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IMMERSIONS OF RIEMANNIAN MANIFOLDS WITH A GIVEN NORMAL BUNDLE STRUCTURE

PART II.

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In the previous part we characterized a maximal isometric immersion of a Riemannian manifold M into a Riemannian space N with the constant curvature by a system of metric tensors $h_1, \ldots, h_r$ on M. Here the "generalized Gaussian equations" formed a complete system of integrability conditions for our problem. (We shall refer to them as to "Gaussian equations of genera $1, \ldots, r$" in the sequel; cf. [1], Formula (24).)

In the present part we consider a new system of tensors $B_1, \ldots, B_r$ on M called *Bompiani forms*. These forms were introduced originally by E. BOMPIANI and we give here a modern interpretation of them. We also give an alternative definition of that structure which was called by E. Bompiani "a Riemannian geometry of genus r". As a basic result, in the maximal case we show the equivalence of our definition with the classical one. We prove this non-trivial fact making use of an immersion theorem due to V. V. RYŽKOV (See [2]): Any sequence $B_1, \ldots, B_r$ of analytic symmetric forms satisfying certain positive definiteness conditions on M is locally realizable as a *partial system* of Bompiani forms of a maximal analytic immersion $M \to N$. (We outline also the direct way how to prove the equivalence of the definitions but an open problem is left in this direction.) As a consequence, we derive a new immersion theorem: The only integrability condition for symmetric forms $B_1, \ldots, B_r$ of class C^∞ to be (globally) realizable as a *full system* of Bompiani forms of a maximal immersion $M \to N$ is that the Gaussian equation of genus r holds in the corresponding Riemanmenian geometry of genus r. (Besides that the positive definiteness conditions must be satisfied by $B_1, \ldots, B_r$. These conditions were mentioned rather vaguely in Paper [2] and they are given precisely here for the first time.)

The last result seems to be much deeper than the classical theorem, saying that the integrability conditions for $B_1, \ldots, B_r$ are the equations of Gauss, Mainardi-Codazzi and Ricci-Kühne. (Cf. [4] and [5].)

We shall keep all the notation of [1] if not otherwise stated. Particularly, let us remind the concept of a graded Riemannian vector bundle (Definition 1) and that of a sequence of canonical connections (Definition 2, ibid.).

RIEMANNIAN GEOMETRIES OF HIGHER GENUS

Proposition 1. *Let $\{E^k, P_k\}^r$ be a graded Riemannian vector bundle over a Riemannian manifold M. Suppose that a sequence of canonical connections $\nabla^{(1)}, \ldots, \nabla^{(r)}$ exists in $\{E^k, P_k\}^r$. Then, under the canonical identification $E^1 \equiv T(M)$, we have the following identities*:

$$\text{(1a)} \qquad R^{(1)}_{UT}V + R^{(1)}_{VU}T + R^{(1)}_{TV}U = 0\,,$$

$$\text{(1b)} \qquad R^{(k+1)}_{UT}P_k(V, X^{(k)}) + R^{(k+1)}_{VU}P_k(T, X^{(k)}) + R^{(k+1)}_{TV}P_k(U, X^{(k)}) =$$
$$= P_k(V, R^{(k)}_{UT}X^{(k)}) + P_k(T, R^{(k)}_{VU}X^{(k)}) + P_k(U, R^{(k)}_{TV}X^{(k)})\,.$$

for $k = 1, 2, \ldots, r - 1$.

Proof. (1a): Directly by Formula (2), [1].

(1b): We express the left-hand side by Formula (2), [1] and then apply in two steps Formula (18), [1].

Remark. Formula (1a) is classical, (1b) is its generalization of "higher genus".

Definition 1. By an equivalence of two graded Riemannian bundles $E = \{E^k, P_k\}^r$, $E' = \{E'^k, P'_k\}^r$ (with the same base M) we mean a bundle morphism $\Phi : E \to E'$ with the following properties:

(i) For any $x \in M$, Φ maps E_x isometrically onto E'_x.

(ii) For any $k = 1, \ldots, r$, Φ maps E^k into E'^k.

(iii) We have $\Phi \circ P_k = P'_k \circ \Phi$ for $k = 1, \ldots, r - 1$.

(iv) For the corresponding solder maps $j : T(M) \to E$, $j' : T(M) \to E'$ we have $j' = \Phi \circ j$.

Definition 2. A graded Riemannian bundle $\{E^k, P_k\}^r \to M$ is called strictly realizable if it is equivalent to an induced vector bundle $\psi^*T(N) \to M$ where ψ is an isometric immersion of M into a Riemannian space N with constant curvature.

The bundle $\{E^k, P_k\}^r \to M$ is called realizable, if it is a graded subbundle of a strictly realizable graded Riemannian bundle $\{E^k, P_k\}^s$, $s \geqq r$.

Definition 3. A Riemannian geometry $G_{r,C}$, of genus r and of exterior curvature C, on a manifold M is a graded Riemannian vector bundle $E = \{E^k, P_k\}^r$ over M such that

(i) a sequence $\nabla^{(1)}, \ldots, \nabla^{(r)}$ of canonical connections exists in E,

(ii) the Gaussian equations of genera $1, \ldots, r - 1$ and with the parameter C hold.

Proposition 2. *Let* $\mathbf{G}_{r,C} = \{E^k, P_k\}^r$ *be a Riemannian geometry of genus r over M. Put*

$$(2) \qquad Q_k(U, T, X^{(k)}, Y^{(k)}) = -L_k(U, T, X^{(k)}, Y^{(k)}) + R^{(k)}(U, T, X^{(k)}, Y^{(k)}) + \\ + C\{\langle U, Y^{(k)}\rangle \langle T, X^{(k)}\rangle - \langle U, X^{(k)}\rangle \langle T, Y^{(k)}\rangle\}$$

for $k = 1, \ldots, r$.

Then we have the following identities:

$$(3a) \qquad Q_1(U, T, V, Y) + Q_1(V, U, T, Y) + Q_1(T, V, U, Y) = 0\,,$$

$$(3b) \qquad Q_{k+1}(U, T, P_k(V, X^{(k)}), Y^{(k+1)}) + Q_{k+1}(V, U, P_k(T, X^{(k)}), Y^{(k+1)}) + \\ + Q_{k+1}(T, V, P_k(U, X^{(k)}), Y^{(k+1)}) = 0$$

for $k = 1, 2, \ldots, r - 1$.

Proof. (3a) follows directly from (2) and (1a) (here $L_1 \equiv 0$).
(3b): Substituting from (2) we obtain the left-hand side of (3b) in the form

$$(4) \qquad -L_{k+1}(U, T, P_k(V, X^{(k)}), Y^{(k+1)}) - L_{k+1}(V, U, P(T, X^{(k)}), Y^{(k+1)}) - \\ - L_{k+1}(T, V, P_k(U, X^{(k)}), Y^{(k+1)}) + \\ + R^{(k+1)}(U, T, P_k(V, X^{(k)}), Y^{(k+1)}) + R^{(k+1)}(V, U, P_k(T, X^{(k)}, Y^{(k+1)}) + \\ + R^{(k+1)}(T, V, P_k(U, X^{(k)}), Y^{(k+1)}).$$

We apply Formula (20), [1], to the first half of (4) and after a simple re-arrangement we obtain the expression

$$(5) \qquad \langle L_{k+1}(U, P_k(V, X^{(k)})) - L_{k+1}(V, P_k(U, X^{(k)})), L_{k+1}(T, Y^{(k+1)})\rangle + \\ + \langle L_{k+1}(V, P_k(T, X^{(k)})) - L_{k+1}(T, P_k(V, X^{(k)})), L_{k+1}(U, Y^{(k+1)})\rangle + \\ + \langle L_{k+1}(T, P_k(U, X^{(k)})) - L_{k+1}(U, P_k(T, X^{(k)})), L_{k+1}(V, Y^{(k+1)})\rangle\,.$$

Now we use the Gaussian equation of genus $k \leqq r - 1$ in the form

$$L_{k+1}(U, P_k(T, X^{(k)})) - L_{k+1}(T, P_k(U, X^{(k)})) = -R^{(k)}_{UT}X^{(k)} + \\ + P_{k-1}(T, L_k(U, X^{(k)})) - P_{k-1}(U, L_k(T, X^{(k)}))\,.$$

We substitute into (5) and detach the vectors $Y^{(k+1)}$ in all scalar products using the duality formula (15) from [1]. Taking into account the symmetry of the mapping $P_{k+1} \circ P_k$ we obtain finally the expression

$$\langle P_k(U, R^{(k)}_{VT}X^{(k)}) + P_k(T, R^{(k)}_{UV}X^{(k)}) + P_k(V, R^{(k)}_{TU}X^{(k)}), Y^{(k+1)}\rangle\,.$$

Now, let us form the scalar product of each side of (1b) with the vector $Y^{(k+1)}$. Then we see immediately that the expression (4) is zero.

Remark. For $k \leqq r-2$ the proof is trivial because we can use the Gaussian equation of genus $k+1$ in the form

$$P_{k+1}(U, T, X^{(k+1)}, Y^{(k+1)}) = Q_{k+1}(U, T, X^{(k+1)}, Y^{(k+1)}) \tag{6}$$

and then substitute P_{k+1} instead of Q_{k+1} into (3b).

Proposition 3. *Any realizable bundle $\{E^k, P_k\}^r \to M$ is a Riemannian geometry of genus r.*

Proof is clear from the considerations of the first part. Now, the Gaussian equation of genus r is a necessary condition for a Riemannian geometry of genus r to be strictly realizable. (This condition is also sufficient in the case that M is simply connected, cf. Theorem 2, [1].)

We derive now a weaker condition which is necessary for a Riemannian geometry $\mathbf{G}_{r,C}$ to be realizable. In fact, let $\mathbf{G}_{r,C} = \{E^k, P_k\}^r$ be a realizable but not strictly realizable Riemannian geometry. Then $\{E^k, P_k\}^r$ is a subbundle of a realizable Riemannian geometry $\{E^k, P_k\}^{r+1}$ of genus $r+1$ (and of the same exterior curvature C). Thus the Gaussian equation of genus r holds in the bundle $\{E^k, P_k\}^{r+1}$:

$$P_r(U, T, X^{(r)}, Y^{(r)}) = Q_r(U, T, X^{(r)}, Y^{(r)})\,. \tag{7}$$

Put

$$h_k(X_1, \ldots, X_k \mid Y_1, \ldots, Y_k) = \langle P^k(X_1, \ldots, X_k), P^k(Y_1, \ldots, Y_k)\rangle\,,$$

and consider the following identity on $\{E^k, P_k\}^{r+1}$:

$$\sum_{i=1}^{r+1} \{h_{r+1}(X_i, \ldots, X_{r+1}, Y_1, \ldots, Y_{i-1} \mid Y_i, \ldots, Y_{r+1}, X_1, \ldots, X_{i-1}) -$$
$$- h_{r+1}(Y_i, X_{i+1}, \ldots, X_{r+1}, Y_1, \ldots, Y_{i-1} \mid X_i, Y_{i+1}, \ldots, Y_{r+1}, X_1, \ldots, X_{i-1})\} = 0\,.$$

Hence we have

$$\sum_{i=1}^{r+1} \{\langle P_r(X_i, P^r(X_{i+1}, \ldots, X_{r+1}, Y_1, \ldots, Y_{i-1})),$$
$$P_r(Y_i, P^r(Y_{i+1}, \ldots, Y_{r+1}, X_1, \ldots, X_{i-1}))\rangle -$$
$$- \langle P_r(Y_i, P^r(X_{i+1}, \ldots, X_{r+1}, Y_1, \ldots, Y_{i-1})),$$
$$P_r(X_i, P^r(Y_{i+1}, \ldots, Y_{r+1}, X_1, \ldots, X_{i-1}))\rangle\} =$$
$$= \sum_{i=1}^{r+1} P_r(Y_i, X_i, P^r(X_{i+1}, \ldots, Y_{i-1}), P^r(Y_{i+1}, \ldots, X_{i-1})) = 0\,.$$

From (7) we obtain a tensor equation on M depending only on the Riemannian geometry $\{E^k, P_k\}^r$:

$$(8)\qquad \sum_{i=1}^{r+1} Q_r(Y_i, X_i, P^r(X_{i+1}, \ldots, X_{r+1}, Y_1, \ldots, Y_{i-1}), P^r(Y_{i+1}, \ldots, Y_{r+1}, X_1, \ldots, X_{i-1})) = 0$$

which we call *the cyclic condition of genus r.*

Remark. If $\mathbf{G}_{r,C} = \{E^k, P_k\}^r$ is strictly realizable, then we have the Gaussian equation of genus r, $Q_r(U, T, X^{(r)}, Y^{(r)}) = 0$. Hence (8) is satisfied trivially.

We can summarize:

Proposition 4. *Any realizable Riemannian geometry* $\mathbf{G}_{r,C}$ *satisfies the cyclic condition of genus r.*

Definition 4. Let $h_k(X_1, \ldots, X_k \mid Y_1, \ldots, Y_k)$, $k = 1, \ldots, r$, be the metric tensors of a graded Riemannian vector bundle $\{E^k, P_k\}^r \to M$, $h_k(X_1, \ldots, X_k \mid Y_1, \ldots, Y_k) = \langle P^k(X_1, \ldots, X_k), P^k(Y_1, \ldots, Y_k)\rangle$. The Bompiani forms of the bundle $\{E^k, P_k\}^r$ are symmetric covariant tensors on M defined for $k = 1, \ldots, r$ by

$$(9)\qquad B_k(X_1, \ldots, X_k, X_{k+1}, \ldots, X_{2k}) = \frac{1}{(2k)!} \sum_{\sigma} h_k(X_{\sigma(1)}, \ldots, X_{\sigma(k)} \mid X_{\sigma(k+1)}, \ldots, X_{\sigma(2k)})$$

where $\sum_{\sigma}$ indicates the summation over all permutations of indices $1, \ldots, 2k$.

In a similar way we define *the Bompiani forms of an immersion* $\psi : M \to N$ to be symmetrizations of the corresponding metric tensors.

Proposition 5. *Let* $\mathbf{G}_{r-1,C} = \{E^k, P_k\}^{r-1}$ *be a Riemannian geometry of genus* $r - 1$ *on a manifold M. Let* $B_r(X_1, \ldots, X_{2r})$ *be a symmetric covariant tensor ("symmetric 2r-form") on M. Then there is, exact up to an equivalence, at most one Riemannian geometry* $\mathbf{G}_{r,C} = \{E^k, P_k\}^r \to M$ such that

a) $\mathbf{G}_{r,C}$ is a prolongation of $\mathbf{G}_{r-1,C}$,

b) B_r is the r-th Bompiani form of $\mathbf{G}_{r,C}$.

Proof. Let $\mathbf{G}_{r,C}$ be a Riemannian geometry with properties *a*), *b*) (provided that it exists). We can write the Gaussian equation of genus $(r - 1)$ in the form

$$\langle P_{r-1}(T, X^{(r-1)}), P_{r-1}(U, Y^{(r-1)})\rangle - \langle P_{r-1}(U, X^{(r-1)}), P_{r-1}(T, Y^{(r-1)})\rangle = Q_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)}) .$$

Using the standard notation

$$P(X_1, \ldots, X_r) = P^r(X_1, \ldots, X_r),$$

$$h_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = \langle P(X_1, \ldots, X_r), P(Y_1, \ldots, Y_r)\rangle$$

we can write

$$\begin{aligned}(10)\qquad & h_r(X_1, \ldots, X_\alpha, Y_{\alpha+1}, \ldots, Y_r \mid Y_1, \ldots, Y_\alpha, X_{\alpha+1}, \ldots, X_r) - \\ & - h_r(X_1, \ldots, X_\alpha, X_{\alpha+1}, Y_{\alpha+2}, \ldots, Y_r \mid Y_1, \ldots, Y_{\alpha+1}, X_{\alpha+2}, \ldots, X_r) = \\ & = Q_{r-1}(X_{\alpha+1}, Y_{\alpha+1}, P(X_1, \ldots, X_\alpha, Y_{\alpha+2}, \ldots, Y_r), P(Y_1, \ldots, Y_\alpha, X_{\alpha+2}, \ldots, X_r)).\end{aligned}$$

Further, we can see easily that (putting for a moment $X_{r+i} = Y_i$)

$$\begin{aligned}(11)\quad B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r) &= \frac{1}{(2r)!} \sum_\sigma h_r(X_{\sigma(1)}, \ldots, X_{\sigma(r)} \mid X_{\sigma(r+1)}, \ldots, X_{\sigma(2r)}) = \\ &= \frac{(r!)^2}{(2r)!} \sum_{\alpha=0}^{r} \sum_{(i\cup I, j\cup J, <)}^{\alpha} h_r(X_{i_1}, \ldots, X_{i_\alpha}, Y_{J_1}, \ldots, Y_{J_{r-\alpha}} \mid Y_{j_1}, \ldots, Y_{j_\alpha}, X_{I_1}, \ldots, X_{I_{r-\alpha}})\end{aligned}$$

where $\sum\limits_{(i\cup I, j\cup J, <)}^{\alpha}$ indicates the summation over *all increasing chains* $i_1 < i_2 < \ldots < i_\alpha$, $j_1 < j_2 < \ldots < j_\alpha$, $J_1 < J_2 < \ldots < J_{r-\alpha}$, $I_1 < I_2 < \ldots < I_{r-\alpha}$ selected from the index set $\{1, 2, \ldots, r\}$ and such that $\{i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}\} = \{j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha}\} = \{1, \ldots, r\}$.

From (10) we obtain

$$\begin{aligned} & h_r(X_{i_1}, \ldots, X_{i_\alpha}, Y_{J_1}, \ldots, Y_{J_{r-\alpha}} \mid Y_{j_1}, \ldots, Y_{j_\alpha}, X_{I_1}, \ldots, X_{I_{r-\alpha}}) = \\ & = \sum_{\beta=1}^{r-\alpha} \{h_r(X_{i_1}, \ldots, X_{i_\alpha}, X_{I_1}, \ldots, X_{I_{\beta-1}}, Y_{J_\beta}, \ldots, Y_{J_{r-\alpha}} \mid Y_{j_1}, \ldots, Y_{j_\alpha}, Y_{J_1}, \ldots \\ & \qquad \ldots, Y_{J_{\beta-1}}, X_{I_\beta}, \ldots, X_{I_{r-\alpha}}) - \\ & - h_r(X_{i_1}, \ldots, X_{i_\alpha}, X_{I_1}, \ldots, X_{I_\beta}, Y_{J_{\beta+1}}, \ldots, Y_{J_{r-\alpha}} \mid Y_{j_1}, \ldots, Y_{j_\alpha}, Y_{J_1}, \ldots \\ & \qquad \ldots, Y_{J_\beta}, X_{I_{\beta+1}}, \ldots, X_{I_{r-\alpha}})\} + \\ & + h_r(X_{i_1}, \ldots, X_{i_\alpha}, X_{I_1}, \ldots, X_{I_{r-\alpha}} \mid Y_{j_1}, \ldots, Y_{j_\alpha}, Y_{J_1}, \ldots, Y_{J_{r-\alpha}}) = \\ & = \sum_{\beta=1}^{r-\alpha} Q_{r-1}(X_{I_\beta}, Y_{J_\beta}, P(Y_{J_{\beta+1}}, \ldots, Y_{J_{r-\alpha}}, X_{i_1}, \ldots, X_{i_\alpha}, X_{I_1}, \ldots, X_{I_{\beta-1}}), \\ & \qquad P(X_{I_{\beta+1}}, \ldots, X_{I_{r-\alpha}}, Y_{j_1}, \ldots, Y_{j_\alpha}, Y_{J_1}, \ldots, Y_{J_{\beta-1}})) + h_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r).\end{aligned}$$

Now we can write (11) in the form

$$
(12)\qquad B_r(X_1,\dots,X_r,Y_1,\dots,Y_r) = h_r(X_1,\dots,X_r \mid Y_1,\dots,Y_r) + \\
+ \frac{(r!)^2}{(2r)!}\sum_{\alpha=0}^{r-1}\ \sum^{\alpha}_{(i\cup I,\,j\cup J,\,<)} S(i_1,\dots,i_\alpha,I_1,\dots,I_{r-\alpha},j_1,\dots,j_\alpha,J_1,\dots,J_{r-\alpha})
$$

where we put for the sake of brevity

$$
(13)\qquad S(i_1,\dots,i_\alpha,I_1,\dots,I_{r-\alpha},j_1,\dots,j_\alpha,J_1,\dots,J_{r-\alpha}) = \\
= \sum_{\beta=1}^{r-\alpha} Q_{r-1}(X_{I\beta}, Y_{J\beta}, P(Y_{J_{\beta+1}},\dots,Y_{J_{r-\alpha}},X_{i_1},\dots,X_{i_\alpha},X_{I_1},\dots,X_{I_{\beta-1}}), \\
P(X_{I_{\beta+1}},\dots,X_{I_{r-\alpha}},Y_{j_1},\dots,Y_{j_\alpha},Y_{J_1},\dots,Y_{J_{\beta-1}}))\,.
$$

Hence we see that h_r is uniquely determined by B_r and $\mathbf{G}_{r-1,C}$. Now, the mapping P^r induces an epimorphism $\varrho^r : \bigodot_r T(M) \to E^r$ and consequently, an isomorphism $j^r : \bigodot_r T(M)/Z \to E^r$, where Z is the kernel of ϱ^r. The metric tensor $h_r(X_1,\dots,X_r \mid Y_1,\dots,Y_r)$ induces a symmetric bilinear positively semi-definite form H_r on $\bigodot_r T(M)$ and Z is the kernel of H_r. Hence, H_r induces a unique Riemannian product on $\bigodot_r T(M)/Z$ and then j^r is an isometry. Finally, we have a unique bundle epimorphism $\tilde{P}_{r-1} : T(M) \otimes E^{r-1} \to \bigodot_r T(M)/Z$ such that the following commutative diagram holds:

$$
\begin{tikzcd}
\bigotimes_r T(M) \arrow[r] \arrow[d] & \bigodot_r T(M) \arrow[dd] \\
T(M) \otimes \left(\bigotimes_{r-1} T(M)\right) \arrow[d, "id\otimes P^{r-1}"] & \\
T(M) \otimes E^{r-1} \arrow[r, "\tilde{P}_{r-1}"] \arrow[d, "P_{r-1}"] & \bigodot_r T(M)/Z \\
E^r &
\end{tikzcd}
$$

Moreover, we have $P_{r-1} = j^r \circ \tilde{P}_{r-1}$. Hence we see that $\mathbf{G}_{r,C}$ is, exact up to an equivalence, uniquely determined by $\mathbf{G}_{r-1,C}$ and h_r. This completes the proof.

*

Let $\sum_{(i\cup I,\,j\cup J)}$ denote the summation over *all finite sequences* $(i_1,\dots,i_\alpha)$, $(I_1,\dots,I_{r-\alpha})$, $(j_1,\dots,j_\alpha)$, $(J_1,\dots,J_{r-\alpha})$ selected from the index set $\{1,\dots,r\}$ and such that $\{i_1,\dots,i_\alpha,I_1,\dots,I_{r-\alpha}\} = \{j_1,\dots,j_\alpha,J_1,\dots,J_{r-\alpha}\} = \{1,\dots,r\}$. Then we have

$$
\sum^{\alpha}_{(i\cup I,\,j\cup J,\,<)} h_r(X_{i_1},\dots,X_{i_\alpha},Y_{J_1},\dots,Y_{J_{r-\alpha}} \mid Y_{j_1},\dots,Y_{j_\alpha},X_{I_1},\dots,X_{I_{r-\alpha}}) = \\
= \frac{1}{(\alpha!)^2\,[(r-\alpha)!]^2}\sum^{\alpha}_{(i\cup I,\,j\cup J)} h_r(X_{i_1},\dots,X_{i_\alpha},Y_{J_1},\dots,Y_{J_{r-\alpha}} \mid Y_{j_1},\dots,Y_{j_\alpha},X_{I_1},\dots,X_{I_{r-\alpha}})
$$

and hence

$$\sum_{(i\cup I, j\cup J, <)}^{\alpha} S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha}) =$$

$$= \frac{1}{(\alpha!)^2\,[(r-\alpha)!]^2} \sum_{(i\cup I, j\cup J)}^{\alpha} S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha})\,.$$

Then we can re-write (12) in the form

$$h_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r) - \tag{14}$$

$$- \frac{1}{(2r)!} \sum_{\alpha=0}^{r-1} (C_r^\alpha)^2 \sum_{(i\cup I, j\cup J)}^{\alpha} S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha})\,.$$

Let us remind that the metric tensor h_r is completely characterized by the following properties:

(i) *h_r is symmetric in both groups of variables $(X_1, \ldots, X_r)$, $(Y_1, \ldots, Y_r)$, and $h_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = h_r(Y_1, \ldots, Y_r \mid X_1, \ldots, X_r)$,*

(ii) *the induced bilinear form H_r on $\bigcirc_r T(M)$ is positively semi-definite,*

(iii) $\dfrac{1}{(2r)!} \sum_\sigma h_r(X_{\sigma(1)}, \ldots, X_{\sigma(r)} \mid X_{\sigma(r+1)}, \ldots, X_{\sigma(2r)}) = B_r(X_1, \ldots, X_r, X_{r+1}, \ldots, X_{2r})$,

(iv) *the Gaussian equation of genus r holds, i.e. we have*

$$h_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) - h_r(X_1, \ldots, X_{r-1}, Y_r \mid Y_1, \ldots, Y_{r-1}, X_r) = \tag{15}$$
$$= Q_{r-1}(Y_r, X_r, P(X_1, \ldots, X_{r-1}), P(Y_1, \ldots, Y_{r-1}))\,.$$

Proposition 6. *Let $\{E^k, P_k\}^{r-1}$ be a Riemannian geometry of genus $r-1$, and let $B_r(X_1, \ldots, X_{2r})$ be a symmetric $2r$-form on M. Define a tensor $\tilde{h}_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r)$ on M by*

$$\tilde{h}_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r) - \tag{16}$$

$$- \frac{1}{(2r)!} \sum_{\alpha=0}^{r-1} (C_r^\alpha)^2 \sum_{(i\cup I, j\cup J)}^{\alpha} S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha})$$

where $S(\ldots)$ is the tensor given by (13) and $C_r^\alpha = r!/\alpha!\,(r-\alpha)!$. Then

1. *$\tilde{h}_r$ is symmetric in both groups of variables $(X_1, \ldots, X_r)$, $(Y_1, \ldots, Y_r)$ and satisfies the identity $\tilde{h}_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = \tilde{h}_r(Y_1, \ldots, Y_r \mid X_1, \ldots, X_r)$.*

   Thus $\tilde{h}_r$ determines a symmetric 2-form $\tilde{H}_r$ on $\bigcirc_r T(M)$.

2. *We have $1/(2r)! \sum_\sigma \tilde{h}_r(X_{\sigma(1)}, \ldots, X_{\sigma(r)} \mid X_{\sigma(r+1)}, \ldots, X_{\sigma(2r)}) = B_r(X_1, \ldots, X_r, X_{r+1}, \ldots, X_{2r})$.*

Proof. From the symmetry of our definition we see that $\tilde{h}_r$ is symmetric in the variables X_i and also in the variables Y_j. Further, if we make the transpositions $X_1 \leftrightarrow Y_1, \ldots, X_r \leftrightarrow Y_r$, then each partial sum $S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots \ldots, J_{r-\alpha})$ passes into $S(j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha}, i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha})$. This follows from (13) and from the identity $Q_{r-1}(T, U, Y^{(r-1)}, X^{(r-1)}) = Q_{r-1}(U, T, X^{(r-1)}, Y^{(r-1)})$. Thus $\tilde{h}_r(Y_1, \ldots, Y_r \mid X_1, \ldots, X_r) = \tilde{h}_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r)$ and the assertion 1. follows.

As for 2., let us remark that the tensor $Q_{r-1}(X_1, X_{r+1}, P(X_{r+2}, \ldots, X_{2r}), P(X_2, \ldots \ldots, X_r))$ is antisymmetric with respect to X_1, X_{r+1}, and hence

$$\frac{1}{(2r)!} \sum_\sigma Q_{r-1}(X_{\sigma(1)}, X_{\sigma(r+1)}, P(X_{\sigma(r+2)}, \ldots, X_{\sigma(2r)}), P(X_{\sigma(2)}, \ldots, X_{\sigma(r)})) = 0 .$$

From (16) and (13) we obtain

$$\frac{1}{(2r)!} \sum_\sigma \tilde{h}_r(X_{\sigma(1)}, \ldots, X_{\sigma(r)} \mid X_{\sigma(r+1)}, \ldots, X_{\sigma(2r)}) =$$
$$= \frac{1}{(2r)!} \sum_\sigma B_r(X_{\sigma(1)}, \ldots, X_{\sigma(r)}, X_{\sigma(r+1)}, \ldots, X_{\sigma(2r)}) = B_r(X_1, \ldots, X_{2r}) .$$

According to Proposition 5, a Riemannian geometry $\mathbf{G}_{r-1,C}$ is uniquely determined by its Bompiani forms $B_1, \ldots, B_{r-1}$. Hence we can introduce the following definition:

Definition 5. Let $\mathbf{G}_{r-1,C} = \{E^k, P_k\}^{r-1}$ be a Riemannian geometry, the Bompiani forms of which are $B_1, \ldots, B_{r-1}$. A symmetric $2r$-form $B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r)$ on M is called relatively positive with respect to $\mathbf{G}_{r-1,C}$, or else, with respect to the forms $B_1, \ldots, B_{r-1}$ involving the parameter C, if the corresponding symmetric 2-form $\tilde{H}_r$ on $\bigcirc_r T(M)$ defined by Proposition 6, is positively definite at each point $x \in M$.

Remark. A relatively positive form $B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r)$, $r \geqq 2$ need not be positively definite as a symmetric bilinear form on $\bigcirc_r T(M)$. On the other hand, if B_r is relatively positive and $\tilde{B}_r$ positively definite on $\bigcirc_r T(M)$, then a) $B_r + \tilde{B}_r$ is relatively positive, b) if f is a function on M with great positive values, then $f \,.\, \tilde{B}_r$ is relatively positive.

MAXIMAL RIEMANNIAN GEOMETRIES

Let us remind that a maximal graded Riemannian bundle $\{E^k, P_k\}^r \to M$ is equivalent to a "normal form" $\{\bigcirc_k T(M), P_k^0\}^r$ (see [1]). As the normal operators P_k^0 are uniquely determined, we shall prefer the symbol $\{\bigcirc_k T(M), H_k\}^r$ referring to a given

Riemannian product $H_k = \langle\ ,\ \rangle_k$ on each subbundle $\underset{k}{\bigcirc} T(M)$. Now, *according to Corollary of Theorem* 3, [1], we can re-write Definition 3 for the maximal case:

Definition 6. A maximal Riemannian geometry $\mathbf{G}_{r,C}$ of genus r on M is a maximal graded Riemannian bundle $\{\underset{k}{\bigcirc} T(M), H_k\}^r \to M$ satisfying the Gaussian equations of genera $k = 1, \dots, r-1$ with the parameter C.

We see that a Riemannian geometry $\mathbf{G}_{r,C}$ is equivalent to a maximal one if and only if each Bompiani form B_k, $k = 1, \dots, r$ is relatively positive with respect to $B_1, \dots$ $\dots, B_{k-1}$, involving the parameter C.

Theorem 1. *Let* $\mathbf{G}_{r-1,C} = \{\underset{k}{\bigcirc} T(M), H_k\}^{r-1}$ *be a maximal Riemannian geometry of genus* $r-1$ *satisfying the corresponding cyclic condition* (8). *Then for any symmetric* $2r$*-form* $B_r(X_1, \dots, X_r, Y_1, \dots, Y_r)$ *relatively positive with respect to* $\mathbf{G}_{r-1,C}$ *there is a unique maximal Riemannian geometry* $\mathbf{G}_{r,C} = \{\underset{k}{\bigcirc} T(M), H_k\}^r$ *such that*

a) $\mathbf{G}_{r,C}$ *is a prolongation of* $\mathbf{G}_{r-1,C}$,

b) B_r *is the* r*-th Bompiani form of* $\mathbf{G}_{r,C}$.

Proof. The uniqueness of the prolongation geometry follows from Proposition 5. It remains to prove the existence. Let $B_r(X_1, \dots, X_r, Y_1, \dots, Y_r)$ be a form as required and let us suppose that the cyclic condition holds. Define the tensor $\tilde{h}_r(X_1, \dots, X_r \mid Y_1, \dots, Y_r)$ by (16). According to Proposition 6 the proof is reduced to showing that the Gaussian equation holds:

$$\tag{17} \begin{aligned} &\tilde{h}_r(X_1, \dots, X_r \mid Y_1, \dots, Y_r) - \tilde{h}_r(X_1, \dots, X_{r-1}, Y_r \mid Y_1, \dots, Y_{r-1}, X_r) = \\ &= Q_{r-1}(Y_r, X_r, P(X_1, \dots, X_{r-1}), P(Y_1, \dots, Y_{r-1}))\,. \end{aligned}$$

Lemma 1. *Each partial sum*

$$S(i_1, \dots, i_\alpha, I_1, \dots, I_{r-\alpha}, j_1, \dots, j_\alpha, J_1, \dots, J_{r-\alpha}) \quad \text{(see (13))}$$

is symmetric in all groups of indices $(i_1, \dots, i_\alpha)$, $(I_1, \dots, I_{r-\alpha})$, $(j_1, \dots, j_\alpha)$, $(J_1, \dots$ $\dots, J_{r-\alpha})$.

Proof. Let us write the cyclic condition (8) for the sequence of variables $(X_{i_1}, \dots$ $\dots, X_{i_\alpha}, X_{I_1}, \dots, X_{I_{r-\alpha}}, Y_{j_1}, \dots, Y_{j_\alpha}, Y_{J_1}, \dots, Y_{J_{r-\alpha}})$:

$$\begin{aligned} &\sum_{\beta=1}^{r-\alpha} Q_{r-1}(X_{I_\beta}, Y_{J_\beta}, P(Y_{J_{\beta+1}}, \dots, Y_{J_{r-\alpha}}, X_{i_1}, \dots, X_{i_\alpha}, X_{I_1}, \dots, X_{I_{\beta-1}}), \\ &\qquad P(X_{I_{\beta+1}}, \dots, X_{I_{r-\alpha}}, Y_{j_1}, \dots, Y_{j_\alpha}, Y_{J_1}, \dots, Y_{J_{\beta-1}})) + \\ &+ \sum_{\gamma=1}^{\alpha} Q_{r-1}(X_{i_\gamma}, Y_{j_\gamma}, P(Y_{j_{\gamma+1}}, \dots, Y_{j_\alpha}, Y_{J_1}, \dots, Y_{J_{r-\alpha}}, X_{i_1}, \dots, X_{i_{\gamma-1}}), \\ &\qquad P(X_{i_{\gamma+1}}, \dots, X_{i_\alpha}, X_{I_1}, \dots, X_{I_{r-\alpha}}, Y_{j_1}, \dots, Y_{j_{\gamma-1}})) = 0\,. \end{aligned}$$

Here $S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha})$ is equal to the first sum, which is symmetric in the groups of variables $(X_{i_1}, \ldots, X_{i_\alpha})$, $(Y_{j_1}, \ldots, Y_{j_\alpha})$, and to the opposite value of the second sum, which is symmetric in the groups of variables $(X_{I_1}, \ldots \ldots, X_{I_{r-\alpha}})$, $(Y_{J_1}, \ldots, Y_{J_{r-\alpha}})$, q.e.d.

Using Lemma 1 we obtain from (16) easily

$$(18) \qquad \tilde{h}_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r) - \\ - \frac{(r!)^2}{(2r)!} \sum_{\alpha=0}^{r-1} \sum_{(i \cup I, j \cup J, <)}^{\alpha} S(i_1, \ldots, i_\alpha, I_1, \ldots, I_{r-\alpha}, j_1, \ldots, j_\alpha, J_1, \ldots, J_{r-\alpha})$$

or else, if we substitute (13),

$$(19) \qquad \tilde{h}_r(X_1, \ldots, X_r \mid Y_1, \ldots, Y_r) = B_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r) - \\ - \frac{(r!)^2}{(2r)!} \sum_{\alpha=0}^{r-1} \sum_{(i \cup I, j \cup J, <)}^{\alpha} \sum_{\beta=1}^{r-\alpha} Q(X_{I_\beta}, Y_{J_\beta}, P(Y_{J_{\beta+1}}, \ldots, Y_{J_{r-\alpha}}, X_{i_1}, \ldots, X_{i_\alpha}, X_{I_1}, \ldots, X_{I_{\beta-1}}), \\ P(X_{I_{\beta+1}}, \ldots, X_{I_{r-\alpha}}, Y_{j_1}, \ldots, Y_{j_\alpha}, Y_{J_1}, \ldots, Y_{J_{\beta-1}}))$$

where we write simply Q instead of Q_{r-1}.

Here we shall distinguish formal *terms* and its *values*, i.e., tensors determined by the terms. The summands $Q(*, *, P(\ldots), P(\ldots))$ of (19) having neither X_r nor Y_r on the places marked by stars will be called *terms of the* 1-*st kind*. The other summands will be called *terms of* 2-*nd kind*.

Lemma 2. *The sum of all terms of the* 1-*st kind is invariant with respect to the transposition* $X_r \leftrightarrow Y_r$.

Proof. Consider a 1-st kind term of "degree" α

$$Q(X_{I_\beta}, Y_{J_\beta}, P(Y_{J_{\beta+1}}, \ldots, Y_{J_{r-\alpha}}, X_{i_1}, \ldots, X_{i_\alpha}, X_{I_1}, \ldots, X_{I_{\beta-1}}), \\ P(X_{I_{\beta+1}}, \ldots, X_{I_{r-\alpha}}, Y_{j_1}, \ldots, Y_{j_\alpha}, Y_{J_1}, \ldots, Y_{J_{\beta-1}})).$$

Let us discuss the possible locations of the variables X_r, Y_r in this term. (Remember that we admit only *increasing* sequences of indices i, j, I, and J.)

a) $i_\alpha = r$, $J_{r-\alpha} = r$, $(\beta < r - \alpha)$, or $j_\alpha = r$, $I_{r-\alpha} = r$, $(\alpha \geqq 1, \beta < r - \alpha)$.

The transformed term has the same value as the given one.

b) $i_\alpha = r$, $j_\alpha = r$, $\alpha \geqq 1$.

The transformed term has the same value as a 1-st kind term of degree $\alpha - 1$:

$$Q(X_{I_\beta}, Y_{J_\beta}, P(Y_{J_{\beta+1}}, \ldots, Y_{J_{r-\alpha}}, Y_{J_{r-\alpha+1}}, X_{i_1}, \ldots, X_{i_{\alpha-1}}, X_{I_1}, \ldots, X_{I_{\beta-1}}), \\ P(X_{I_{\beta+1}}, \ldots, X_{I_{r-\alpha}}, X_{I_{r-\alpha+1}}, Y_{j_1}, \ldots, Y_{j_{\alpha-1}}, Y_{J_1}, \ldots, Y_{J_{\beta-1}}))$$

where we put $J_{r-\alpha+1} = r$, $I_{r-\alpha+1} = r$.

c) $I_{r-\alpha} = r$, $J_{r-\alpha} = r$, $\alpha \leqq r - 2$, $\beta < r - \alpha$.

The transformed term has the same value as a 1-st kind term of degree $\alpha + 1$

$$Q(X_{I_\beta}, Y_{J_\beta}, P(Y_{J_{\beta+1}}, \dots, Y_{J_{r-\alpha-1}}, X_{i_1}, \dots, X_{i_{\alpha+1}}, X_{I_1}, \dots, X_{I_{\beta-1}}),$$
$$P(X_{I_{\beta+1}}, \dots, X_{I_{r-\alpha-1}}, Y_{j_1}, \dots, Y_{j_{\alpha+1}}, Y_{J_1}, \dots, Y_{J_{\beta-1}}))$$

where we put $i_{\alpha+1} = r$, $j_{\alpha+1} = r$.

We see that the mappings defined in b) and c) are inverse operations on terms. Hence Lemma 2 follows.

Now, let us consider the 2-nd kind terms and their transforms by the transposition $X_r \leftrightarrow Y_r$.

a) If $I_\beta = r$, $J_\beta = r$, we have $\beta = r - \alpha \geqq 1$ and there is exactly $(C^\alpha_{r-1})^2$ terms of this form and of degree $\alpha = 0, 1, \dots, r-1$. The value of each term is $Q(X_r, Y_r, P(X_1, \dots, X_{r-1}), P(Y_1, \dots, Y_{r-1}))$. Hence the difference of each term and its transform is equal to $2Q(X_r, Y_r, P(X_1, \dots, X_{r-1}), P(Y_1, \dots, Y_{r-1}))$.

b) If $I_\beta = r$, $J_\beta \neq r$, we have $\beta = r - \alpha$, $j_\alpha = r$, $\alpha > 1$. There is exactly $C^\alpha_{r-1}C^{\alpha-1}_{r-1}$ terms of this form and of degree α. The difference of each term and its transform is equal to

$$Q(X_r, Y_{r-\alpha}, P(X_{i_1}, \dots, X_{i_\alpha}, X_{I_1}, \dots, X_{I_{r-\alpha-1}}), P(Y_{j_1}, \dots, Y_{j_{\alpha-1}}, Y_r, Y_{J_1}, \dots, Y_{J_{r-\alpha-1}})) -$$
$$- Q(Y_r, Y_{r-\alpha}, P(X_{i_1}, \dots, X_{i_\alpha}, X_{I_1}, \dots, X_{I_{r-\alpha-1}}), P(Y_{j_1}, \dots, Y_{j_{\alpha-1}}, X_., Y_{J_1}, \dots, Y_{J_{r-\alpha-1}})) =$$
$$= Q(Y_{r-\alpha}, X_r, P_{r-2}(Y_r, P(Y_1, \dots, \hat{Y}_{r-\alpha}, \dots, Y_{r-1})), P(X_1, \dots, X_{r-1})) +$$
$$+ Q(Y_r, Y_{r-\alpha}, P_{r-2}(X_r, P(Y_1, \dots, \hat{Y}_{r-\alpha}, \dots, Y_{r-1})), P(X_1, \dots, X_{r-1})) =$$
$$= -Q(X_r, Y_r, P_{r-2}(Y_{r-\alpha}, P(Y_1, \dots, \hat{Y}_{r-\alpha}, \dots, Y_{r-1})), P(X_1, \dots, X_{r-1})) =$$
$$= Q(X_r, Y_r, P(X_1, \dots, X_{r-1}), P(Y_1, \dots, Y_{r-1})) \quad \text{(see (3b))}.$$

c) If $J_\beta = r$, $I_\beta \neq r$, we have $\beta = r - \alpha$, $i_\alpha = r$. There is exactly $C^\alpha_{r-1}C^{\alpha-1}_{r-1}$ terms of this form and of degree α. The difference of each term and its transform is again equal to $Q(X_r, Y_r, P(X_1, \dots, X_{r-1}), P(Y_1, \dots, Y_{r-1}))$.

Consider the difference

$$\tilde{h}_r(X_1, \dots, X_r \mid Y_1, \dots, Y_r) - \tilde{h}_r(X_1, \dots, X_{r-1}, Y_r \mid Y_1, \dots, Y_{r-1}, X_r).$$

The contribution of the 1-st kind terms is zero according to Lemma 2. Now, the contribution of the 2-nd kind terms is equal to

$$-\frac{(r!)^2}{(2r)!}\left(\sum_{\alpha=0}^{r-1} 2(C^\alpha_{r-1})^2 + \sum_{\alpha=1}^{r-1} 2C^\alpha_{r-1}C^{\alpha-1}_{r-1}\right) . \, Q(X_r, Y_r, P(X_1, \dots, X_{r-1}), P(Y_1, \dots, Y_{r-1}))$$

where $2\{\sum_{\alpha=0}^{r-1}(C^\alpha_{r-1})^2 + \sum_{\alpha=1}^{r-1} C^\alpha_{r-1}C^{\alpha-1}_{r-1}\} = C^r_{2r}$. Hence (17) follows.

Theorem 2. *The cyclic condition* (8) *is satisfied by any maximal Riemannian geometry* $\mathbf{G}_{r,C} = \{\underset{k}{\bigcirc} T(M), H_k\}^r$.

Proof. We have two different ways how to prove formula (8), namely indirect and direct ones.

A) Indirect proof. Here we use some basic results of papers [2], [3] due to V. V. Ryžkov. These results, translated into our language, can be formulated as follows:

Theorem. *Let M be a real analytic manifold of dimension n and $G_{r,C} = \{\underset{k}{\bigcirc} T(M), H_k\}^r$ a maximal analytic Riemannian geometry of genus r on M. Then the geometry $G_{r,C}$ is realizable, in a neighbourhood U of any point $x \in M$, by an analytic immersion of U into a complete Riemannian space N with constant curvature C and of dimension $\sum_{s=1}^{r} C_{n+2s-1}^{2s}$.* (Cf. Theorems 1 and 3 from [2].) Hence we obtain

Proposition 7. *Let $\mathbf{G}_{r-1,C}$ be an analytic maximal Riemannian geometry of genus $r-1$ over an analytic manifold M. Let $B_r(X_1, \ldots, X_{2r})$ be an analytic, symmetric $2r$-form on M which is relatively positive with respect to $\mathbf{G}_{r-1,C}$. Then there is exactly one analytic Riemannian geometry $\mathbf{G}_{r,C}$ such that*

a) *$\mathbf{G}_{r,C}$ is a prolongation of $\mathbf{G}_{r-1,C}$,*

b) *B_r is the r-th Bompiani form of $\mathbf{G}_{r,C}$.*

Proof of Proposition 7. For any point $x \in M$ the geometry $\mathbf{G}_{r-1,C}$ is realizable on a neighbourhood U of x and according to Proposition 4 the cyclic condition of genus $r-1$ holds at x. Hence the cyclic condition holds on the whole M and we can apply Theorem 1.

Now, let $B_1(X_1, X_2), \ldots, B_r(X_1, \ldots, X_{2r})$ be symmetric analytic forms on an analytic manifold M. Let C be a real number. If B_1 is positive definite on M, then it determines an analytic Riemannian geometry $\mathbf{G}_{1,C}$. If B_2 is relatively positive with respect to B_1, involving the parameter C, then B_1, B_2 determine an analytic Riemannian geometry $\mathbf{G}_{2,C}$ prolonging $\mathbf{G}_{1,C}$, etc .... Finally, if B_r is relatively positive with respect to $B_1, \ldots, B_{r-1}$, involving C, then $B_1, \ldots, B_r$ determine an analytic Riemannian geometry $\mathbf{G}_{r,C}$, which is maximal and unique.

Let M be a manifold of class C^∞ and $\mathbf{G}_{r,C}$ a maximal Riemannian geometry on M. Denote by $j_x^s f$ the s-jet with the source x of a function f defined on M. If T is a tensor on M, then the s-jet $j_x^s T$ is determined by the s-jets of its components with respect to an arbitrary local coordinate system $(u^1, \ldots, u^n)$ in a neighbourhood of x. Now we can see easily that the tensor $Q_r(U, T, X^{(r)}, Y^{(r)})$ at the point x is uniquely determined by the jets $j_x^{2r-2} B_1, \ldots, j_x^2 B_{r-1}, j_x^0 B_r$ of the corresponding Bompiani forms. Choose a coordinate neighbourhood $U(u^1, \ldots, u^n)$ of x and a system of symmetric forms $\tilde{B}_1(X_1, X_2), \ldots, \tilde{B}_r(X_1, \ldots, X_{2r})$ such that

a) $\tilde{B}_1, \ldots, \tilde{B}_r$ are defined in $U \subset M$,

b) $\tilde{B}_1, \ldots, \tilde{B}_r$ have analytic components with respect to the coordinate system $(u^1, \ldots, u^n)$,

c) $j_x^{2r-2}\tilde{B}_1 = j_x^{2r-2}B_1, \ldots, j_x^2\tilde{B}_{r-1} = j_x^2 B_{r-1}, j_x^0\tilde{B}_r = j_x^0 B_r$.

We know that each form B_k is relatively positive with respect to $B_1, \ldots, B_{k-1}$, involving C, at the point x. Moreover, if a form $\tilde{B}_k$ is relatively positive with respect to a Riemannian geometry $\tilde{\mathbf{G}}_{k-1,C}$ at a point x, then this property is preserved in a neighbourhood of x. Hence we see by induction: for $k = 1, \ldots, r-1$ the forms $\tilde{B}_1, \ldots, \tilde{B}_{k-1}$ determine a Riemannian geometry $\tilde{\mathbf{G}}_{k-1,C}$ on a neighbourhood $U_{k-1} \subset \subset U$ and $\tilde{B}_k$ is relatively positive with respect to $\tilde{\mathbf{G}}_{k-1,C}$ in a neighbourhood $U_k \subset \subset U_{k-1}$. Finally, there is a Riemannian geometry $\tilde{\mathbf{G}}_{r,C}$ with the given Bompiani forms $\tilde{B}_1, \ldots, \tilde{B}_r$ on U_r. Let us make U_r an analytic manifold so that $(u^1, \ldots, u^n)$ are its analytic coordinates. Then $\tilde{\mathbf{G}}_{r,C}$ is an analytic maximal Riemannian geometry over U_r. Moreover, we have $\tilde{Q}_r(U, T, X^{(r)}, Y^{(r)}) = Q_r(U, T, X^{(r)}, Y^{(r)})$ at the point x. The geometry $\tilde{\mathbf{G}}_{r,C}$ is realizable in a neighbourhood $V \subset U_r$ of x and hence it satisfies the cyclic condition (8) at x. Consequently, the Riemannian geometry $\mathbf{G}_{r,C}$ satisfies the cyclic condition at x. Theorem 2 is proved.

B) Direct proof. To tell the truth, the author has no idea how to proceed in general. For $r = 2$ a "routine" process proved to be successful but it leads to very tedious calculations. For $\dim M = 2$ we have an interesting, purely combinatorial proof; the maximality of $\mathbf{G}_{r,C}$ is not required here.

Theorem 3. *Let $\{E^k, P_k\}^r$ be a Riemannian geometry (equivalent or not to a maximal geometry) over a 2-dimensional manifold M. Then the cyclic condition holds.*

Proof of Theorem 3. Consider an oriented $(2r + 2)$-sided polygon $\mathcal{Q}$ with vertices $X_1, X_2, \ldots, X_{r+1}, Y_1, \ldots, Y_{r+1}$. For any $P \in \mathcal{Q}$ let us denote by $\tilde{P}$ the opposite vertex to P. Let be given a fixed map f adj oining one of two symbols X or Y to each vertex of $\mathcal{Q}$. Let $\alpha(P)$ denote the number of vertices Q placed between P and $\tilde{P}$ in the positive direction and such that $f(Q) = X$.

A vertex $P \in \mathcal{Q}$ will be called *essential* if $f(\tilde{P}) \neq f(P)$,

principal if $f(\tilde{P}) \neq f(P)$, $\alpha(\tilde{P}) \neq \alpha(P)$.

Two principal vertices P, Q will be called:

a) *congruent* if either $f(Q) = f(P)$, $\alpha(Q) = \alpha(P)$
or $f(Q) = f(\tilde{P})$, $\alpha(Q) = \alpha(\tilde{P})$,

b) *conjugate* if either $f(Q) = f(P)$, $\alpha(Q) = \alpha(\tilde{P})$
or $f(Q) = f(\tilde{P})$, $\alpha(Q) = \alpha(P)$.

An essential vertex Q will be called *the successor* of an essential vertex P (and P *the predecessor* of Q) if Q is the first essential vertex following P in the positive direction.

Lemma 1. *Let Q be the successor of P. If $f(Q) \neq f(P)$, then $\alpha(Q) = \alpha(P)$. If $f(Q) = f(P) = X$, then $\alpha(Q) = \alpha(P) - 1$. If $f(Q) = f(P) = Y$, then $\alpha(Q) = \alpha(P) + 1$.*

Proof is clear from the definition.

Lemma 2. *Between any two congruent vertices P, Q there is a principal vertex R which is conjugate to P and Q.*

Proof. **a)** Let $f(P) = f(Q) = X$, then $\alpha(P) = \alpha(Q) = m$. Let R be the successor of P, then $\alpha(R) \leqq m$ (Lemma 1). If $\alpha(R) = m$, then Lemma 1 implies $f(R) = Y$ and R is conjugate to P (and to Q). If $\alpha(R) < m$, then let S be the first essential vertex following R and such that $\alpha(S) = m$. Then for the predecessor S' of S we must have $\alpha(S') = m - 1$. Hence $\alpha(S) = \alpha(S') + 1$ and Lemma 1 implies $f(S) = f(S') = Y$. Thus S is conjugate to P and Q.

b) Let $f(P) = X$, $f(Q) = Y$, then $\alpha(P) = m$ and $\alpha(Q) = m' \neq m$. For the successor R of P we have $\alpha(R) \leqq m$. If $m < m'$, let S be the last essential vertex between P and Q such that $\alpha(S) = m$. Then for the successor S^* of S we must have $\alpha(S^*) = m + 1$. Hence $f(S) = f(S^*) = Y$ and S is conjugate to P and Q. If $m > m'$, let T be the first essential vertex following P such that $\alpha(T) = m'$. Then if T' is the predecessor of T, we have $\alpha(T') = \alpha(T) + 1$ and hence $f(T') = f(T) = X$. Consequently, T is conjugate to P and Q.

c) $f(P) = Y$, $f(Q) = X$.

d) $f(P) = f(Q) = Y$.

These two cases can be discussed in a similar way. Hence Lemma 2 follows.

Now we see that if P is an essential (principal) vertex, then $\tilde{P}$ is also essential (principal). In this case we say that the diagonal $P\tilde{P}$ is essential (principal). Further, if P, Q are congruent (conjugate), then $\tilde{P}$, $\tilde{Q}$ are also congruent (conjugate). Then we speak about congruent (conjugate) diagonals $P\tilde{P}$, $Q\tilde{Q}$. (Remember that P and $\tilde{P}$ are always congruent if P is principal.) Now we obtain from Lemma 2: *The principal diagonals of $\mathcal{Q}$ can be distributed into pairs of mutually conjugate diagonals.*

Let $\{E^k, P_k\}^r \to M$ be a Riemannian geometry over a 2-dimensional manifold and consider the "cyclic sum"

$$\sum_{i=1}^{r+1} Q_r(Y_i, X_i, P(X_{i+1}, \ldots, X_{r+1}, Y_1, \ldots, Y_{i-1}), P(Y_{i+1}, \ldots, Y_{r+1}, X_1, \ldots, X_{i-1})) .$$

Choose $x \in M$ and let $X, Y \in T_x(M)$ form a basis. As the cyclic sum is a multilinear function on $T_x(M)$, it suffices to prove (8) in the case that some of the variables X_i, Y_j are substituted by X and some of them by Y. Then we obtain a polygon $\mathcal{Q}$ as above with a given mapping f of vertices. Each term of the cyclic sum corresponds to a diagonal of $\mathcal{Q}$. If the diagonal is not essential, then the corresponding term has the form $Q_r(X, X, X^{(r)}, Y^{(r)})$ or $Q_r(Y, Y, X^{(r)}, Y^{(r)})$, which is zero. If the diagonal is essential but not principal, then the corresponding term has the form $Q_r(X, Y, X^{(r)}, X^{(r)})$ or $Q_r(Y, X, X^{(r)}, X^{(r)})$, which is zero again. Finally, any two terms corresponding to a pair of conjugate diagonals cancel each other. Hence Theorem follows.

Conjecture. The cyclic condition (8) is satisfied by any Riemannian geometry $\mathbf{G}_{r,C} = \{E^k, P_k\}^r$, equivalent or not to a maximal one.

CONSEQUENCES

Theorems 1 and 2 together imply the following:

Theorem 4. *Let $\mathbf{G}_{r-1,C} \to M$ be a maximal Riemannian geometry of genus $r - 1$. Let $B_r(X_1, \ldots, X_{2r})$ be a symmetric $2r$-form on M which is relatively positive with respect to $\mathbf{G}_{r-1,C}$. Then there is exactly one maximal Riemannian geometry $\mathbf{G}_{r,C}$ such that*

a) *$\mathbf{G}_{r,C}$ is a prolongation of $\mathbf{G}_{r-1,C}$,*

b) *B_r is the r-th Bompiani form of $\mathbf{G}_{r,C}$.*

Hence an assertion follows which is sensible only if understood as an inductive process. It says that *our definition of a Riemannian geometry of genus r is eguivalent with the classical one* (in the maximal case). (Cf. [2] and [6].)

Theorem 5. *Let $B_1(X_1, X_2), \ldots, B_r(X_1, \ldots, X_{2r})$ be symmetric forms on a differentiable manifold M and let C be a real number. If for $k = 1, \ldots, r$ B_k is relatively positive with respect to $B_1, \ldots, B_{k-1}$ involving C, then there is exactly one maximal Riemannian geometry $\mathbf{G}_{r,C} = \{\bigcirc_k T(M), H_k\}^r$* such that $B_1, \ldots, B_r$ are the Bompiani forms of $\mathbf{G}_{r,C}$.

Finally, using Theorem 4 from [1] we obtain the following Immersion Theorem:

Theorem 6. *Let $B_1(X_1, X_2), \ldots, B_r(X_1, \ldots, X_{2r})$ be symmetric covariant tensors of orders $2, 4, \ldots, 2r$ respectively on a simply connected manifold M and let C be a real number. Let N be a complete Riemannian space with the constant curvature C, of dimension $d = \sum_{s=1}^{r} C^s_{n+s-1}$, and such that its isometry group acts transitively on orthonormal frames. Suppose that for $k = 1, \ldots, r$ the form B_k is relatively positive with respect to $B_1, \ldots, B_{k-1}$, involving C, and that the Gaussian equation of genus r holds in the corresponding Riemannian geometry $\mathbf{G}_{r,C}$:*

$$L_r(U, T, X^{(r)}, Y^{(r)}) = R^{(r)}(U, T, X^{(r)}, Y^{(r)}) + C\{\langle U, X^{(r)}\rangle \langle T, Y^{(r)}\rangle - \langle U, Y^{(r)}\rangle \langle T, X^{(r)}\rangle\}\,.$$

Then there is, exact up to an isometry of N, a unique immersion $\psi : M \to N$ such that $B_1, \ldots, B_r$ are the Bompiani forms of ψ. Particularly, the immersion ψ is isometric and maximal.

Remark. We can prove Theorem 4 more shortly using the immersion theorem by V. V. Ryžkov in the following interpretation: *Let M be a real analytic manifold of dimension n and $\mathbf{G}_{r-1,C} = \{\bigcirc_k T(M), H_k\}^{r-1}$ a maximal analytic Riemannian geometry of genus $r - 1$ on M with Bompiani forms $B_1, \ldots, B_{r-1}$. Let $B_r(X_1, \ldots, X_{2r})$ be an analytic, symmetric $2r$-form on M which is relatively positive with respect to $\mathbf{G}_{r-1,C}$. Then for a neighbourhood U of any point $x \in M$ there is an analytic immersion of U into a complete Riemannian space N with the constant curvature C and of dimension $\sum_{s=1}^{r} C_{n+2s-1}^{2s}$ such that the restrictions of $B_1, \ldots, B_r$ to U are the first r Bompiani forms of the immersion.*

The proof of Theorem 4 is then similar to that of Theorem 2.

THE EXISTENCE OF PROLONGATIONS

Let $(E, j) \to M$ be a soldered Riemannian vector bundle with a Riemannian connection ∇ and consider the canonical orthogonal splitting $E = E^1 \oplus E^2 \oplus \ldots \ldots \oplus E^r \oplus Z$ as defined by Formula (13) in [1]. We allow here that the subbundle Z be non-trivial. We shall call the number r *the genus of E*. If $\psi : M \to N$ is an isometric immersion we put by definition: *genus of $\psi \overset{\text{def}}{=}$ genus of $\psi^* T(N)$*.

Now, let us present another variant of the Ryžkov's immersion theorem (cf. Theorems 2 and 4, [2]): *Let M be a real analytic manifold of dimension n and $\mathbf{G}_{r,C} = \{\bigcirc_k T(M), H_k\}$ a maximal analytic Riemannian geometry of genus r on M. Then the geometry $\mathbf{G}_{r,C}$ is realizable in a neighbourhood U of any point $x \in M$ by a maximal analytic immersion of U into a complete Riemannian space N^d with the constant curvature C and of dimension $d = \sum_{s=1}^{2r} C_{n+s-1}^{s} = C_{n+2r}^{2r} - 1$. All such maximal immersions are of genus $2r$ and depend (for a fixed U) on $(C_{n+2r}^{2r} - - 1 - \sum_{s=1}^{r} C_{n+2s-1}^{2s})$ arbitrary functions of n arguments.*

Hence we see that any maximal analytic Riemannian geometry $\mathbf{G}_{r,C}$ can be locally prolonged to a maximal analytic Riemannian geometry $\mathbf{G}_{2r,C}$ (and hence to a geometry $\mathbf{G}_{r+1,C}$). Particularly, let M be a small piece of the coordinate space R^n and let us construct a maximal analytic immersion ψ of genus r of M into a Euclidean space E^d, $d = \sum_{s=1}^{r} C_{n+s-1}^{s}$. Then we have $Q_r = 0$ (the r-th Gaussian equation) for the corresponding Riemannian geometry $\mathbf{G}_{r,C} = \psi^* T(E^d)$ over M. If $\mathbf{G}_{r+1,C}$ is a maximal analytic prolongation geometry of $\mathbf{G}_{r,C}$, we can see that the Bompiani form B_{r+1}

coincides with the metric tensor h_{r+1} and hence B_{r+1} defines a positively definite bilinear form on $\bigodot_{r+1} T(M)$. If we limit ourselves to a point $x \in M$ and put $V = T_x(M)$, we obtain the following purely algebraic theorem:

Let V be a finite dimensional vector space over R and $\bigodot_r V$ its r-th symmetric tensor power. Then on $\bigodot_r V$ there is a positively definite symmetric bilinear form $\langle\, ,\rangle$ such that the induced 2r-linear form on V, $B(X_1, \ldots, X_r, X_{r+1}, \ldots, X_{2r}) = \langle X_1 \odot \ldots \odot X_r, X_{r+1} \odot \ldots \odot X_{2r}\rangle$ is symmetric with respect to all arguments.

The last result seems to be *non-trivial* and it would be of interest to seek its direct proof for the following reason: the direct proof of this algebraic assertion together with a direct proof of the cyclic identity (see the discussion of Theorem 2) will provide a direct proof of the following *prolongation theorem*:

Theorem 7. *Any maximal Riemannian geometry* $\mathbf{G}_{r,C}$ (of class C^∞) *can be prolonged to a maximal Riemannian geometry* $\mathbf{G}_{r+1,C}$.

Proof. Let M be a manifold of class C^∞. From the algebraic lemma we see that for any sufficiently small coordinate neighbourhood $U \subset M$ there is a $(2r + 2)$-form $B^U_{r+1}(X_1, \ldots, X_{r+1}, Y_1, \ldots, Y_{r+1})$ on U, symmetric in all $2r + 2$ arguments and defining a positively definite symmetric 2- form on $\bigodot_{r+1} T(U)$. Let $\mathbf{G}_{r,C}$ be a maximal Riemannian geometry defined on M. Then, multiplying B^U_{r+1} by a convenient positive function f_U on U we can make it relatively positive with respect to $\mathbf{G}_{r,C}$ on U. Let us choose a locally finite covering $\{U^\alpha\}_{\alpha\in A}$ on M by coordinate neighbourhoods of the above property. Using properly the differential variant of Uryson's Lemma, we can construct a family $\{B^\alpha_{r+1}\}_{\alpha\in A}$ of global symmetric $(2r + 2)$-forms such that each B^α_{r+1} is relatively positive with respect to $\mathbf{G}_{r,C}$ on U^α and the family $\{\text{supp}\, B^\alpha_{r+1}\}_{\alpha\in A}$ of supports forms a locally finite covering of M. Then the expression $\sum_{\alpha\in A} B^\alpha_{r+1}$ defines a global form which is relatively positive with respect to $\mathbf{G}_{r,C}$ on the whole M. Finally, the prolongation geometry exists according to Theorem, 1, q.e.d.

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APPENDIX

1. As for Theorem 1, [1], an equivalent result was obtained by R. H. SZCZARBA by different methods. ("On existence and rigidity of isometric immersions", Bull. Amer. Math. Soc. 75 (1969), pp. 783–787.)

2. As for Theorem 2, [1], it is a modern interpretation of an earlier result by C. B. ALLENDOERFER. ("The imbedding of Riemann spaces in the large", Duke Math. J. 3. (1937), pp. 317–333).

3. Corollary of Theorem 3, [1], can be also derived as a consequence of Ryžkov's immersion theorems.

4. A formally similar theorem (for order 2 only!) was obtained by C. B. Allendoerfer under different assumptions (the maximality property is replaced by the assumption that the *type number* of the subbundle E^2 is $\geqq 4$). Cf. "Rigidity for spaces of class greater than one", Amer. J. Math. 61 (1939), pp. 633–644.

5. Theorem 3, [1], has the following consequence:

Theorem. *Let $\{E^k, P_k\}^r$ be a maximal graded Riemannian bundle over a two-dimensional manifold M. Then a sequence $\nabla^{(1)}, \ldots, \nabla^{(r)}$ of canonical connections exists in $\{E^k, P_k\}^r$.*

Proof: Each function $P_l(U, T, X^{(l)}, Y^{(l)})$, $1 \leqq l \leqq r-1$ trivially satisfies the Bianchi identity.

6. **Errata.** In the statement of Theorem 4, [1], insert: Let N be a complete... of dimension $d = \dim(E^1 \oplus \ldots \oplus E^r)$ *and such that its isometry group acts transitively on orthonormal frames.*

7. *The cyclic condition* (8) appears in a coordinate form in the Allendoerfer's paper "The imbedding..." (Formula (5.6)). The author asserts that this formula "actually is an identity" but no proof is given to justify the assertion.

8. The tensors $Q_k(U, T, X^{(k)}, Y^{(k)})$ (see Formula (2)) have the following geometrical significance: Let $\psi : M \to N$ be a stable isometric immersion of genus r of a Riemannian manifold M into a Riemannian space N with the constant curvature, and let ∇ be the induced Riemannian connection in $\psi_* T(N) \to M$. Let S^k, $k \leqq r$ be the k-th osculation bundle of M, $S^k \subset \psi_* T(N)$, $S^k = E^1 \oplus \ldots \oplus E^k$, and denote by $\nabla^{[k]}$ the orthogonal projection of the induced connection ∇ into S^k. (See [1], p. 680.) Finally, let us denote by $R^{[k]}$ the curvature tensor of $\nabla^{[k]}$ in S^k. Then we have $Q_k(U, T, X^{(k)}, Y^{(k)}) = R^{[k]}(U, T, X^{(k)}, Y^{(k)})$. In fact, in the paper of C. B. Allendoerfer "The imbedding..." the tensors Q_k are presented as "higher curvature tensors".

9. **Problems.**

1. Is any analytic Riemannian geometry $\boldsymbol{G}_{r,C}$ locally realizable in a Riemannian space of the constant curvature C?

2. Does any Riemannian geometry $\boldsymbol{G}_{r,C}$ of class C^∞ have a nontrivial prolongation $\boldsymbol{G}_{r+1,C}$?

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