we see that any element $\psi \in S^*$ can be identified via the cut $\{L\psi, U\psi\}$ as belonging to the equivalence class of a characteristic function of the form $\chi_{[e_\alpha, e_m]}$, or as being the natural extension of a semicharacter τ on a thread associated with some initial point of E.

We now return to the proof of Theorem 4. Suppose that ψ_e and τ_f are two elements of S^* where $e, f \in E$ and $\psi_e \in (S'_e)^*$, $\tau_f \in (S'_f)^*$. Then we can define the pointwise product $\psi_e \circ \tau_f$ as follows

$$\begin{aligned} \psi_e \circ \tau_f &= \text{usual product} \qquad \text{if} \quad e = f \\ &= \psi_e \circ \tau_f = \tau_f \circ \psi_e = \tau_f \quad \text{if} \quad e < f \,. \end{aligned}$$

However, since $S^* = \bigcup_{e \in E} (S'_e)^*$ where $\chi_{[e_\alpha, e_m]}$ is the only element of (S'_{e_α}) if $S'_{e_\alpha} = \{e_\alpha\}$ we conclude that, as a semigroup, S^* is the ordinal sum over E^0 of the duals $(S'_e)^*$; that is,

$$S^* = \sum_{e \in E^0} (S'_e)^*$$

and thus

$$S^{**} = \sum_{e \in (E^0)^0} (S'_e)^{**} \,.$$

However, from Corollary 2 and the trivial relation $(E^0)^0 = E$ it follows that

$$S^{**} = \sum_{e \in E} (S'_e) = S \,.$$

The proof of the converse is much easier since the existence of a non-idempotent nilpotent implies the existence of a nil thread in the ordinal sum. Since the dual of a nil thread is a two point semigroup, it follows that the second dual is not isomorphic to the thread.

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