Štefan Schwarz

On idempotent binary relations on a finite set

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 4, 696–702

Persistent URL: http://dml.cz/dmlcz/100991

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

ON IDEMPOTENT BINARY RELATIONS ON A FINITE SET

ŠTEFAN SCHWARZ, Bratislava

(Received August 14, 1969)

Let $\Omega = \{a_1, a_2, \ldots, a_n\}$ be a finite set with $n \geqq 2$ different elements. By a binary relation on Ω we mean a subset of $\Omega \times \Omega$. The diagonal is denoted by $\Delta_\Omega = \Delta$. The empty relation will be denoted by z.

Let B_Ω be the set of all binary relations on Ω. If $\varrho \in B_\Omega$, we denote

$$a_i\varrho = \{x \in \Omega \mid (a_i, x) \in \varrho\}, \quad \varrho a_i = \{y \in \Omega \mid (y, a_i) \in \varrho\}.$$

If M is a subset of Ω, then $M\varrho = \bigcup_{a_i \in M} a_i\varrho$. Further we denote

$$\mathrm{pr}_1(\varrho) = \bigcup_{i=1}^{n} \varrho a_i, \quad \mathrm{pr}_2(\varrho) = \bigcup_{j=1}^{n} a_j\varrho.$$

Introducing in B_Ω the usual multiplication of relations (see, e.g., [1]), B_Ω becomes a finite semigroup having 2^{n^2} elements (with z as zero element).

To any $\varrho \in B_\Omega$ we can associate a "matrix" $M(\varrho) = (e_{ij})$ with elements 0 and 1 by writting $e_{ij} = 1$ on the place (i, j) if $(a_i, a_j) \in \varrho$ and $e_{ij} = 0$ if $(a_i, a_j) \notin \varrho$. We call $M(\varrho)$ the *matrix representation* of ϱ. We define the product $M(\varrho) M(\sigma)$ by the usual matrix multiplication, where for the elements 0 and 1 the addition and multiplication is defined by the rules: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$; $0 . 0 = 0 . 1 = 1 . 0 = 0$, $1 . 1 = 1$.

The correspondence $\varrho \to M(\varrho)$ is an isomorphism of the semigroup B_Ω onto the semigroup of all such "matrices".

A relation $\varrho \in B_\Omega$ is called *irreducible* if and only if $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^m$ (for some $m \geqq 1$) is a square, i.e. there is a subset $A \subset \Omega$ such that $\varrho \cup \ldots \cup \varrho^m = A \times A$. (It is easy to see that $m \leqq n$.) (See, e.g., [2].)

If $\varrho \in B_\Omega$, we define ϱ^{-1} by the requirement $(a_i, a_j) \in \varrho^{-1} \Leftrightarrow (a_j, a_i) \in \varrho$. A relation $\pi \in B_\Omega$ is called a permutation relation if and only if $\pi\pi^{-1} = \pi^{-1}\pi = \Delta$.

It is well known that to any relation ϱ there is a permutation relation π such that the matrix representation of $\pi\varrho\pi^{-1}$ is of the form

$$(1)\qquad \begin{pmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \cdots\cdots & & & \\ \cdots\cdots\cdots & & & \\ \cdots\cdots\cdots\cdots & & & \\ A_{s1} & A_{s2} & \ldots & A_{ss} \end{pmatrix}.$$

Here A_{ii} is either a "matrix" which corresponds to an irreducible relation or A_{ii} is a "zero matrix" of order 1.

For the study of the structure of the semigroup B_Ω it is of greatest importance to know *the idempotents* $\in B_\Omega$, i.e. the binary relations ϱ satisfying $\varrho^2 = \varrho$.

Clearly ϱ is idempotent if and only if $\pi\varrho\pi^{-1}$ is idempotent (where π is a permutation relation on Ω). Hence, for further purposes, we may restrict our attention to relations having a matrix representation of the form (1).

We shall briefly say that a relation ϱ is of the form (1) if its matrix representation is of the form (1). (No ambiguity can arise by this abuse of language.) Since no misunderstanding can arise, we shall also use the word matrix to denote a $0-1$ matrix of the kind introduced above.

If ϱ is an idempotent relation of the form (1), it is easy to see that A_{ii} is either a (full) square (a *positive* square matrix) or A_{ii} is a zero matrix of order 1.

In this paper we give a complete description of all idempotent relations of the form (1). In fact we give a non-tentative method how to construct all of them.

In section I we prove some necessary conditions which any idempotent relation of the form (1) must satisfy.

In section II we use these conditions to the construction of idempotents of the form (1). It will turn out that the necessary conditions obtained in section I are in an obvious sense sufficient.

We shall often use the following trivial remark. The relation ϱ is idempotent if and only if $a_i\varrho = (a_i\varrho)\,\varrho$ for every $a_i \in \Omega$.

I

Let ϱ be an idempotent $\in B_\Omega$ of the form (1). As remarked above, A_{ii} is either a positive square matrix or a zero matrix of order 1.

Denote

$$A_i = \begin{cases} \mathrm{pr}_2\,(A_{ii}) & \text{if } A_{ii} \text{ is positive}\,, \\ a_k & \text{if } A_{ii} \text{ is the zero matrix on the place } (k, k)\,. \end{cases}$$

Hence $\Omega = A_1 \cup A_2 \cup \ldots \cup A_s$.

We shall write $A_i = A_i^+$ if A_{ii} is positive and $A_i = A_i^0$ if A_{ii} is the zero matrix of order 1. Further $(A_1 \cup A_2 \cup \ldots \cup A_i)^+$ will denote the union of those A_l, $1 \leqq l \leqq i$, for which A_{ll} is positive and $(A_1 \cup A_2 \cup \ldots \cup A_i)^0$ the union of those A_l, $1 \leqq l \leqq i$, for which A_{ll} is the zero matrix of order 1. Hence $A_1 \cup \ldots \cup A_i = (A_1 \cup \ldots \cup A_i)^+ \cup \cup (A_1 \cup \ldots \cup A_i)^0$.

A) Let be first $a_j \in A_i^+$. Then $a_j\varrho = \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}, A_i\}$, where $\{a_{j_1}, \ldots, a_{j_l}\} \subset \subset A_1 \cup A_2 \cup \ldots \cup A_{i-1}$. The idempotency implies:

(2) $$a_j\varrho = (a_j\varrho)\,\varrho = \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}, A_i\}\,\varrho = \{a_{j_1}\varrho, \ldots, a_{j_l}\varrho, A_i\varrho\}\,.$$

Suppose that A_i has more than one element: $A_i = \{a_j, a_j', a_j'', \ldots\}$. Then

(3) $$a_j\varrho = \{a_{j_1}\varrho, \ldots, a_{j_l}\varrho, a_j\varrho, a_j'\varrho, a_j''\varrho, \ldots\}$$

implies $a_j\varrho \supset a_j'\varrho$. By changing the role of a_j and a_j', we have $a_j'\varrho \supset a_j\varrho$. Hence $a_j\varrho = a_j'\varrho$. This means that $a_j\varrho$ is the same set for all $a_j \in A_i$; hence $A_i\varrho = a_j\varrho$.

In other words: All rows corresponding to a given A_i^+ are identical. (Note, by the way, that an analogous argument shows that all columns corresponding to A_i^+ are also identical.)

Denote $B = \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}\}$. Then $B \subset A_1 \cup \ldots \cup A_{i-1}$, $B \cap A_i = \emptyset$. The relation (2) implies $B\varrho \subset a_j\varrho = B \cup A_i$. Since

$$B\varrho \subset (A_1 \cup \ldots \cup A_{i-1})\,\varrho \subset A_1 \cup \ldots \cup A_{i-1}\,,$$

we have $B\varrho \cap A_i = \emptyset$, and finally $B\varrho \subset B$.

We have proved that $a_j\varrho$ has the following property: If $a_m \in a_j\varrho$, then also the whole "row" $a_m\varrho$ is contained in $a_j\varrho$. Moreover, $a_j\varrho$ is a union of two disjoint sets $a_j\varrho = B \cup A_i$, where $B\varrho \subset B$.

For simplicity in further investigations we introduce the following notion:

Definition. Let σ be any binary relation on Ω. A subset $B \subset \Omega$ is called *an R-set* (*with respect to* σ) if it has the following property: If $a_m \in B$, then the whole set $a_m\sigma$ is contained in B.

Remark. The set $B = \{a_m, a_m\sigma\}$ need not be an R-set. But if σ is idempotent, then $\{a_m, a_m\sigma\}\,\sigma = a_m\sigma \subset B$, so that $\{a_m, a_m\sigma\}$ is an R-set. Clearly if B is an R-set, then $B\sigma \subset B$. If, e.g., a_m is such that $a_m\sigma$ is empty, then $B = \{a_m\}$ is an R-set and $B\sigma = \emptyset$.

It is convenient in the following to consider the empty set as an R-set. Clearly, if B_1 and B_2 are R-sets, also $B_1 \cup B_2$ is an R-set.

Using this terminology we can summarise our result as follows.

Theorem 1. *Suppose that ϱ is idempotent of the form (1). If $a_j \in A_i^+$, then $a_j\varrho$ is a union of two disjoint sets $a_j\varrho = A_i \cup B$, where B is an R-set.*

B) Let now $a_j \in A_i^0$ (i.e. $a_j = A_i^0$). Let $a_j\varrho = \{a_{j_1}. a_{j_2}, \ldots, a_{j_l}\}$. Here again $\{a_{j_1}, \ldots, a_{j_l}\} \subset A_1 \cup A_2 \cup \ldots \cup A_{i-1}$. In this case $a_j\varrho$ may be empty. Suppose in the following that $a_j\varrho \neq \emptyset$. The idempotency implies

$$(4) \qquad a_j\varrho = \{a_{j_1}, \ldots, a_{j_l}\} = \{a_{j_1}\varrho, \ldots, a_{j_l}\varrho\} .$$

This implies that there are integers $\{i_1, i_2, \ldots, i_l\} \subset \{j_1, j_2, \ldots, j_l\}$ such that

$$a_{j_1} \in a_{i_1}\varrho,\ a_{i_1} \in a_{i_2}\varrho,\ \ldots,\ a_{i_{l-1}} \in a_{i_l}\varrho .$$

The $l + 1$ integers $j_1 = i_0, i_1, i_2, \ldots, i_l$ cannot be all different. There are therefore two integers $s \neq s + h$ $(0 \leqq s \leqq l - 1,\ 2 \leqq s + h \leqq l)$ such that $i_s = i_{s+h}$. We then have

$$a_{i_s} \in a_{i_{s+1}}\varrho,\ a_{i_{s+1}} \in a_{i_{s+2}}\varrho,\ \ldots,\ a_{i_{s+h-1}} \in a_{i_{s+h}}\varrho = a_{i_s}\varrho ,$$

i.e.

$$a_{i_s} \in a_{i_{s+1}}\varrho \subset a_{i_{s+2}}\varrho \subset \ldots \subset a_{i_{s+h}}\varrho = a_{i_s}\varrho .$$

The relation $a_{i_s} \in a_{i_s}\varrho$ says that there is an element $a_{i_s} \in A_1 \cup \ldots \cup A_{i-1}$ such that $(a_{i_s}, a_{i_s}) \in \varrho$ and that there is a subset $A_l = A_l^+$, $l < i$, such that $a_{i_s} \in A_l$. Finally, (4) implies that $a_{i_s}\varrho \subset a_j\varrho$.

We may write therefore,

$$a_j\varrho = \{a_{i_s}\varrho, a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_v}\} ,$$

where $\{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_v}\}$ is either empty or it has an empty intersection with $a_{i_s}\varrho$. Hereby $\{a_{\alpha_1}, \ldots, a_{\alpha_v}\} \subsetneqq \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}\}$.

Suppose that the set just considered is not empty. The idempotency again implies

$$(5) \qquad a_j\varrho = \{a_{i_s}\varrho, a_{\alpha_1}, \ldots, a_{\alpha_v}\} = \{a_{i_s}\varrho, a_{\alpha_1}\varrho, \ldots, a_{\alpha_v}\varrho\} .$$

By supposition

$$\{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_v}\} \subset \{a_{\alpha_1}\varrho, a_{\alpha_2}\varrho, \ldots, a_{\alpha_v}\varrho\} .$$

This implies that there are integers $\{\beta_1, \beta_2, \ldots, \beta_v\} \subset \{\alpha_1, \alpha_2, \ldots, \alpha_v\}$ such that

$$a_{\alpha_1} \in a_{\beta_1}\varrho,\ a_{\beta_1} \in a_{\beta_2}\varrho,\ \ldots,\ a_{\beta_{v-1}} \in a_{\beta_v}\varrho .$$

The $v + 1$ integers $\alpha_1 = \beta_0, b_1, \ldots, \beta_v$ cannot be all different and we obtain analogously as above that there is an a_{α_t} with $a_{\alpha_t} \in a_{\alpha_t}\varrho$ and hence an $A_\alpha = A_\alpha^+$ with $\alpha < i$

such that $a_{\alpha_t} \in A_\alpha$. (Clearly $A_\alpha \neq A_i$.) Finally we conclude from (5) that $a_{\alpha_t}\varrho \subset a_j\varrho$.

Summarily: $a_j\varrho = \{a_{i_s}\varrho, a_{\alpha_t}\varrho, a_{\gamma_1}, \ldots, a_{\gamma_u}\}$, where $\{a_{\gamma_1}, \ldots, a_{\gamma_u}\} \subsetneqq \{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_v}\}$.

Repeating this argument we obtain (by changing slightly the notation) $a_j\varrho = \{a_\alpha\varrho, a_\beta\varrho, \ldots, a_v\varrho\}$, where each of the $a_\alpha, a_\beta, \ldots, a_v$, is contained in a suitably chosen A_l^+ with $l < i$.

We have proved:

Theorem 2. *If ϱ is an idempotent of the form (1) and $a_j = A_i^0$, then $a_j\varrho$ is either empty or a union of some "rows" $a_l\varrho$, where $a_l \in A_h^+$ with $h < i$.*

II

We now give an inductive method how to construct all idempotent relations of the form (1).

Choose first arbitrarily the positive square matrices and zero matrices of order 1 $A_{11}, A_{22}, \ldots, A_{ss}$ such that $A_1 \cup A_2 \cup \ldots \cup A_s = \Omega$.

It is clear that to any such choice there is at least one idempotent relation of the form (1). This is the relation ε_0 with

$$M(\varepsilon_0) = \begin{pmatrix} A_{11} & & & 0 \\ & A_{22} & & \\ 0 & & \ddots & \\ & & & A_{ss} \end{pmatrix}.$$

Suppose that in order to build up an idempotent relation ε of the form (1) we have already constructed the rows corresponding to $A_1, A_2, \ldots, A_{i-1}$ ($i \geqq 2$). We shall construct the rows corresponding to A_i.

A) Suppose first that $A_i = A_i^+$ and let $a_j \in A_i$. Take any R-set B contained in $A_1 \cup A_2 \cup \ldots \cup A_{i-1}$ and put

$$(6) \qquad a_j\varepsilon = B \cup A_i \quad \text{for every} \quad a_j \in A_i\,.$$

If $A_i = \{a_j, a_j', a_j'', \ldots\}$, we have $a_j\varepsilon = B \cup A_i$, $a_j'\varepsilon = B \cup A_i$, $a_j''\varepsilon = B \cup A_i, \ldots$ Summing these relations we get $A_i\varepsilon = B \cup A_i$, and comparing with (6) we have $A_i\varepsilon = a_j\varepsilon$.

Now $(a_j\varepsilon)\,\varepsilon = (B \cup A_i)\,\varepsilon = B\varepsilon \cup A_i\varepsilon$. Since $B\varepsilon \subset B$, we have $B\varepsilon \subset B \subset a_j\varepsilon = A_i\varepsilon$. Hence $a_j\varepsilon^2 = A_i\varepsilon = a_j\varepsilon$.

B) Suppose next that $a_j \in A_i^0$. If $(A_1 \cup \ldots \cup A_{i-1})^+ = \emptyset$, put $a_j\varepsilon = \emptyset$. If $(A_1 \cup \ldots$

$\ldots \cup A_{i-1})^+ \neq \emptyset$, consider all "rows" $a_l\varepsilon$ with $a_l \in (A_1 \cup \ldots \cup A_{i-1})^+$. Take any "rows" of them, say $a_l\varepsilon, a'_l\varepsilon, a''_l\varepsilon, \ldots$ and put $a_j\varepsilon = \{a_l\varepsilon, a'_l\varepsilon, a''_l\varepsilon, \ldots\}$.

We have:

$$a_j\varepsilon^2 = \{a_l\varepsilon^2, a'_l\varepsilon^2, a''_l\varepsilon^2, \ldots\} = \{a_l\varepsilon, a'_l\varepsilon, a''_l\varepsilon, \ldots\} = a_j\varepsilon\,.$$

Since by this proceeding (by applying successively A) and B)) we obtain $a_l\varepsilon = a_l\varepsilon^2$ for every $a_l \in \Omega$, the relation ε is an idempotent, and it follows from Theorems 1 and 2 that any idempotent relation of the form (1) is obtained in this manner.

Remark. Suppose that we have already constructed the rows corresponding to all $a_l \in A_1 \cup \ldots \cup A_{i-1}$. The question arises how to find all R-sets which we have to use for the construction of $a_j\varepsilon$, where $a_j \in A_i^+$.

a) If $a_l \in (A_1 \cup \ldots \cup A_{i-1})^+$, then clearly $a_l\varepsilon$ is an R-set and $a_l \in a_l\varepsilon$.

b) If $a_l = A_h^0$, $h < i$, and $a_l\varepsilon \neq \emptyset$, then $\{a_l, a_l\varepsilon\}$ is an R-set but $a_l \notin a_l\varepsilon$. [Note that $a_l\varepsilon$ is a union of some of the preceding rows, so that $a_l\varepsilon$ itself is also an R-set.]

c) If $a_l = A_h^0$, $h < i$, and $a_l\varepsilon = \emptyset$, then $\{a_l\}$ itself is an R-set.

Any R-set contained in $A_1 \cup \ldots \cup A_{i-1}$ is a union of these three types of R-sets [We shall call them in section III "*elementary R-sets*".]

III

We now illustrate our proceeding on the following example.

We have to find all idempotent relations of the form (1) on the set $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$ if the diagonal matrices are prescribed as indicated below:

$$(7) \qquad \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ . & . & 0 & & \\ . & . & . & 0 & \\ . & . & . & . & 1 \end{pmatrix}.$$

Since $A_2 = A_2^0$, the third row is either empty or it contains the whole preceding row. We have therefore two possibilities:

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ 0 & 0 & 0 & & \\ . & . & . & 0 & \\ . & . & . & . & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ 1 & 1 & 0 & & \\ . & . & . & 0 & \\ . & . & . & . & 1 \end{pmatrix}.$$

In the fourth row we have an analogous situation, so that we get four possibilities:

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ . & . & . & . & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ 0 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & \\ . & . & . & . & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ 1 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & \\ . & . & . & . & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ 1 & 1 & 0 & & \\ 1 & 1 & 0 & 0 & \\ . & . & . & . & 1 \end{pmatrix}.$$

In the first case the elementary R-sets contained in $\{a_1, a_2, a_3, a_4\}$ are: $\emptyset$, $\{a_1, a_2\}$, $\{a_3\}$, $\{a_4\}$. There exist therefore eight possible R-sets which lead to the following possibilities for the last row:

$$(00001),\ (11001),\ (00101),\ (00011),$$
$$(11101),\ (11011),\ (00111),\ (11111).$$

In the second case the elementary R-sets are: $\emptyset$, $\{a_1, a_2\}$, $\{a_3\}$, $\{a_4, a_1, a_2\}$. We get only six different possibilities for the last row, namely: (00001), (11001), (00101), (11011), (11101), (11111).

In the third case the elementary R-sets are: $\emptyset$, $\{a_1, a_2\}$, $\{a_3, a_1, a_2\}$, $\{a_4\}$. We get the following six possibilities for the last row: (00001), (11001), (11101), (00011), (11011), (11111).

Finally, in the last case the elementary R-sets are: $\emptyset$, $\{a_1, a_2\}$, $\{a_3, a_1, a_2\}$, $\{a_4, a_1, a_2\}$. They lead to the following 5 possibilities for the last row: (00001), (11001), (11101), (11011), (11111).

We have obtained altogether $8 + 6 + 6 + 5 = 25$ idempotent relations and these are exactly all idempotent relations of the form (1) having the diagonal matrices prescribed in (7). [Note that there are $2^9 = 512$ different relations of the form (1) having the diagonal matrices prescribed in (7).]

References

[1] *A. H. Clifford* and *G. B. Preston:* The Algebraic Theory of Semigroups, Vol. 1., Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1962.

[2] *Š. Schwarz:* On the semigroup of binary relations on a finite set. Czech. math. J. *20* (1970), 632—679.

Author's address: Bratislava, Gottwaldovo nám. 2, ČSSR (Slovenská vysoká škola technická).