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COMPLEX VELOCITIES ON REAL MANIFOLDS

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The paper starts with the definitions of complex velocities and of complex contact elements of order r on a real manifold of class C^r. These concepts are introduced inevitably in a formal way, nevertheless, they represent a theoretical basis for the use of imaginary elements in many problems in differential geometry. Our main conclusion is that one may consider some kinds of imaginary elements up to order r on every real manifold of class C^r. To give a concrete illustration of our ideas, we discuss the asymptotic directions at an elliptic point of a surface of class C^2 in the real projective 3-dimensional space. In the remaining part of the paper, the general concept of a complex r-jet of a real manifold of class C^r into a real manifold of the same class is introduced and the complex prolongations of real fibered manifolds are treated.

1. Let g be a real analytic function defined in a neighbourhood of the origin $0 \in \mathbf{R}^n$, then g determines a power series of $(x^1, \ldots, x^n)$ convergent in a domain $U \subset \mathbf{R}^n$ of the form $|x^i| < r_i$, $r_i > 0$, $i = 1, \ldots, n$. Denote by CU the set of all $(z^1, \ldots, z^n) \in \mathbf{C}^n$ satisfying $|z^i| < r_i$, then the power series converges also for every $(z^1, \ldots, z^n) \in CU$ and defines a complex analytic function Cg, which will be called the *complexification* of g. Similarly, let f be a real analytic mapping of a neighbourhood of $0 \in \mathbf{R}^n$ into $\mathbf{R}^m$, then f is an m-tuple $(f^1, \ldots, f^m)$ of real analytic functions and we define its complexification Cf by $Cf = (Cf^1, \ldots, Cf^m)$. Evidently, Cf is a complex analytic mapping of a domain $CU \subset \mathbf{C}^n$ into $\mathbf{C}^m$.

Denote by $CL^r_{m,n}$ the space of all r-jets of complex analytic mappings of $\mathbf{C}^n$ into $\mathbf{C}^m$ with source 0 and target 0. Any r-jet of $\mathbf{R}^n$ into $\mathbf{R}^m$ can be considered as an r-jet of a complexification of a real analytic mapping, which is the reason why the space $L^r_{m,n}$ of all real r-jets of $\mathbf{R}^n$ into $\mathbf{R}^m$ with source 0 and target 0 is a subspace of $CL^r_{m,n}$. ($CL^r_{m,n}$ or $L^r_{m,n}$ is canonically identified with the set of all m-tuples of polynomials of degree r of n variables without absolute terms with complex or real coefficients respectively.) The usual composition of jets defines the product XY, $X \in CL^r_{k,m}$, $Y \in CL^r_{m,n}$. CL^r_n means the group of all invertible elements of $CL^r_{n,n}$, so that L^r_n (i.e. the group of all invertible elements of $L^r_{n,n}$) is a subgroup of CL^r_n. In particular, L^r_m acts from the left and L^r_n acts from the right on $CL^r_{m,n}$ and $L^r_{m,n}$ is invariant with respect to both actions.

Remark 1. In the semi-holonomic and non-holonomic cases, cf. [2], one can introduce analogously $C\bar{L}^r_{m,n}$, $C\tilde{L}^r_{m,n}$, $C\bar{L}^r_n$, $C\tilde{L}^r_n$.

2. Let M be a real manifold of class C^r, $\dim M = n$. The real m^r-velocity on M at $x \in M$ is an r-jet of $\mathbf{R}^m$ into M with source 0 and target x; the space of all real m^r-velocities on M is denoted by $T^r_m(M)$. The well-known analysis shows, see [1], that $T^r_m(M)$ is a fibre bundle associated with the r-th principal prolongation $H^r(M)$ of M with standard fibre $L^r_{n,m}$, i.e. $T^r_m(M)$ has the symbol $(M, L^r_{n,m}, L^r_n, H^r(M))$. In other words, every element of $T^r_m(M)$ can also be considered as an equivalence class $\{(u, x)\}$, $u \in H^r(M)$, $x \in L^r_{n,m}$, with respect to the equivalence relation $(u, x) \sim \sim (ug, g^{-1}x)$, $g \in L^r_n$.

Definition 1. *The fibre bundle $CT^r_m(M)$ associated with $H^r(M)$ with standard fibre $CL^r_{n,m}$ will be called the bundle of the complex m^r-velocities on M; i.e. $CT^r_m(M)$ has the symbol $(M, CL^r_{n,m}, L^r_n, H^r(M))$.*

Thus, a complex m^r-velocity on M is an equivalence class $\{(u, z)\}$, $u \in H^r(M)$, $z \in CL^r_{n,m}$ with respect to the equivalence relation $(u, z) \sim (ug, g^{-1}z)$, $g \in L^r_n$. Since $L^r_{n,m}$ is an invariant subspace of $CL^r_{n,m}$, we have $T^r_m(M) \subset CT^r_m(M)$. The elements of $CT^r_m(M) \setminus T^r_m(M)$ will be called *imaginary m^r-velocities* on M.

If $Y \in CL^r_{m,k}$ and $X \in CT^r_m(M)$, $X = \{(u, z)\}$, then the product $XY \in CT^r_k(M)$ is defined by $XY = \{(u, zY)\}$.

Remark 2. $T^1_1(M) = T(M)$ is the vector bundle over M and it is easy to see that $CT^1_1(M) = T(M) \otimes \mathbf{C}$. This is the bundle of all complex tangent vectors on M.

Remark 3. To justify Definition 1 in a deeper way, we shall show that a direct definition of complex m^r-velocities on a real manifold M of class C^ω can be given provided one introduces the germ of complexification of M at a point $x \in M$ as follows. Denote by $H^\lambda_x(M)$ the set of all germs of local charts of $\mathbf{R}^n$ into M with source 0 and target x. Let $\varphi_a \in H^\lambda_x(M)$ and let φ_a be the germ of a local chart f_a, $a = 1,2$, then φ_{21} means the germ of $f_{21} = f_2^{-1} f_1$ at 0. The germ μ_x of M at x can also be defined as the equivalence class $\{(\varphi, \varrho^n_0)\}$ with respect to the equivalence relation $(\varphi_1, \varrho^n_0) \sim \sim (\varphi_2, \varphi_{21}(\varrho^n_0))$, where $\varphi, \varphi_1, \varphi_2 \in H^\lambda_x(M)$ and ϱ^n_0 means the germ of $\mathbf{R}^n$ at 0. Since M is of class C^ω, φ_{21} is a germ of a real analytic mapping and one can construct its complexification $C\varphi_{21}$. Now, the *germ $C\mu_x$ of complexification of M at x* is introduced as the equivalence class $\{(\varphi, \gamma^n_0)\}$ with respect to the equivalence relation $(\varphi_1, \gamma^n_0) \sim (\varphi_2, C\varphi_{21}(\gamma^n_0))$, where $\varphi, \varphi_1, \varphi_2 \in H^\lambda_x(M)$ and γ^n_0 means the germ of $\mathbf{C}^n$ at 0. Then it is easy to see that a complex m^r-velocity at $x \in M$ can be considered as an r-jet of a germ of a complex analytic mapping of γ^m_0 into $C\mu_x$ and the real m^r-velocities at x can be considred as r-jets of those germs which are complexifications of germs of real analytic mappings of ϱ_0^m into μ_x. — On the other hand, we underline that the formal approach of Definition 1 can be applied to every real manifold of class C^r.

3. Let X be a real m^r-velocity on M, then the real contact m^r-element determined by X is the set XL_m^r, see [1]; the fibre bundle of all regular real contact m^r-elements on M will be denoted by $K_m^r(M)$, cf. [3]. *An m-dimensional submanifold S of M determines canonically a real contact m^r-element $k_p^r S$ at each point $p \in S$.* Evidently, the elements of $K_1^1(M)$ are the directions in $T(M)$, or, shortly, the directions on M. Now, let X be a complex m^r-velocity on M, then by the *complex contact m^r-element* determined by X, the set XCL_m^r will be meant. A complex contact m^r-element is called real, if there exists a real m^r-velocity belonging to this set; in the opposite case, the contact element is called imaginary. (If a real m^r-velocity X is considered as a special complex m^r-velocity, then it determines the contact element XCL_m^r, but the difference between XL_m^r and XCL_m^r is only formal and may be neglected.) The fibre bundle of all regular complex contact m^r-elements on M will be denoted by $CK_m^r(M)$; we have $K_m^r(M) \subset CK_m^r(M)$. The elements of $CK_1^1(M)$ are the complex directions on M.

If M is a real manifold of class C^ω, then one can consider germs of m-dimensional imaginary submanifolds in $C\mu_x$, but if M is of class C^r only, one must replace these germs by imaginary contact m^r-elements at x. Nevertheless, this often suffices for different kinds of geometric constructions with imaginary elements in differential geometry. As an example, we shall discuss the asymptotic directions at an elliptic point from this point of view in § 5. In the next paragraph, we present some further necessary definitions.

4. In the real case, a k^r-velocity X' is said to *be contained* in an m^r-velocity X, if there exists an element $Y \in L_{m,k}^r$ such that $X' = XY$. Furthermore, a contact k^r-element $X'L_k^r$ is said to be contained in a contact m^r-element XL_m^r, if X' is contained in X, see [1]. Analogously, a complex k^r-velocity X' will be said to be contained in a complex m^r-velocity X, if there exists an element $Y \in CL_{m,k}^r$ such that $X' = XY$. Of course, an imaginary k^r-velocity can be contained in a real m^r-velocity. In the same way, a complex contact k^r-element $X'CL_k^r$ will be said to be contained in a complex contact m^r-element XCL_m^r, if X' is contained in X. Let S be an m-dimensional submanifold of M and let $p \in S$. If a contact k^r-element on M at p is contained in $k_p^r S$, then we also say that this contact k^r-element lies on S.

Let j_r^s, $s < r$ denote the canonical projection of r-jets into s-jets, i.e. $j_r^s X$ means the s-th part (i.e. the underlying s-jet) of an r-jet X. One sees directly that j_r^s is canonically extended to real contact elements, to complex velocities and to complex contact elements. In particular, every contact 1^r-element is projected by j_r^1 into a direction on the corresponding manifold; we shall also say that this contact 1^r-element lies in this direction.

5. Let S be a surface of class C^2 in the real projective 3-dimensional space P_3. If $p \in S$ is a hyperbolic point, then the following assertion is well-known: There are exactly two directions on S at p such that every curve on S in some of these directions has the second order contact with the tangent plane τ of S at p; these directions are called asymptotic. If p is an elliptic point and if S is of class C^{ω}, then we obtain an analogous assertion replacing the real curves of S by germs of imaginary curves in the corresponding germ of complexification of S. But if S is of class C^2 only, one must further replace the germs of imaginary curves by imaginary contact 1^2-elements, which yields

Proposition 1. *Let p be an elliptic point of a surface S of class C^2 in P_3, then there exist exactly two imaginary directions on S at p such that every contact 1^2-element on S in some of these directions is contained in the contact 2^2-element $k_p^2\tau$ of the tangent plane τ of S at p.*

For purely technical reasons, we postpone the proof up to § 9.

6. Let M, M', M'' be real manifolds of class C^r, $\dim M = n$, $\dim M' = m$, $\dim M'' = k$.

Definition 2. *A complex r-jet X of M into M' is an equivalence class $X = \{(h_2, z, h_1)\}$, $h_1 \in H^r(M)$, $h_2 \in H^r(M')$, $z \in CL^r_{m,n}$, with respect to the equivalence relation $(h_2, z, h_1) \sim (h_2 g_2, g_2^{-1} z g_1, h_1 g_1)$, $g_1 \in L^r_n$, $g_2 \in L^r_m$. The space of all complex r-jets of M into M' will be denoted by $CJ^r(M, M')$; we put $\alpha X = \beta h_1$, $\beta X = \beta h_2$. If $Y \in CJ^r(M', M'')$ and if $\alpha Y = \beta X$, then Y can be written in the form $Y = \{(h_3, w, h_2)\}$, $h_3 \in H^r(M'')$, $w \in CL^r_{k,m}$, and we define the composition of complex r-jets by*

$$YX = \{(h_3, wz, h_1)\}\,. \tag{1}$$

Evidently, $J^r(M, M')$ is a subspace of $CJ^r(M, M')$.

Remark 4. Let f be a real mapping of class C^r of M' into M'' and let $X \in CJ^r(M, M')$, then the image of X by f is defined as the product of $j^r_{\beta(X)} f$ and X, $fX = (j^r_{\beta(X)} f)\, X$, so that it is reduced to formula (1).

Remark 5. The space $C\bar{J}^r(M, M')$ or $C\tilde{J}^r(M, M')$ of all complex semi-holonomic or non-holonomic r-jets of M into M' can be introduced analogously as the set of all equivalence classes $\{(h_2, z, h_1)\}$ with respect to the equivalence relation $(h_2, z, h_1) \sim (h_2 g_2, g_2^{-1} z g_1, h_1 g_1)$, $h_1 \in H^r(M)$, $h_2 \in H^r(M')$, $g_1 \in L^r_n$, $g_2 \in L^r_m$ and $z \in C\bar{L}^r_{m,n}$ or $z \in C\tilde{L}^r_{m,n}$ respectively. The composition of complex semi-holonomic and non-holonomic jets is also given by (1).

Remark 6. $J^1(M, M')$ is a vector bundle over $M \times M'$ and it is easy to see that $CJ^1(M, M') = J^1(M, M') \otimes \mathbf{C}$.

Remark 7. If M and M' are real manifolds of class C^ω, then one can give a direct definition of complex r-jets of M into M' based on the same idea as in Remark 3. Consider the germs μ_x and $\mu'_{x'}$ of M and M' at $x \in M$ and $x' \in M'$ as well as their complexifications $C\mu_x$ and $C\mu'_{x'}$. The complex r-jets of M into M' with source x and target x' can be identified with r-jets of germs of complex analytic mappings of $C\mu_x$ into $C\mu'_{x'}$ and the corresponding real r-jets can be identified with r-jets of those germs which are complexifications of germs of real analytic mappings of μ_x into $\mu'_{x'}$.

7. Let (E, p, B), $\dim B = n$ be a real fibered manifold of class C^r and let $J^r(E, p, B)$ mean its holonomic prolongation of order r. It is well known that the elements of $J^r(E, p, B)$ can be identified with the r-jets of $J^r(B, E)$ satisfying $pX = j^r_{\alpha(X)}$ $(= j^r_{\alpha(X)}\, id_B)$.

Definition 3. *The complex holonomic prolongation of order r of a real fibered manifold (E, p, B) is defined by*

$$CJ^r(E, p, B) = \{X \in CJ^r(B, E);\ pX = j^r_{\alpha(X)}\}\,.$$

Remark 8. Analogously, the complex semi-holonomic or non-holonomic prolongation of order r of (E, p, B) can be defined by

$$C\bar{J}^r(E, p, B) = \{X \in C\bar{J}^r(B, E);\ pX = j^r_{\alpha(X)}\}$$

or

$$C\tilde{J}^r(E, p, B) = \{X \in C\tilde{J}^r(B, E);\ pX = j^r_{\alpha(X)}\}\,.$$

Remark 9. If (E, p, B) is a vector bundle, then $J^r(E, p, B)$ is also a vector bundle and it holds $CJ^r(E, p, B) = J^r(E, p, B) \otimes \mathbf{C}$; the same is also true for the semi-holonomic and non-holonomic cases.

8. An element $X \in CT^r_n(E)$ is said to *be in general position with respect to fibres* if pX is an invertible complex n^r-velocity on B. A complex contact n^r-element XCL^r_n on E is said to be in general position with respect to fibres if the same holds for X.

Proposition 2. *$CJ^r(E, p, B)$ is identified canonically with those complex contact n^r-elements on E which are in general position with respect to fibres.*

Proof. Denote by $CH^r(B)$ the space of all complex invertible n^r-velocities on B. $CH^r(B)$ is a principal fibre bundle over B with the structure group CL^r_n; it can also be constructed as the extension of the principal fibre bundle $H^r(B)$ to the structure group $CL^r_n \supset L^r_n$. I. Let $X \in CJ^r(E, p, B)$ and let $h \in CH^r(B)$ such that $\alpha X = \beta h$, then $XhCL^r_n \in CK^r_n(E)$ and $pXh = h$. II. Let $XCL^r_n \in CK^r_n(E)$ and let X be in general position with respect to fibres. Then $pX \in CH^r(B)$ and $pX(pX)^{-1} = j^r_{\alpha(X)}$, hence $X(pX)^{-1} \in$ $\in CJ^r(E, p, B)$.

Remark 10. Proposition 2 holds also for the semi-holonomic or non-holonomic case provided a complex semi-holonomic or non-holonomic contact n^r-element is defined as the set $X C\bar{L}_n^r$ or $Y C\tilde{L}_n^r$, where X or Y is a complex semi-holonomic or non-holonomic n^r-velocity respectively.

9. We are now in a position to prove Proposition 1. Choose a plane v not passing through p and introduce an affine coordinate system in $P_3 \setminus v$ with the origin at p and such that $z = 0$ is the equation of τ; then $P_3 \setminus v$ is identified with $\mathbf{R}^3$. If $\mathbf{R}^3$ is considered as fibered manifold $(\mathbf{R}^3, pr_3, \mathbf{R}^2)$, then $k_p^2 S$ is given by $z = ax^2 + 2bxy + cy^2$, $a, b, c \in \mathbf{R}$, $ac - b^2 < 0$, and $k_p^2 \tau$ is given by $z = 0$. If one considers $\mathbf{R}^3$ as fibered manifold $(\mathbf{R}^3, (pr_2, pr_3), \mathbf{R}^1)$, then every complex contact 1^2-element at p lying on S is given by $y = \alpha x + \beta x^2$, $z = (a + 2b\alpha + c\alpha^2) x^2$, $\alpha, \beta \in \mathbf{C}$. This contact 1^2-element is contained in $k_p^2 \tau$ if and only if $a + 2b\alpha + c\alpha^2 = 0$, β arbitrary, QED.

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