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ON INTEGRATION IN BANACH SPACES, I

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INTRODUCTION

Let T be a non empty set and $\mathscr{P}$ a δ-ring (a ring closed with respect to the forming of countable intersections) of subsets of T. Let further X and Y be (real or complex) Banach spaces and denote by $L(X, Y)$ the Banach space of all bounded linear operators from X to Y. We say that a set function $\boldsymbol{m} : \mathscr{P} \to L(X, Y)$ is an operator valued measure countably additive in the strong operator topology, if for every $\boldsymbol{x} \in X$ the set function $E \to \boldsymbol{m}(E)\, \boldsymbol{x}$, $E \in \mathscr{P}$ is a countably additive vector measure. Our purpose is to develop a Lebesgue type integration theory for functions on T with values in X with respect to such measure. This part I contains the theory of so called integrable functions. In part II, see [8], we develop the theory of L_p spaces. The need of this integration theory arises, among other, from [9], where we represent a wide class of bounded linear operators, including weakly compact operators, on a space $C_0(T, X)$ as integrals with respect to such measures. Here T is a locally compact Hausdorff topological space and $C_0(T, X)$ denotes the Banach space of all continuous X valued functions on T tending to zero at infinity with the usual supremum norm.

§ 1 is preparatory. An important continuation of section 1.1 is § 1 of [9]. The basic quantity of the theory is the semivariation $\hat{\boldsymbol{m}}$ of the measure $\boldsymbol{m}$. This concept has its origin in [11]. Suppose that the semivariation $\hat{\boldsymbol{m}}$ is finite on $\mathscr{P}$. Then the $\mathscr{P}$-simple functions on T with values in X are called simple integrable functions, and a function $\boldsymbol{f} : T \to X$ is called measurable, if there is a sequence of simple integrable functions converging pointwise to it.

In § 2 with the help of Egoroff-Lusin's Theorem, see section 1.4, we extend the notion of the integral from simple integrable functions to a substantially wider class of functions, which we call integrable, see Definition 2, and investigate their properties. The obtained class of integrable functions satisfies the following characterization: *The class of all integrable functions is the smallest class of functions containing the class of simple integrable functions for which the fundamental Theorem 16 on interchange of limit and integral is valid.*

In § 3 we compare the introduced integral with some well known integrals. Our integral substantially generalizes the countably additive case of the general bilinear vector integral of R. G. BARTLE from [1]. The basic generalization is that our measure is countably additive in general only in the strong operator topology, while Bartle supposes countable additivity in the uniform operator topology. Further, our approach, which is quite different from that of Bartle, gives in general a wider class of integrable functions also in the case when the measure is countably additive in the uniform operator topology, see Example 7″ and the last paragraph of the paper. On the other hand, in the important special case when $\mathscr{P}$ is a σ-algebra and the measure $\boldsymbol{m}$ satisfies the $*$-condition of Definition 2 in [1], which always happens when $\boldsymbol{Y}$ is a weakly complete Banach space, see the $*$-Theorem in section 1.1 below, our and the Bartle's integral coincides.

Much material about vector and operator valued measures and on integration with respect to them, however, under the assumption of finite variation, can be found in the book [6]. To the important problem of extending a vector measure from a ring of subsets to the generated σ-ring we refer the reader to papers [15], [16] and [17].

§ 1. PRELIMINARY

1.1 OPERATOR VALUED MEASURES AND SIMPLE INTEGRABLE FUNCTIONS

Let $\mathscr{P}_0$ be a δ-ring of subsets of a set T and let $\boldsymbol{m} : \mathscr{P}_0 \to L(\boldsymbol{X}, \boldsymbol{Y})$ be an operator valued measure countably additive in the strong operator topology. Let us note that by the Orlicz-Pettis theorem, see IV. 10.1 in [10], the countable additivity in the strong and in the weak operator topology are equivalent conditions, and as Example 6 below demonstrates, they do not imply the countable additivity in the uniform operator topology.

Let us denote by $\mathfrak{S}(\mathscr{P}_0)$ the smallest σ-ring containing $\mathscr{P}_0$. It is easy to show, see [6, § 1 Prop. 9 and Cor. 1], that each set $E \in \mathfrak{S}(\mathscr{P}_0)$ can be written as a union of a countably many disjoint elements of $\mathscr{P}_0$. From here immediately follows that if $A \in \mathscr{P}_0$ and $E \in \mathfrak{S}(\mathscr{P}_0)$ then $A \cap E \in \mathscr{P}_0$.

If $E \subset T$ then χ_E always will denote its characteristic function on T. By a $\mathscr{P}_0$-simple function on T with values in $\boldsymbol{X}$ we mean a function of the form

$$f = \sum_{i=1}^{r} \boldsymbol{x}_i \cdot \chi_{E_i}$$

where $\boldsymbol{x}_i \in \boldsymbol{X}$, $E_i \in \mathscr{P}_0$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, r$. Its integral on a set $E \in \mathfrak{S}(\mathscr{P}_0)$ with respect to the measure $\boldsymbol{m}$ is defined by equality

$$\int_E f \,\mathrm{d}\boldsymbol{m} = \sum_{i=1}^{r} \boldsymbol{m}(E \cap E_i)\, \boldsymbol{x}_i .$$

If Z is a Banach space and $z \in Z$, then $|z|$ denotes the norm of z. Naturally the question arises how to extend the integral to a wider class of functions. For this purpose for a function $f: T \to X$ and a set $A \subset T$ put $\|f\|_A = \sup_{t \in A} |f(t)|$, and define on $\mathfrak{S}(\mathscr{P}_0)$ a non negative set function $\hat{m}$, which will be called the semivariation of the measure m, by equality

$$\hat{m}(E) = \sup \left\{ \left| \int_E f \, \mathrm{d}m \right| , f \text{ is } \mathscr{P}_0\text{-simple}, \ \|f\|_E \leqq 1 \right\},$$

$E \in \mathfrak{S}(\mathscr{P}_0)$. From this definition we immediately have, see [6, § 4 Prop. 3], that $\hat{m}(\emptyset) = 0$, that $\hat{m}$ is monotone and countably subadditive, and further, that for each $\mathscr{P}_0$-simple function f and each set $E \in \mathfrak{S}(\mathscr{P}_0)$ the following inequality holds:

$$\left| \int_E f \, \mathrm{d}m \right| \leqq \|f\|_E \, . \, \hat{m}(E) \, .$$

If now $\hat{m}(E)$ is finite, then by this inequality we may extend the integral on E from $\mathscr{P}_0$-simple functions to their closure in the norm $\|\,.\,\|_T$ in the Banach space of bounded X valued functions on T. As we shall see later, it is in this simple inequality where the importance of the semivariation for our theory of integration lies.

From now on $\mathscr{P}_1$ will denote the class of those sets from $\mathfrak{S}(\mathscr{P}_0)$ which have finite semivariation. It is easy to verify that $\mathscr{P}_1$ is a δ-ring and that for each sets $A \in \mathscr{P}_1$ and $E \in \mathfrak{S}(\mathscr{P}_0)$, $A \cap E \in \mathscr{P}_1$. In examples below we demonstrate that between $\mathscr{P}_0$ and $\mathscr{P}_1$ all possible set relations may occur.

Definition 1. *Put $\mathscr{P} = \mathscr{P}_0 \cap \mathscr{P}_1$. Elements of $\mathscr{P}$ will be called integrable sets. A $\mathscr{P}$-simple function on T with values in X will be called simple integrable function. The set of all simple integrable functions will be denoted by $\mathfrak{I}_s$.*

From this definition it is obvious that $\mathscr{P}$ is a δ-ring and that for each sets $A \in \mathscr{P}$ and $E \in \mathfrak{S}(\mathscr{P}_0)$, $A \cap E \in \mathscr{P}$. The basic properties of simple integrable functions and their integrals will be collected in Lemma 2 below. Now we prove

Lemma 1. $\mathfrak{S}(\mathscr{P}) = \mathfrak{S}(\mathscr{P}_1)$, *and for each set $E \in \mathfrak{S}(\mathscr{P})$*

$$\hat{m}(E) = \sup \left\{ \left| \int_E f \, \mathrm{d}m \right| , f \in \mathfrak{I}_s, \|f\|_E \leqq 1 \right\}. \tag{1}$$

Proof. If $A \in \mathscr{P}_1$, then A can be written as a union of countably many disjoint $A_n \in \mathscr{P}_0$, $n = 1, 2, \ldots$. Since $\hat{m}$ is monotone, $A_n \in \mathscr{P}_1$ and thus $A_n \in \mathscr{P}$ for each n. Therefore $A \in \mathfrak{S}(\mathscr{P})$, which proves that $\mathfrak{S}(\mathscr{P}_1) \subset \mathfrak{S}(\mathscr{P})$. The inclusion $\mathfrak{S}(\mathscr{P}) \subset \mathfrak{S}(\mathscr{P}_1)$ is obvious since $\mathscr{P} = \mathscr{P}_0 \cap \mathscr{P}_1$.

If $E \in \mathscr{P}_1$, then $f . \chi_E \in \mathfrak{I}_s$ for each $\mathscr{P}_0$-simple function f, so in this case (1) is proved. Suppose now that $E \in \mathfrak{S}(\mathscr{P})$ and $\hat{m}(E) = +\infty$. Then there is a sequence f_n, $n = 1, 2, \ldots$ of $\mathscr{P}_0$-simple functions with $|\int_E f_n \, \mathrm{d}m| > n$ for each n. Since E can be written as a union of countably many disjoint $E_k \in \mathscr{P}$, $k = 1, 2, \ldots$, and since for each n the integral $\int f_n \, \mathrm{d}m$ as a set function is a countably additive vector measure on the σ-ring $\mathfrak{S}(\mathscr{P})$, for each n there is a positive integer k_n such that $|\int_E f_n . \chi_{F_n} \, \mathrm{d}m| > n$ where $F_n = \bigcup_{k=1}^{k_n} E_k$. But $F_n \in \mathscr{P}$ for each n, and therefore $f_n . \chi_{F_n} \in \mathfrak{I}_s$ for each n. Since $\mathfrak{I}_s$ is a subset of the set of all $\mathscr{P}_0$-simple functions, the lemma is proved.

From now on we shall be interested only in the δ-rings $\mathscr{P}$ and $\mathscr{P}_1$ and in the σ-ring $\mathfrak{S}(\mathscr{P}) = \mathfrak{S}(\mathscr{P}_1)$.

Let now T be a locally compact Hausdorff topological space. By $\mathscr{B}_0$ and $\mathscr{B}$ we shall denote the δ-rings of relatively compact Baire and Borel subsets of T respectively. In other words, $\mathscr{B}_0$ and $\mathscr{B}$ are those Baire and Borel subsets of T in the sence of § 51 in [12] whose closures are compact subsets of T. $\mathfrak{S}(\mathscr{B}_0)$ and $\mathfrak{S}(\mathscr{B})$ denote the corresponding σ-rings. We say that m is a Baire operator valued measure if T is a locally compact Hausdorff topological space and $\mathscr{P} = \mathscr{B}_0$ or $\mathscr{P} = \mathfrak{S}(\mathscr{B}_0)$. Let us note that each Baire operator valued measure is regular in the strong operator topology, see [7, Theor. 4]. We say that m is a regular Borel operator valued measure if T is a locally compact Hausdorff topological space, $\mathscr{P} = \mathscr{B}$ or $\mathscr{P} = \mathfrak{S}(\mathscr{B})$, and the measure m is regular in the strong (equivalently weak, see [18, Cor. 2 of Lemma 1]) operator topology.

The scalar semivariation of the measure m, which will be denoted by $\|m\|$, is the non negative set function on $\mathfrak{S}(\mathscr{P})$ defined by the equality

$$\|m\| (E) = \sup \left| \sum_{i=1}^{r} a_i . m(E \cap E_i) \right| , \quad E \in \mathfrak{S}(\mathscr{P})$$

where the supremum is taken over all finite collections of scalars a_i, $|a_i| \leqq 1$, and over all finite collections of disjoint $E_i \in \mathscr{P}$, see [10, IV. 10.3]. Clearly $\|m\| (\emptyset) = 0$ and $\|m\|$ is monotone. Since for each $x \in X$ the set function $\|m(.)\, x\|$ is countably subadditive on $\mathfrak{S}(\mathscr{P})$, see [10, IV.10.4] the scalar semivariation $\|m\|$ is also countably subadditive on $\mathfrak{S}(\mathscr{P})$. If the measure m is countably additive in the uniform operator topology then moreover, for each set $A \in \mathscr{P}$ there exists a finite non negative countably additive measure λ_A on $\mathfrak{S}(\mathscr{P})$ such that $\lambda_A(E) \leqq \|m\| (A \cap E)$ and $\lim_{\lambda_A(E) \to 0} \|m\| (A \cap E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$, see [10, IV.10.5].

It is obvious from the definitions that $\|m\| (E) \leqq \hat{m}(E)$ for each set $E \in \mathfrak{S}(\mathscr{P})$. It may happen, see Example 5 below, that $\|m\| (E) < +\infty$ and at the same time $\hat{m}(E) = +\infty$. A set $N \in \mathfrak{S}(\mathscr{P})$ is called a m zero set if $\|m\| (N) = 0$. Clearly $\|m\| (N) = 0$ if and only if $\hat{m}(N) = 0$, $N \in \mathfrak{S}(\mathscr{P})$, and the collection of all m zero sets is a σ-ideal in the σ-ring $\mathfrak{S}(\mathscr{P})$.

Further let us introduce a concept of absolute continuity. We say that an additive set function $\mathbf{v} : \mathfrak{S}(\mathscr{P}) \to \boldsymbol{Z}$ is absolutely continuous with respect to a set function $\boldsymbol{\mu} : \mathfrak{S}(\mathscr{P}) \to \boldsymbol{Z}_1$ where $\boldsymbol{Z}$ and $\boldsymbol{Z}_1$ are Banach spaces, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\sup \{|\boldsymbol{\mu}(A)|, A \in \mathfrak{S}(\mathscr{P}), A \subset E\} < \delta$ implies $\|\mathbf{v}\| (E) < \varepsilon$, $E \in \mathfrak{S}(\mathscr{P})$.

The basic properties of simple integrable functions and their integrals are collected in the following lemma, the proof of which is evident.

Lemma 2. *The set $\mathfrak{I}_s$ of all simple integrable functions is a linear space. For a fixed set $E \in \mathfrak{S}(\mathscr{P})$ the mapping $\boldsymbol{f} \to \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$, $\boldsymbol{f} \in \mathfrak{I}_s$, is linear, and for a fixed function $\boldsymbol{f} \in \mathfrak{I}_s$ the set function $E \to \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$, $E \in \mathfrak{S}(\mathscr{P})$ is a countably additive absolutely $\|\boldsymbol{m}\|$ and $\hat{\boldsymbol{m}}$ continuous vector measure on $\mathfrak{S}(\mathscr{P})$ with values in $\boldsymbol{Y}$. If $\boldsymbol{m}$ is a Baire or a regular Borel operator valued measure, then this measure is regular on $\mathfrak{S}(\mathscr{B}_0)$ or on $\mathfrak{S}(\mathscr{B})$, respectively. Further, for each set $E \in \mathfrak{S}(\mathscr{P})$ and for each function $\boldsymbol{f} \in \mathfrak{I}_s$, $\int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{f} \cdot \chi_E \, \mathrm{d}\boldsymbol{m}$.*

In [5, Theorem 5 and section 6] the validity of the theorem of Orlicz, see 3.2.1 in [14], was extended from weakly complete Banach spaces to spaces containing no subspace isomorphic to the space $\boldsymbol{c}_0$. We use this result in the proof of the first assertion of the below stated important $*$-Theorem. The second assertion immediately follows from Lemma 2 in [9]. Let us note that the first assertion of this theorem for the particular case $T = [0, 1]$, $\mathscr{P} = \mathscr{B}$ was recently proved in a different way by J. Batt in [3, Satz 6]. However, the construction used in our proof is needed for the proof of the interesting Theorem 18 in part II, see [8].

$*$-Theorem. *Let $\boldsymbol{Y}$ contain no subspace isomorphic to the space $\boldsymbol{c}_0$, for example let $\boldsymbol{Y}$ be a weakly complete Banach space, see pp. 160 and 161 in [5]. Then the semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathscr{P}_1$, i.e., if $E_n \in \mathscr{P}_1$, $E_n \searrow \emptyset$, $n = 1, 2, \ldots$, then $\lim_{n \to \infty} \hat{\boldsymbol{m}}(E_n) = 0$. In general the continuity of the semivariation $\hat{\boldsymbol{m}}$ on $\mathscr{P}_1$ is equivalent to the following property: for each set $A \in \mathscr{P}_1$ there is a finite non negative countably additive measure λ_A on $\mathfrak{S}(\mathscr{P})$ such that $\lambda_A(E) \leqq \|\boldsymbol{m}\| (E \cap A)$ and $\lim_{\lambda_A(E) \to 0} \hat{\boldsymbol{m}}(A \cap E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$.*

Proof. Let $\boldsymbol{Y}$ contain no subspace isomorphic to the space $\boldsymbol{c}_0$ and suppose that the semivariation $\hat{\boldsymbol{m}}$ is not continuous on $\mathscr{P}_1$. Then there is an $\varepsilon > 0$ and a decreasing sequence of sets $A_k \searrow \emptyset$, $A_k \in \mathscr{P}_1$ with $\hat{\boldsymbol{m}}(A_k) > \varepsilon$ for every $k = 1, 2, \ldots$. Owing to the definition of the semivariation $\hat{\boldsymbol{m}}$, for A_1 there are $\boldsymbol{x}_i \in \boldsymbol{X}$, $|\boldsymbol{x}_i| \leqq 1$, and disjoint $E_i \in \mathscr{P}$, $i = 1, 2, \ldots, r_1$ with $|\sum_{i=1}^{r_1} \boldsymbol{m}(A_1 \cap E_i) \, \boldsymbol{x}_i| > \varepsilon$. Since $A_k \searrow \emptyset$, the countable additivity of the measure $\boldsymbol{m}$ in the strong operator topology implies $\lim_{k \to \infty} \boldsymbol{m}(A_k \cap E_i) \, \boldsymbol{x}_i = 0$ for every $i = 1, 2, \ldots, r_1$. Hence there is a k_1 with $|\sum_{i=1}^{r_1} \boldsymbol{m}[(A_1 - A_{k_1}) \cap E_i] \, \boldsymbol{x}_i| > \varepsilon$. Continuing in this manner, for every $n = 1, 2, \ldots$

there is a $y_n \in Y$ of the form $y_n = \sum_{i=r_{n-1}+1}^{r_n} m[(A_{k_{n-1}} - A_{k_n}) \cap E_i]\, x_i$, $x_i \in X$, $|x_i| \leqq 1$, disjoint $E_i \in \mathscr{P}$, with $|y_n| > \varepsilon$. On the other hand, since for every sequence of real numbers $\{a_n\}$, $|a_n| \leqq 1$, $n = 1, 2, \ldots$ by the definition of the semivariation $\hat{m}$ we have $|\sum_{n=1}^{\infty} a_n y_n| \leqq \hat{m}(T) < +\infty$, Lemma 2 in [5] implies that the series $\sum_{n=1}^{\infty} y_n$ is weakly unconditionally convergent. But by the assumption of the theorem Y contains no subspace isomorphic to the space c_0, so according to the extended Orlicz theorem (see Theorem 5 in [5]) the series $\sum_{n=1}^{\infty} y_n$ is (strongly) unconditionally convergent, and thus $\lim_{n\to\infty} |y_n| = 0$, a contradiction. This proves the first assertion of the theorem. The second assertion of the theorem immediately follows from Lemma 2 in [9].

The latter property in the just proved theorem is in fact a localization on $\mathscr{P}_1$ of the $*$-property od Definition 2 in [1]. Thus we see its equivalence with the continuity of the semivariation $\hat{m}$ on $\mathscr{P}_1$, and also that for a weakly complete Banach space it is always fulfilled. In Example 7 below we construct a measure m countably additive in the uniform operator topology whose semivariation $\hat{m}$ is not continuous on the corresponding $\mathscr{P}_1$.

Examples. Let us have now few examples of operator valued measures. Examples 1, 2, 3 and 4 are starting points of well known integration theories and will be met again in § 3 below. Examples 5, 6 and 7 are more or less illustrative and will be used in § 2 in [9].

1. Let $X = Y$ be the space of real or complex numbers, let m be a real or a complex countably additive measure on a δ-ring $\mathscr{P}_0$, and let $m(E)\, x = x \,.\, m(E)$ for $E \in \mathscr{P}_0$ and $x \in X$. Then clearly $\|m\|\,(E) = \hat{m}(E) = v(m, E) < +\infty$ for each set $E \in \mathscr{P}_0$. Here $v(m, E)$ denotes the variation of m on E. In this way $\mathscr{P}_1 \supset \mathscr{P}_0$. It is not difficult to find cases where $\mathscr{P}_1 \neq \mathscr{P}_0$. From this situation the classical theory of scalar Lebesgue type integration starts, see for example [12].

2. Let $X = Y$ be a Banach space, let m be a countably additive finite scalar measure on $\mathscr{P}_0$ and let $m(E)\, x = x \,.\, m(E)$ for $E \in \mathscr{P}_0$ and $x \in X$. Then again $\|m\|\,(E) = \hat{m}(E) = v(m, E) < +\infty$ for each set $E \in \mathscr{P}_0$, and therefore $\mathscr{P}_1 \supset \mathscr{P}_0$. This is the starting point of the well known Bochner integral, see for example § 3.7 in [14].

3. Let Y be a Banach space, X be the space of scalars of Y, m a countably additive vector measure on $\mathscr{P}_0$ with values in Y, and let $m(E)\, x = x \,.\, m(E)$ for $E \in \mathscr{P}_0$ and $x \in X$. Then for each set $E \in \mathscr{P}_0$, $\hat{m}(E) = \|m\|\,(E) < +\infty$, see [10, IV.10.4], and therefore again $\mathscr{P}_1 \supset \mathscr{P}_0$. From this point the theory of integration of scalar functions with respect to a vector measure starts, see § IV.10 in [10] and [15].

4. Let $\mathscr{P}_0$, X and Y be arbitrary and let the measure m be countably additive in the uniform operator topology on $\mathscr{P}_0$. R. G. Bartle in [1] starts from this point. On page 339 fifth line from above he wrongly asserts that $\mathscr{P}_1 \supset \mathscr{P}_0$, see the following Example 5.

5. Let X be a Banach space, Y be the space of scalars of X, and let $m : \mathscr{P}_0 \to \to L(X, Y) = X^*$ be an operator valued measure countably additive in the strong operator topology of X^*, i.e., in its X-topology. Then from definitions of the variation $v(m, .)$ and the semivariation $\hat{m}$ and from Hahn-Banach's theorem we immediately have the equality $\hat{m}(E) = v(m, E)$ for each set $E \in \mathfrak{S}(\mathscr{P}_0)$. At the same time the variation $v(m\ .)$ is obviously a non negative, not necessarily finite, countably additive measure on $\mathfrak{S}(\mathscr{P}_0)$. Let us demonstrate now on a simple example that the situation $\mathscr{P}_0 \supset \mathscr{P}_1, \mathscr{P}_0 \neq \mathscr{P}_1$ may occur. Let T be the set of natural numbers, $\mathscr{P}_0$ the σ-algebra of all subsets of T and X the Hilbert space l_2. Let further $e_k \in l_2$, $k = 1, 2, \ldots$ be the orthonormal system of vectors in l_2 of form $e_k = [\overbrace{0, 0, \ldots\ 0, 1}^{k}, 0, \ldots]$, and put $m(\{k\}) = 1/k \,.\, e_k$, $m(E) = \sum_{k \in E} m(\{k\})$, $E \in \mathscr{P}_0$. Then by Riesz-Fischer's theorem the measure m is countably additive in the uniform operator topology of $l_2^* = l_2$, i.e., in its norm topology. On the other hand its variation $v(m, E) = \sum_{k \in E} 1/k$ is not finite for $E = T$. Thus $\mathscr{P}_0 \supset \mathscr{P}_1$ and $\mathscr{P}_0 \neq \mathscr{P}_1$.

6. This is an example of an operator valued measure countably additive in the strong operator topology which is not countably additive in the uniform operator topology on the corresponding $\mathscr{P}$.

Let T be the set of natural numbers, $\mathscr{P}_0$ the σ-algebra of all subsets of T, X the real space l_1 and Y the real space c_0. For $x \in l_1$ of the form $x = [x_1, x_2, \ldots, x_k, \ldots]$ and $k = 1, 2, \ldots$, let us put $m(\{k\})\, x = [0, 0, \ldots, 0, x_k, 0, \ldots] \in c_0$, and $m(E) = \sum_{k \in E} m(\{k\})$ for $E \in \mathscr{P}_0$. Then it is obvious that m is an operator valued measure countably additive in the strong operator topology, that $\mathscr{P}_1 = \mathscr{P}_0$ and that for each non empty set $E \in \mathscr{P}_1$, $|m(E)| = \hat{m}(E) = 1$ In this way m is not countably additive in the uniform operator topology on $\mathscr{P}$.

7. This is an example of an operator valued measure countably additive in the uniform operator topology, the semivariation $\hat{m}$ of which is not continuous on the corresponding $\mathscr{P}$.

Let T, $\mathscr{P}_0$, X and Y be the same as in the preceding example 6 and let $y_k \in c_0$, $k = 1, 2, \ldots$ be the following sequence of vectors:

$$\begin{aligned} y_1 &= [1, 0, 0, \ldots\ldots\ldots\ldots], \\ y_2 = y_3 &= [0, \tfrac{1}{2}, 0, 0, \ldots\ldots\ldots], \\ y_4 = y_5 = y_6 &= [0, 0, \tfrac{1}{3}, 0, 0, \ldots\ldots], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

For $x \in l_1$, $x = [x_1, x_2, \ldots, x_k, \ldots]$ and $E \in \mathscr{P}_0$ let us define $m(E)\,x = \sum_{k\in E} x_k \cdot y_k$. Then it is easy to see that for a non empty set $E \in \mathscr{P}_0$, $|m(E)| = \sup_{k\in E} |y_k|$ and $\hat{m}(E) = |\sum_{k\in E} y_k|$. From these equalities it is clear that m is an operator valued measure countably additive in the uniform operator topology on $\mathscr{P}_1 = \mathscr{P}_0$ and that the semivariation $\hat{m}$ is not continuous on $\mathscr{P}$ since $\hat{m}(\{k, k+1, k+2, \ldots\}) = 1$ for each $k = 1, 2, \ldots$.

1.2 MEASURABLE FUNCTIONS

Proofs of the assertions stated below about measurable functions and about the convergence in measure and in semivariation are classical and well known, so they are omitted.

A function $f: T \to X$ is called measurable if there is a sequence of simple integrable functions $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for each $t \in T$. From the properties of scalar measurable functions, see [12, § 20 Theor. B], it is obvious that for each measurable function f there is a sequence of simple integrable functions $\{g_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} g_n(t) = f(t)$ for each $t \in T$ and the sequence $\{|g_n|\}_{n=1}^{\infty}$ is non decreasing for each $t \in T$.

The set of all measurable functions is a linear space and if in addition X is a Banach algebra, then this space is an algebra of functions. It is obvious that if φ is a scalar measurable function and f is a measurable function, then $\varphi \cdot f$ is a measurable function. If f is a measurable function, then $|f|$ is a scalar measurable function and $\{t \in T, |f(t)| > 0\} \in \mathfrak{S}(\mathscr{P})$.

It may be shown, see Theor. 3.5.3 in [14] that a function f is measurable if and only if is separable valued and weakly measurable, i.e., for each functional $x^* \in X^*$ the scalar function x^*f is measurable. As a consequence we immediately have that the set of all measurable functions is closed under the formation of pointwise limits of sequences, i.e., if $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions and for each $t \in T$, $\lim_{n\to\infty} f_n(t) = f(t) \in X$, then f is a measurable function.

Another useful criterium of measurability of a function is the following theorem, see [10, III.6.9]. A function f is measurable if and only if it is separable valued and for each open subset G of X, $f^{-1}(G) \cap \{t \in T, |f(t)| > 0\} \in \mathfrak{S}(\mathscr{P})$. As a consequence we have that if m is a Baire or a Borel operator valued measure, then each weakly continuous function f for which $\{t \in T, |f(t)| > 0\} \in \mathfrak{S}(\mathscr{P})$ is measurable. Thus for such measures each function $f \in C_0(T, X)$ is measurable.

Analogously to the classical definition, we say that a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$ converges in measure or in semivariation to a measurable function f, if for each $\delta > 0$, $\lim_{n\to\infty} \|m\| (\{t \in T, |f_n(t) - f(t)| > \delta\}) = 0$ or $\lim_{n\to\infty} \hat{m}(\{t \in T, |f_n(t) - f(t)| > \delta\}) = 0$ respectively. Similarly as in [12, § 21] we introduce the concept of the almost uniform convergence in measure and in semivariation. In the same way as in § 22 in [12] it may be proved that if a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$ is fundamental in measure or in semivariation, then there is a measurable function and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converging to f almost uniformly in measure or in semivariation respectively, and therefore also almost everywhere m. At the same time the sequence $\{f_n\}_{n=1}^{\infty}$ converges in measure or in semivariation respectively to the function f.

If a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$ converges on each set $E \in \mathscr{P}$ in measure or in semivariation to a measurable function f, then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converging to f on the whole T almost everywhere m. (Put $F = \bigcup_{n=1}^{\infty} \{t \in T, |f_n(t)| > 0\}$. Then $F \in \mathfrak{S}(\mathscr{P})$, so there is a non decreasing sequence of sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$, with $\bigcup_{k=1}^{\infty} F_k = F$. For each k choose from the sequence $\{f_{n_{k-1,i}}\}_{i=1}^{\infty}$, $f_{n_{0,i}} = f_i$, a subsequence $\{f_{n_{k,i}}\}_{i=1}^{\infty}$ converging almost everywhere m on F_k to f and put $f_{n_k} = f_{n_{k,k}}$ for $k = 1, 2, \ldots$).

If the measure m is countably additive in the uniform operator topology on $\mathscr{P}$, and generally only in this case, then on each set $E \in \mathscr{P}$ the Egoroff theorem is valid for $\|m\|$. Similarly, if the semivariation $\hat{m}$ is continuous on $\mathscr{P}_1$, and generally only in this case, then on each set $E \in \mathscr{P}_1$ the Egoroff theorem is valid for $\hat{m}$, see the $*$-Theorem in section 1.1 above. Let us note that the convergence in semivariation always implies the convergence in measure. If the semivariation $\hat{m}$ is continuous on $\mathscr{P}_1$, and generally only in this case, then the converse is also true for each set $E \in \mathscr{P}_1$.

Let us also state analogs of Lusin's Theorem, see [12, § 55 exerc. 3]. Let a Baire or a regular Borel operator valued measure m be countably additive in the uniform operator topology on $\mathscr{P}$ and let f be a measurable function. Then for each set $E \in \mathscr{P}$ and each $\varepsilon > 0$ there is a compact G_δ set C such that $C \subset E$, $\|m\| (E - C) < \varepsilon$ and the restriction of f on C is continuous. At the same time if f is Borel measurable, then there is a Baire measurable function f_1 such that $f = f_1$ almost everywhere m, see [4, § 68 Theor. 1]. Quite analogously, if the semivariation $\hat{m}$ is continuous on $\mathscr{P}_1$, then for each set $A \in \mathscr{P}_1$ and each $\varepsilon > 0$ there is a compact G_δ set $C_1 \subset A$ such that $\hat{m}(A - C_1) < \varepsilon$ and f is continuous on C_1, see the $*$-Theorem in section 1.1 above.

1.4 EGOROFF-LUSIN'S THEOREM

Our procedure of extending the integral from the class of simple integrable functions to the substantially wider class of integrable functions is based on the simple consequence of Egoroff's theorem stated below, which we call Egoroff-Lusin's theorem. This name is borrowed from [19, Chapter II, exerc. 5 and 6]. This theorem is of fundamental importance for our theory of integration.

Egoroff-Lusin's Theorem. *Let* $\boldsymbol{\mu} : \mathfrak{S}(\mathscr{P}) \to Y$ *be a countably additive vector measure, let a sequence of measurable functions* $\{f_n\}_{n=1}^{\infty}$ *with values in* X *converge almost everywhere* $\boldsymbol{\mu}$ *to a measurable function* f *and put* $F = \bigcup_{n=0}^{\infty} \{t \in T, |f_n(t)| > 0\}$ *where* $f_0 = f$. *Then there is a set* $N \in \mathfrak{S}(\mathscr{P})$ *and a non decreasing sequence of sets* $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ *with* $\bigcup_{k=1}^{\infty} F_k = F - N$ *such that* N *is a* $\boldsymbol{\mu}$ *zero set and on each set* F_k *the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to the function* f.

Proof. Since $\boldsymbol{\mu}$ is a countably additive vector measure on the σ-ring $\mathfrak{S}(\mathscr{P})$, there is a finite non negative countably additive measure λ on $\mathfrak{S}(\mathscr{P})$ such that $\lambda(E) \leqq \|\boldsymbol{\mu}\|(E)$ and $\lim_{\lambda(E)\to 0} \|\boldsymbol{\mu}\|(E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$, see [10, IV.10.5]. Hence it is enough to prove the theorem for the measure λ. But for the measure λ the theorem follows easily from Egoroff's theorem, see [12, § 21 exerc. 3], and from the fact that for each set $E \in \mathfrak{S}(\mathscr{P})$, particularly for $E = F$, there is a non decreasing sequence of sets $G_k \in \mathscr{P}$, $k = 1, 2, \ldots$, with $\bigcup_{k=1}^{\infty} G_k = E$.

§ 2. INTEGRABLE FUNCTIONS AND THEIR PROPERTIES

Warning. Since all properties of simple integrable functions and their integrals remain valid also for general integrable functions and their integrals, in theorems we shall use the following formulation: ... (simple) integrable functions .... On one hand by this formulation we want to emphasize that the theorem has an importance by itself for simple integrable functions, while on the other hand we want to emphasize that the theorem is valid for general integrable functions. We prove them first for simple integrable functions and at the same time we point out their proofs for general integrable functions.

Observe that in the procedure of extending the integral from simple integrable functions to integrable functions we substantially use the fact that the integral from a simple integrable function as a set function is a countably additive vector measure on the σ-ring $\mathfrak{S}(\mathscr{P})$. The following theorem is an easy consequence of this fact and of Egoroff-Lusin's Theorem.

Theorem 1. *Let a sequence of (simple) integrable functions* $\{f_n\}_{n=1}^{\infty}$ *converge almost everywhere* $\boldsymbol{m}$ *to a measurable function* f. *Put* $F = \bigcup_{n=0}^{\infty} \{t \in T, |f_n(t)| > 0\}$ *where* $f_0 = f$. *Then there is a set* $N \in \mathfrak{S}(\mathscr{P})$, $N \subset F$, *and a non decreasing sequence of sets* $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ *with* $\bigcup_{k=1}^{\infty} F_k = F - N$, *so that* $\int_E f_n \cdot \chi_N \, \mathrm{d}\boldsymbol{m} = 0$ *for every* $E \in \mathfrak{S}(\mathscr{P})$ *and* $n = 1, 2, \ldots$, *and on each set* F_k, $k = 1, 2, \ldots$ *the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to the function* f. *If the measure* $\boldsymbol{m}$ *is countably additive in the uniform operator topology, then the set* N *can be chosen to be an* $\boldsymbol{m}$ *zero set.*

Proof. For $E \in \mathfrak{S}(\mathscr{P})$ put

$$\boldsymbol{\mu}(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\int_E f_n \, \mathrm{d}\boldsymbol{m}}{1 + \sup_{A \in \mathfrak{S}(\mathscr{P})} \left| \int_A f_n \, \mathrm{d}\boldsymbol{m} \right|}.$$

Then $\boldsymbol{\mu}$ is a countably additive vector measure on the σ-ring $\mathfrak{S}(\mathscr{P})$. From the construction of $\boldsymbol{\mu}$ and from the definition of a $\boldsymbol{\mu}$ zero set it is obvious that for each $\boldsymbol{\mu}$ zero set $N \in \mathfrak{S}(\mathscr{P})$, $\int_E f_n \cdot \chi_N \, \mathrm{d}\boldsymbol{m} = 0$ for every set $E \in \mathfrak{S}(\mathscr{P})$ and every $n = 1, 2, \ldots$. Now the assertion of the theorem follows directly from Egoroff-Lusin's theorem.

The assertion of the theorem for the case when the measure $\boldsymbol{m}$ is countably additive in the uniform operator topology follows from the fact that F can be written as a union of a non decreasing sequence of sets from $\mathscr{P}$ and on each such set instead of the measure $\boldsymbol{\mu}$ we use the measure $\boldsymbol{m}$ in the preceding proof.

Let a sequence of countably additive vector measures $\boldsymbol{v}_n : \mathfrak{S}(\mathscr{P}) \to Y$, $n = 1, 2, \ldots$ be uniformly countably additive on $\mathfrak{S}(\mathscr{P})$. Then from Hahn-Banach's theorem, see Lemma IV.10.4 in [10], and from the definition of the scalar semivariation we immediately obtain that their scalar semivariations $\|\boldsymbol{v}_n\|$, $n = 1, 2, \ldots$, are uniformly continuous on $\mathfrak{S}(\mathscr{P})$. We use this fact in the proof of the next theorem.

Theorem 2. *Let a sequence of (simple) integrable functions* $\{f_n\}_{n=1}^{\infty}$ *converge almost everywhere* $\boldsymbol{m}$ *to a measurable function* f *and let the integrals* $\int f_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$, *be uniformly countably additive on* $\mathfrak{S}(\mathscr{P})$. *Then there exists the limit* $\lim_{n\to\infty} \int_E f_n \, \mathrm{d}\boldsymbol{m} = \boldsymbol{v}(E) \in Y$, *uniformly with respect to* $E \in \mathfrak{S}(\mathscr{P})$. *This limit is unique for the function* f *in the sense that if a sequence of (simple) integrable functions* $\{\boldsymbol{g}_n\}_{n=1}^{\infty}$ *converges almost everywhere* $\boldsymbol{m}$ *to* f *and their integrals* $\int \boldsymbol{g}_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$ *are uniformly countably additive on* $\mathfrak{S}(\mathscr{P})$, *then again* $\lim_{n\to\infty} \int_E \boldsymbol{g}_n \, \mathrm{d}\boldsymbol{m} = \boldsymbol{v}(E)$ *for each set* $E \in \mathfrak{S}(\mathscr{P})$. *Moreover,* $\boldsymbol{v}$ *is a countably additive absolutely* $\|\boldsymbol{m}\|$ *and* $\hat{\boldsymbol{m}}$ *continuous vector measure on* $\mathfrak{S}(\mathscr{P})$. *If* $\boldsymbol{m}$ *is a Baire or a regular Borel operator valued measure, then* $\boldsymbol{v}$ *is a Baire or a regular Borel vector measure on* $\mathfrak{S}(\mathscr{B}_0)$ *or* $\mathfrak{S}(\mathscr{B})$, *respectively.*

Proof. The following proof is based on these facts: a) for each integrable function f the integral $\int f \, \mathrm{d}\boldsymbol{m}$ is a countably additive vector measure on $\mathfrak{S}(\mathscr{P})$, and b) for each integrable function f and each set $E \in \mathfrak{S}(\mathscr{P})$ the inequality $|\int_E f \, \mathrm{d}\boldsymbol{m}| \leqq \|f\|_E \, . \, \hat{\boldsymbol{m}}(E)$ is valid. Until now we have verified these facts only for simple integrable functions. However, Theorems 3 and 14 below will prove their validity for general integrable functions.

The proof itself runs as follows. Let us preserve the notation of Theorem 1. Then for each $n, p, k = 1, 2, \ldots$, and each set $E \in \mathfrak{S}(\mathscr{P})$ the following inequality holds:

$$\left|\int_E f_n \, \mathrm{d}\boldsymbol{m} - \int_E f_p \, \mathrm{d}\boldsymbol{m}\right| \leqq \|f_n - f_p\|_{F_k} \, . \, \hat{\boldsymbol{m}}(F_k) + \left\|\int f_n \, \mathrm{d}\boldsymbol{m}\right\| (F - N - F_k) + \\ + \left\|\int f_p \, \mathrm{d}\boldsymbol{m}\right\| (F - N - F_k) \, .$$

Let us have an $\varepsilon > 0$. Since the sequence F_k, $k = 1, 2, \ldots$ is non decreasing and $\bigcup_{k=1}^{\infty} F_k = F - N$, and since the scalar semivariations $\|\int f_n \, \mathrm{d}\boldsymbol{m}\| \, (.)$, $n = 1, 2, \ldots$ are uniformly continuous on $\mathfrak{S}(\mathscr{P})$, see the paragraph before this theorem, the sum of the second and third term on the right hand side is smaller then $\frac{1}{2}\varepsilon$ for sufficiently large $k = k_0$. Since on each set F_k the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, see Theorem 1, and since for each k, $\hat{\boldsymbol{m}}(F_k) < +\infty$, the existence ($Y$ is complete) and also the uniformity of the limit $\lim_{n \to \infty} \int_E f_n \, \mathrm{d}\boldsymbol{m} = \mathbf{v}(E) \in Y$ is proved.

The unicity of the limit $\mathbf{v}$ follows from the just proved existence by the fact that the sequence of (simple) integrable functions $\{f_1, g_1, f_2, g_2, \ldots, f_n, g_n, \ldots\}$ converges almost everywhere $\boldsymbol{m}$ to the function f and their integrals are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$.

The countable additivity of $\mathbf{v}$ follows from the uniform countable additivity of integrals $\int f_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$, on $\mathfrak{S}(\mathscr{P})$. Let λ be such a countably additive finite non negative measure on $\mathfrak{S}(\mathscr{P})$ that $\lambda(E) \leqq \|\mathbf{v}\| \, (E)$ and $\lim_{\lambda(E) \to 0} \|\mathbf{v}\| \, (E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$, see [10, IV.10.5]. Then the absolute $\|\boldsymbol{m}\|$ continuity of $\mathbf{v}$ is equivalent to the absolute $\|\boldsymbol{m}\|$ continuity of λ. Suppose that $\|\boldsymbol{m}\| \, (N) = 0$, $N \in \mathfrak{S}(\mathscr{P})$. Then from the definition of $\mathbf{v}$ and from the absolute $\|\boldsymbol{m}\|$ continuity of the integrals $\int f_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$, see Lemma 2 and Theorem 3 below, it is obvious that $\|\mathbf{v}\| \, (N) = 0$ and therefore also $\lambda(N) = 0$. Suppose that the measure λ is not absolutely $\|\boldsymbol{m}\|$ continuous. Then there is an $\varepsilon > 0$ and a sequence of sets $E_n \in \mathfrak{S}(\mathscr{P})$, $n = 1, 2, \ldots$ such that for each n, $\|\boldsymbol{m}\| \, (E) < 1/2^n$ and $\lambda(E_n) > \varepsilon$. For each $k = 1, 2, \ldots$ let us put $F_k = \bigcup_{n=k}^{\infty} E_n$, and $F = \bigcap_{k=1}^{\infty} F_k$. Then by the countable subadditivity and monotonocity of $\|\boldsymbol{m}\|$ it is $\|\boldsymbol{m}\| \, (F) = 0$, while by the countable additivity of the measure λ, $\lambda(F) > \varepsilon$. This is a contradiction which proves the absolute $\|\boldsymbol{m}\|$ continuity of λ and therefore also of $\mathbf{v}$.

Since $\|m\|(E) \leqq \hat{m}(E)$ for each set $E \in \mathfrak{S}(\mathscr{P})$, the absolute $\|m\|$ continuity of $\mathbf{v}$ implies its absolute $\hat{m}$ continuity.

If m is a Baire operator valued measure, then the regularity of $\mathbf{v}$ on $\mathfrak{S}(\mathscr{B}_0)$ follows from its countable additivity, see [7, Theor. 4]. Let now m be a regular Borel operator valued measure. Since the integrals $\int f_n \, \mathrm{d}m$, $n = 1, 2, \ldots$ are regular on $\mathfrak{S}(\mathscr{B})$, see Lemma 2 and Theorem 3, their uniform countable additivity implies their uniform regularity on $\mathfrak{S}(\mathscr{B})$ (the measure λ constructed in the proof of Theor. IV.9.2 in [10] is regular). In this way the regularity of $\mathbf{v}$ on $\mathfrak{S}(\mathscr{B})$ and also the theorem is proved.

Owing to this theorem the following definition of the integrable function and its integral is correct.

Definition 2. *A measurable function $f: T \to X$ is called integrable, if there exists a sequence of simple integrable functions $\{f_n\}_{n=1}^{\infty}$ converging almost everywhere m to f for which the integrals $\int f_n \, \mathrm{d}m$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$. In that case the integral of the function f on a set $E \in \mathfrak{S}(\mathscr{P})$ is defined by the equality $\int_E f \mathrm{d}m = \lim_{n \to \infty} \int_E f_n \, \mathrm{d}m$. Here, as we know from Theorem 2, this limit is uniform with respect to $E \in \mathfrak{S}(\mathscr{P})$.*

It is not hard to see that in the preceding Definition 2 we may replace the convergence almost everywhere m by the convergence everywhere without changing the class of all integrable functions (in this case f is automatically measurable).

From Definition 2 and Theorem 2 we immediately obtain the validity of the following theorem which collects the basic properties of integrable functions and their integrals, compare with Lemma 2 from section 1.1 above.

Theorem 3. *The set of all integrable functions, which will be denoted by $\mathfrak{J}$, is a linear space. For a fixed set $E \in \mathfrak{S}(\mathscr{P})$ the mapping $f \to \int_E f \mathrm{d}m$, $f \in \mathfrak{J}$ is linear, and for a fixed function $f \in \mathfrak{J}$ the set function $E \to \int_E f \mathrm{d}m$, $E \in \mathfrak{S}(\mathscr{P})$ is a countably additive absolutely $\|m\|$ and $\hat{m}$ continuous vector measure on $\mathfrak{S}(\mathscr{P})$ with values in Y. If m is a Baire or a regular Borel operator valued measure, then this measure is regular on $\mathfrak{S}(\mathscr{B}_0)$ or on $\mathfrak{S}(\mathscr{B})$ respectively. Further, for each function $f \in \mathfrak{J}$ and each set $E \in \mathfrak{S}(\mathscr{P})$, $\int_E f \mathrm{d}m = \int_E f \cdot \chi_E \, \mathrm{d}m$.*

As a corollary we immediately have the validity of Theorem 1 for general integrable functions.

Let us note that from Theorem 1 § 68 in [4] and from IV.10.5 in [10] we immediately obtain that if m is a regular Borel operator valued measure, then for each simple integrable Borel function f there is a simple integrable Baire function f_1 such that $\int_E (f - f_1) \, \mathrm{d}m = 0$ for every set $E \in \mathfrak{S}(\mathscr{P})$. From this fact, using the proof of Theorem 1 it is easy to see that to each Borel integrable function f there is Baire integrable function f_1 such that $\int_E (f - f_1) \, \mathrm{d}m = 0$ for every $E \in \mathfrak{S}(\mathscr{P})$. In the case when the measure $\boldsymbol{m}$ is countably additive in the uniform operator topology we may find even a function $\boldsymbol{f}_2$ such that $\boldsymbol{f} = \boldsymbol{f}_2$ almost everywhere $\boldsymbol{m}$, see the end of section 1.2.

The following theorem is a useful one.

Theorem 4. *Let $\boldsymbol{f}$ be an integrable function and let φ be a bounded scalar measurable function. Then $\varphi \, . \boldsymbol{f}$ is an integrable function.*

Proof. Without loss of generality we may suppose that $|\varphi(t)| \leqq 1$ for each $t \in T$. Let us choose a sequence of scalar $\mathscr{P}$-simple functions $\{\varphi_n\}_{n=1}^{\infty}$ converging on the whole T to φ, for which $\|\varphi_n\|_T \leqq 1$ for each n. According to Definition 2 choose a sequence of simple integrable functions $\{\boldsymbol{f}_n\}_{n=1}^{\infty}$ converging almost everywhere $\boldsymbol{m}$ to $\boldsymbol{f}$, for which the integrals $\dot{\int} \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$. Then $\{\varphi_n \, . \boldsymbol{f}_n\}_{n=1}^{\infty}$ is a sequence of simple integrable functions converging almost everywhere $\boldsymbol{m}$ to $\varphi \, . \boldsymbol{f}$ and from the definition of the scalar semivariation we have the inequality $\|\dot{\int} \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}\| (E) \geqq \|\dot{\int} \varphi_n \, . \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}\| (E)$ for each n and each set $E \in \mathfrak{S}(\mathscr{P})$. But the uniform countable additivity of the integrals $\dot{\int} \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$, on $\mathfrak{S}(\mathscr{P})$ implies the uniform continuity of their scalar semivariations, see the paragraph before Theorem 2. Thus the integrals $\dot{\int} \varphi_n \, . \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$, which proves the integrability of $\varphi \, . \boldsymbol{f}$.

Let us remind the $*$-Theorem from section 1.1 and proceed to the following important theorem.

Theorem 5. *Let the semivariation $\hat{\boldsymbol{m}}$ be continuous on $\mathscr{P}_1$, let $A \in \mathscr{P}_1$ and let $\boldsymbol{f}$ be a bounded measurable function. Then the function $\boldsymbol{f} \, . \, \chi_A$ is integrable.*

Proof. Let us have a sequence $\{\boldsymbol{f}_n\}_{n=1}^{\infty}$ of simple integrable functions such that $\{\boldsymbol{f}_n \, . \, \chi_A\}_{n=1}^{\infty}$ converges on the whole T to $\boldsymbol{f} \, . \, \chi_A$ and $\|\boldsymbol{f}_n\|_A \leqq \|\boldsymbol{f}\|_A$ for each n. Then the uniform countable additivity of the integrals $\dot{\int} \boldsymbol{f}_n \, . \, \chi_A \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$ on $\mathfrak{S}(\mathscr{P})$ follows from the inequality $|\dot{\int}_E \boldsymbol{f}_n \, . \, \chi_A \, \mathrm{d}\boldsymbol{m}| \leqq \|\boldsymbol{f}\|_A \, . \, \hat{\boldsymbol{m}}(A \cap E)$, $n = 1, 2, \ldots, E \in \mathfrak{S}(\mathscr{P})$ by the continuity of the semivariation $\hat{\boldsymbol{m}}$ on $\mathscr{P}_1$, since $A \in \mathscr{P}_1$. This proves the theorem.

As the following simple example demonstrates, without the assumption of continuity of the semivariation $\hat{\boldsymbol{m}}$ on $\mathscr{P}_1$ the preceding theorem is not valid even when the measure $\boldsymbol{m}$ is countably additive in the uniform operator topology.

Example 7′. Let us consider Example 7 from section 1.1 and define the function $\boldsymbol{f}$ as follows:

$$\boldsymbol{f}(1) = [1, 0, 0, \ldots\ldots\ldots] \in \boldsymbol{l}_1,$$
$$\boldsymbol{f}(2) = \boldsymbol{f}(3) = [0, 1, 0, 0, \ldots\ldots\ldots] \in \boldsymbol{l}_1,$$
$$\boldsymbol{f}(4) = \boldsymbol{f}(5) = \boldsymbol{f}(6) = [0, 0, 1, 0, 0, \ldots\ldots] \in \boldsymbol{l}_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then f is a measurable function and $\|f\|_T = 1$. Clearly for each finite set $E = \{k_1, k_2, \ldots, k_n\}, f \cdot \chi_E$ is a simple integrable function. Further, from the definition of the measure $\boldsymbol{m}$ we have the following equalities

$$\int_{\{1\}} f \, \mathrm{d}\boldsymbol{m} = [1, 0, 0, \ldots\ldots\ldots] \in \boldsymbol{c}_0 ,$$

$$\int_{\{2,3\}} f \, \mathrm{d}\boldsymbol{m} = [0, 1, 0, 0, \ldots\ldots\ldots] \in \boldsymbol{c}_0 .$$

$$\int_{\{4,5,6\}} f \, \mathrm{d}\boldsymbol{m} = [0, 0, 1, 0, 0, \ldots\ldots] \in \boldsymbol{c}_0 ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since the series formed by summing the terms of this sequence is not converging in the space $\boldsymbol{c}_0$, the function f cannot be integrable since the integral of an integrable function is a countably additive vector measure on $\mathfrak{S}(\mathscr{P})$, see Theorem 3.

Let us remind that the variation of the measure $\boldsymbol{m}$, which we denote by $v(\boldsymbol{m}, .)$, is the non negative, not necessarily finite, countably additive measure on $\mathfrak{S}(\mathscr{P})$ defined by the equality

$$v(\boldsymbol{m}, E) = \sup \sum_{i=1}^{r} |\boldsymbol{m}(E \cap E_i)| , \quad E \in \mathfrak{S}(\mathscr{P})$$

where the supremum is taken over all finite collections of disjoint $E_i \in \mathscr{P}$. Clearly for each set $E \in \mathfrak{S}(\mathscr{P})$ the inequality $\hat{\boldsymbol{m}}(E) \leqq v(\boldsymbol{m}, E)$ holds. It is not hard to find examples where $\hat{\boldsymbol{m}}(E) < +\infty$ and $v(\boldsymbol{m}, E) = +\infty$. Similarly as Theorem 5 we can prove:

Theorem 6. *Let f be a measurable function and let $\int_F |f| \, \mathrm{d}v(\boldsymbol{m}, .) < +\infty$ where $F = \{t \in T, |f(t)| > 0\}$. Then f is an integrable function and for each set $E \in \mathfrak{S}(\mathscr{P})$ the inequality $v(\int f \, \mathrm{d}\boldsymbol{m}, E) \leqq \int_E |f| \, \mathrm{d}v(\boldsymbol{m}, E)$ holds.*

Let us note that there are functions f with $v(\int f \, \mathrm{d}\boldsymbol{m}, E) = 0$ for each set $E \in \mathfrak{S}(\mathscr{P})$, and at the same time it may happen that $\int_T |f| \, \mathrm{d}v(\boldsymbol{m}, .) = +\infty$.

Let $\{\mathbf{v}_n\}_{n=1}^{\infty}$ be a sequence of countably additive vector measures on $\mathfrak{S}(\mathscr{P})$ with values in $\boldsymbol{Y}$ and let the limit $\lim_{n \to \infty} \mathbf{v}_n(E) = \mathbf{v}(E) \in \boldsymbol{Y}$ for each set $E \in \mathfrak{S}(\mathscr{P})$ exist. Then the scalar semivariations $\|\mathbf{v}_n\| (.)$, $n = 1, 2, \ldots$ are uniformly continuous on $\mathfrak{S}(\mathscr{P})$, see the proof of Theorem IV.10.6 in [10]. From this fact and from Definition 2 we immediately obtain the following criterium of integrability of a function.

Theorem 7. *A measurable function* $f: T \to X$ *is integrable if and only if there exists a sequence of simple integrable functions* $\{f_n\}_{n=1}^{\infty}$ *converging almost everywhere* m *to* f, *such that for each set* $E \in \mathfrak{S}(\mathscr{P})$ *the limit* $\lim_{n\to\infty} \int_E f_n \, dm = v(E) \in Y$ *exists. In this case* $\int_E f \, dm = \lim_{n\to\infty} \int_E f_n \, dm$ *for each set* $E \in \mathfrak{S}(\mathscr{P})$ *and this limit is uniform with respect to* $E \in \mathfrak{S}(\mathscr{P})$.

Let now Z be a Banach space and let U be a bounded linear operator from Y to Z. Then clearly the set function Um, $Um(E) = U[m(E)]$, $E \in \mathscr{P}$ is an operator valued measure countably additive in the strong operator topology of the space $L(X, Z)$. If m is countably additive in the uniform operator topology, then Um has the same property. Since $\widehat{Um}(E) \leqq |U| . \hat{m}(E)$ for each set $E \in \mathfrak{S}(\mathscr{P})$, each simple integrable function f is a simple integrable function also with respect to the measure Um and $U \int_E f \, dm = \int_E f \, dUm$ for each set $E \in \mathfrak{S}(\mathscr{P})$. From this, using Theorem 7 we immediately obtain that each integrable function f is also integrable with respect to the measure Um and $U \int_E f \, dm = \int_E f \, dUm$ for each set $E \in \mathfrak{S}(\mathscr{P})$.

As we know, $\mathfrak{I}_s$ denotes the set of all simple integrable functions. Denote by $\overline{\mathfrak{I}}_s$ the closure of $\mathfrak{I}_s$ in the norm $\|.\|_T$ in the space of all bounded X valued functions on T. Let us remind further that for a locally compact Hausdorff topological space T, $C_0(T, X)$ denotes the Banach space of continuous X valued functions on T approaching zero at infinity with the usual supremum norm. In other words, a function f is in $C_0(T, X)$ if and only if f is continuous and for each $\varepsilon > 0$ there is a compact set C, by [12, § 50 Theor. D] a compact G_δ set C_1 such that $\|f\|_{T-C_1} < \varepsilon$. Using this notation we have:

Theorem 8. *A function* f *is in* $\overline{\mathfrak{I}}_s$ *if and only if the following conditions are fulfilled:* a) f *is measurable,* b) $\{f(t), t \in T\}$ *is a relatively compact subset of* X, *and* c) *for each* $\varepsilon > 0$ *there is an* $A \in \mathscr{P}$ *such that* $\|f\|_{T-A} < \varepsilon$. *Particularly, if* m *is a Baire or a Borel operator valued measure, then* $C_0(T, X) \subset \overline{\mathfrak{I}}_s$.

Proof. Let conditions a), b) and c) be fulfilled for a function f. For each $n = 1, 2, \ldots$ take $A_n \in \mathscr{P}$ such that $\|f\|_{T-A_n} < 1/n$, and let $x_1, x_2, \ldots, x_{k_n}$ be a $1/n$ – net for the relatively compact set $\{f(t), t \in A_n\}$. For $k = 1, 2, \ldots, k_n$ put $B_k = \{t \in A_n, |f(t) - x_k| < 1/n\}$. Then by the measurability of f $B_k \in \mathscr{P}$ for each k. If we now put $B'_k = B_k - \bigcup_{i=1}^{k-1} B_i$, $k = 1, 2, \ldots, k_n$, and $f_n = \sum_{k=1}^{k_n} x_k \cdot \chi_{B'_k}$, then for each n it is $f_n \in \mathfrak{I}_s$ and $\|f - f_n\|_T < 1/n$. Hence $f \in \overline{\mathfrak{I}}_s$. The other assertions of the theorem are obvious.

Denote by $B\mathfrak{I}$ the set of all bounded integrable functions and let $\overline{B\mathfrak{I}}$ be its closure in the norm $\|.\|_T$ in the space of all bounded X valued functions on T. Then we have:

Theorem 9. *Let $A \in \mathscr{P}_1$ and let $f \in \overline{\mathfrak{I}}_s$, or more generally, let $f \in \overline{B\mathfrak{I}}$. Then $f \cdot \chi_A \in B\mathfrak{I}$. Particularly, if $\sup_{A \in \mathscr{P}_1} \hat{m}(A) < +\infty$, then $\overline{\mathfrak{I}}_s \subset B\mathfrak{I}$ and $\overline{B\mathfrak{I}} = B\mathfrak{I}$.*

Proof. If $f \in \overline{\mathfrak{I}}_s$, then the integrability of $f \cdot \chi_A$ follows from the inequality $|\int_E f \,\mathrm{d}\boldsymbol{m}| \leqq \|f\|_E \cdot \hat{m}(E)$ by Theorem 7. If $f \in \overline{B\mathfrak{I}}$, then the integrability of $f \cdot \chi_A$ follows from the same inequality, which is proved for general integrable functions in Theorem 14 below, by Theorem 16 below. The other assertions of the theorem are now obvious.

In the following simple example we construct such a bounded integrable function f that $f \notin \overline{\mathfrak{I}}_s$. At the same time $T \in \mathscr{P}$ and $v(\boldsymbol{m}, T) < +\infty$. Thus in general the contents of the preceding theorem cannot be reduced to its contents for simple integrable functions.

Example. Let T be the set of natural numbers, let $\mathscr{P}_0$ be the set of all subsets of T and let $X = Y = l_1$. Define $\boldsymbol{m}(\{k\})\,x = 1/k^2 \cdot x$, $x \in X$, $k = 1, 2, \ldots$, and $\boldsymbol{m}(E)\,x = \sum_{k \in E} \boldsymbol{m}(\{k\})\,x$ for $E \in \mathscr{P}_0$. Then $\mathscr{P}_0 = \mathscr{P}$ and the function f defined by $f(1) = [1, 0, 0, \ldots]$, $f(2) = [0, 1, 0, 0, \ldots]$, $f(3) = [0, 0, 1, 0, 0, \ldots]$, ... is a bounded integrable function, and clearly $f \notin \overline{\mathfrak{I}}_s$.

Let us note that if $\sup_{A \in \mathscr{P}_1} \hat{m}(A) < +\infty$, then for the extension of the integral from $\mathfrak{I}_s$ to $\overline{\mathfrak{I}}_s$ the elementary theory based on the inequality $|\int_E f \,\mathrm{d}\boldsymbol{m}| \leqq \|f\|_E \cdot \hat{m}(E)$, $f \in \mathfrak{I}_s$, i.e , on the definition of the semivariation $\hat{m}$ is completely sufficient. At the same time this elementary theory, due to the inclusion $C_0(T, X) \subset \overline{\mathfrak{I}}_s$ for a Baire measure $\boldsymbol{m}$, is sufficient to represent a wide class of bounded linear operators $U : C_0(T, X) \to Y$ in the form of an integral $Uf = \int_T f \,\mathrm{d}\boldsymbol{m}$, see § 2 in [9]. But for the investigation of the properties of such operators the general theory, which we are developing, is necessary.

From definition of the measurable function and from Theorems 1 and 8 we immediately have:

Theorem 10. *Let f be a measurable function and put $F = \{t \in T, |f(t)| > 0\}$. Then there is a set $N \in \mathfrak{S}(\mathscr{P})$, $N \subset F$ and a non decreasing sequence of sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ with $\bigcup_{k=1}^{\infty} F_k = F - N$, so that $\int_E f \cdot \chi_N \,\mathrm{d}\boldsymbol{m} = 0$ for every $E \in \mathfrak{S}(\mathscr{P})$ and for each k, $f \cdot \chi_{F_k} \in \overline{\mathfrak{I}}_s$. In other words, each measurable function f can be written in the form $f = f \cdot \chi_{T-N} + f \cdot \chi_N$ where $f \cdot \chi_{T-N}$ has a relatively σ-compact range in X and $\int_E f \cdot \chi_N \,\mathrm{d}\boldsymbol{m} = 0$ for every $E \in \mathfrak{S}(\mathscr{P})$. When the measure $\boldsymbol{m}$ is countably additive in the uniform operator topology, then the set N can be taken to be an $\boldsymbol{m}$ zero set.*

In connection with the following theorem see also Example 7″ below and Theorem 3 in [1].

Theorem 11. *Let $A \in \mathscr{P}_1$ and let f be a bounded measurable function. If there exists a sequence of (simple) integrable functions $\{f_n\}_{n=1}^{\infty}$ converging in the semivariation $\hat{m}$ to the function $f \cdot \chi_A$, then $f \cdot \chi_A$ is an integrable function.*

Proof. For each $k = 1, 2, \ldots$ take an n_k so that $\hat{m}(\{t \in A, |f_{n_k}(t) - f(t)| > 1/2^k\}) < 1/2^k$, and denote the set in brackets by A_k. Let us put $B_k = \bigcup_{i=k}^{\infty} A_i$, $k = 1, 2, \ldots$, and let $B = \bigcap_{k=1}^{\infty} B_k$. Then by the countable subadditivity of the semivariation $\hat{m}$ B is an m zero set, and therefore $\int_E f \cdot \chi_B \, \mathrm{d}m = 0$ for every set $E \in \mathfrak{S}(\mathscr{P})$.

For each $k = 1, 2, \ldots$ let us put $f_k' = f_{n_k} \cdot \chi_{A-B_k}$. Then $\{f_k'\}_{k=1}^{\infty}$ is a sequence of (simple) integrable functions converging at each point $t \in T$ to the function $f \cdot \chi_{A-B}$ and at the same time $\|f_k'\|_T \leqq \|f\|_T + 1$ for each $k = 1, 2, \ldots$. Further, (for general integrable functions by Theorem 14 below), for each set $E \in \mathfrak{S}(\mathscr{P})$ and each natural numbers $k_1 \leqq k_2$ the following inequalities hold:

$$\left|\int_E f_{k_1}' \, \mathrm{d}m - \int_E f_{k_2}' \, \mathrm{d}m\right| \leqq \left|\int_{E \cap (A - B_{k_1})} (f_{k_1}' - f_{k_2}') \, \mathrm{d}m\right| +$$

$$+ \left|\int_{E \cap B_{k_1}} f_{k_1}' \, \mathrm{d}m\right| + \left|\int_{E \cap B_{k_1}} f_{k_2}' \, \mathrm{d}m\right| \leqq \frac{1}{2^{k_1 - 2}} \cdot (\hat{m}(A) + \|f\|_T + 1).$$

From these inequalities, since $\hat{m}(A) + \|f\|_T$ is by the assumption of the theorem finite and since Y is complete, the existence of the limit $\lim_{k \to \infty} \int_E f_k' \, \mathrm{d}m = \mathbf{v}(E) \in Y$ follows for each set $E \in \mathfrak{S}(\mathscr{P})$. From here the integrability of $f \cdot \chi_{A-B}$ and consequently of $f \cdot \chi_A$ follows in the case of simple integrable f_n, $n = 1, 2, \ldots$ from Theorem 7, while in the general case from Theorem 16 below.

We say that a measurable function f has an $\hat{m}$ almost relatively compact range, if for each $\varepsilon > 0$ there is a set $A \in \mathfrak{S}(\mathscr{P})$ such that $\hat{m}(A) < \varepsilon$ and the function $f \cdot \chi_{T-A}$ has a relatively compact range in X. If m is a Baire or a Borel operator valued measure, then for the measurable function the concepts of the $\hat{m}$ almost continuity and of the $\hat{m}$ almost compact support are defined in a similar way. With these concepts, as a corollary of Theorems 8, 9 and 11 we immediately have.

Theorem 12. *Let $A \in \mathscr{P}$ and let f be a bounded measurable function. If the function $f \cdot \chi_A$ has $\hat{m}$ almost relatively compact range, then it is integrable. Particularly, if m is a Baire or a Borel operator valued measure and the function $f \cdot \chi_A$ is $\hat{m}$ almost continuous and has an $\hat{m}$ almost compact support then it is integrable.*

Let $A \in \mathscr{P}_1$, let the semivariation $\hat{m}$ be continuous on $\mathscr{P}_1$, see the $*$-Theorem in section 1.1 and let f be a bounded measurable function. Then owing to Egoroff's theorem for the semivariation $\hat{m}$ there is a sequence of simple integrable functions converging in the semivariation $\hat{m}$ to the function $f \,.\, \chi_A$. In connection with this fact, with Theorem 11 and also with the integral of R. G. Bartle in [1], see in detail the end of the paper, the following example is interesting.

Example 7″. In this example we construct a bounded integrable function for which there is no sequence of simple integrable functions converging to it in semivariation $\hat{m}$. At the same time the measure m is countably additive in the uniform operator topology and $T \in \mathscr{P}$.

Let us modify the Example 7 from section 1.1 as follows: T, $\mathscr{P}_0$, X, Y and the sequence $\{y_k\}_{k=1}^{\infty}$ remain as before. On the other hand we define a new measure m: for $x \in X = l_1$, $x = [x_1, x_2, \ldots, x_n, \ldots]$ we put

$$1\left\{ m(\{1\})\,x = [x_1\,, \quad 0 \quad , \quad 0 \quad , \ldots]\,, \right.$$

$$2\begin{cases} m(\{2\})\,x = \left[0\,, \dfrac{1}{2}\,x_2 + \left(\dfrac{1}{2^3} - \dfrac{1}{2}\right)x_3\,, \quad 0 \quad , \ldots\right], \\ m(\{3\})\,x = \left[0\,, \dfrac{1}{2}\,x_4 + \left(\dfrac{1}{2^3} - \dfrac{1}{2}\right)x_5\,, \quad 0 \quad , \ldots\right], \end{cases}$$

$$3\begin{cases} m(\{4\})\,x = \left[0\,, \quad 0 \quad , \dfrac{1}{3}\,x_6 + \left(\dfrac{1}{3^3} - \dfrac{1}{3}\right)x_7\,, \ldots\right], \\ m(\{5\})\,x = \left[0\,, \quad 0 \quad , \dfrac{1}{3}\,x_8 + \left(\dfrac{1}{3^3} - \dfrac{1}{3}\right)x_9\,, \ldots\right], \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

It is again easy to see that for a non empty set $A \in \mathfrak{S}(\mathscr{P}_0) = \mathscr{P}_0$, $|m(A)| = \sup_{k \in A} |y_k|$ and $\hat{m}(A) = |\sum_{k \in A} y_k|$ just as in Example 7 in section 1.1. Thus the measure m is countably additive in the uniform operator topology on $\mathscr{P} = \mathscr{P}_0$ and its semivariation $\hat{m}$ is not continuous on $\mathscr{P}_1 = \mathscr{P}$.

Define now the function f as follows:

$$f(1) = [1, 0, 0, \ldots\ldots\ldots\ldots\ldots\ldots]\,,$$

$$f(2) = [0, 1, 1, 0, 0, \ldots\ldots\ldots\ldots\ldots]\,,$$

$$f(3) = [0, 0, 0, 1, 1, 0, 0, \ldots\ldots\ldots]\,,$$

$$f(4) = [0, 0, 0, 0, 0, 1, 1, 0, 0, \ldots\ldots]\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Then f is a bounded measurable function and the following equalities holds:

$$1\left\{\int_{\{1\}} f\,\mathrm{d}\boldsymbol{m} = [1, 0, 0, \ldots\ldots\ldots],\right.$$

$$2\left\{\begin{aligned} \int_{\{2\}} f\,\mathrm{d}\boldsymbol{m} &= \left[0, \frac{1}{2^3}, 0, 0, \ldots\right], \\ \int_{\{3\}} f\,\mathrm{d}\boldsymbol{m} &= \left[0, \frac{1}{2^3}, 0, 0, \ldots\right], \end{aligned}\right.$$

$$3\left\{\begin{aligned} \int_{\{4\}} f\,\mathrm{d}\boldsymbol{m} &= \left[0, 0, \frac{1}{3^3}, 0, 0, \ldots\right], \\ \int_{\{5\}} f\,\mathrm{d}\boldsymbol{m} &= \left[0, 0, \frac{1}{3^3}, 0, 0, \ldots\right], \end{aligned}\right.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since the series formed by summing the terms of this sequence is absolutely convergent in $\boldsymbol{Y} = \boldsymbol{c}_0$, f is a bounded integrable function. Let us demonstrate now that there exists no sequence of simple integrable functions converging to the function f in the semivariation $\hat{\boldsymbol{m}}$.

Let $\boldsymbol{g} = \sum_{i=1}^{r} \boldsymbol{x}_i \,.\, \chi_{E_i}$ be a simple integrable function, put $A = \{t \in T, |f(t) - g(t)| > \frac{1}{3}\}$ and suppose that $\hat{\boldsymbol{m}}(A) < \frac{1}{3}$. Then from definitions of A and f it is clear that $\bigcup_{i=1}^{r} E_i \supset T - A$. If some $E_i - A$ contained two distinct points t_1 and t_2, then we should have $1 = |f(t_1) - f(t_2)| \leqq |f(t_1) - \boldsymbol{x}_i| + |f(t_2) - \boldsymbol{x}_i| \leqq \frac{2}{3}$, which is impossible. Hence $T - A$ is a subset of some set $\{t_1, t_2, \ldots, t_s\}$, $s \leqq r$. Let us put $t_0 = \max t_i$, $i = 1, 2, \ldots, s$. Then $A \supset \{t_0 + 1, t_0 + 2, \ldots\}$, and therefore $\hat{\boldsymbol{m}}(A) = 1$ which is a contradiction with the assumption $\hat{\boldsymbol{m}}(A) < \frac{1}{3}$. Thus we proved that there exists no sequence of simple integrable functions converging in the semivariation $\hat{\boldsymbol{m}}$ to the function f.

Let us note that in a similar way, starting from Example 6 in section 1.1, the measure $\boldsymbol{m}$ being countably additive only in the strong operator topology, it is possible to construct a bounded integrable function that there exists no sequence of simple integrable functions converging in measure to it. This fact shows that if we had extended the integral by means of the Theorem 13 below, then in general, when the measure $\boldsymbol{m}$ is not countably additive in the uniform operator topology, we should not have obtained all integrable functions.

Theorem 13. *Let f be a measurable function and let a sequence of (simple) integrable functions $\{f_n\}_{n=1}^{\infty}$ converge on each set $E \in \mathfrak{S}(\mathscr{P})$ in measure $\boldsymbol{m}$ to the function f. Then the following conditions are equivalent:* a) *The integrals $\int f_n\,\mathrm{d}\boldsymbol{m}$,*

$n = 1, 2, \ldots$ *are uniformly countably additive on* $\mathfrak{S}(\mathscr{P})$. b) *For each set* $E \in \mathfrak{S}(\mathscr{P})$ *there exists the limit* $\lim_{n\to\infty} \int_E f_n \, \mathrm{d}\boldsymbol{m} = \mathbf{v}(E) \in Y$. *If these conditions are fulfilled, then the function* f *is integrable,* $\int_E f \mathrm{d}\boldsymbol{m} = \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\boldsymbol{m}$ *for each set* $E \in \mathfrak{S}(\mathscr{P})$ *and this limit is uniform with respect to* $E \in \mathfrak{S}(\mathscr{P})$.

Proof. From each sequence of measurable functions converging in measure $\boldsymbol{m}$ on each set $E \in \mathscr{P}$ to the measurable function f we may select a subsequence converging almost everywhere $\boldsymbol{m}$ on the whole T to the function f, see section 1.3. From this fact, by indirect proof we easily obtain the validity of the theorem, following in the case of simple integrable functions from Definition 2 and Theorem 7, while in the general case from Theorems 15 and 16 stated below.

By the definition of the semivariation $\hat{\boldsymbol{m}}$, for each set $E \in \mathfrak{S}(\mathscr{P})$, $\hat{\boldsymbol{m}}(E) = \sup \{ |\int_E f \mathrm{d}\boldsymbol{m}|, f \in \mathfrak{I}_s, \|f\|_E \leqq 1 \}$, see Lemma 1 in section 1.1. The following important theorem asserts that this supremum does not increase when we replace simple integrable functions by general integrable functions.

Theorem 14. *For each set* $E \in \mathfrak{S}(\mathscr{P})$, $\hat{\boldsymbol{m}}(E) = \sup \{ |\int_E f \mathrm{d}\boldsymbol{m}|, f \in \mathfrak{I}, \|f\|_E \leqq 1 \}$. *Hence for each integrable function* f *and each set* $E \in \mathfrak{S}(\mathscr{P})$ *the inequality*

$$\left| \int_E f \mathrm{d}\boldsymbol{m} \right| \leqq \|f\|_E \cdot \hat{\boldsymbol{m}}(E)$$

holds.

Proof. Let us have an $E \in \mathfrak{S}(\mathscr{P})$ and let f be an integrable function with $\|f\|_E \leqq 1$. Since the function f is measurable, there is a sequence of simple integrable functions $\{f_n\}_{n=1}^{\infty}$ converging on the whole T to f and such that for each $t \in T$, $|f_n(t)| \leqq |f(t)|$ for each $n = 1, 2, \ldots$, see section 1.2. Let us use the notation of Theorem 1. Then for each $n, k = 1, 2, \ldots$,

$$\left| \int_E f \mathrm{d}\boldsymbol{m} \right| \leqq \left| \int_{E \cap (F - N - F_k)} f \mathrm{d}\boldsymbol{m} \right| + \left| \int_{E \cap F_k} (f - f_n) \, \mathrm{d}\boldsymbol{m} \right| + \left| \int_{E \cap F_k} f_n \mathrm{d}\boldsymbol{m} \right| .$$

Let us have an $\varepsilon > 0$. Since the integral $\int f \mathrm{d}\boldsymbol{m}$ is a countably additive set function on $\mathfrak{S}(\mathscr{P})$, for sufficiently large $k = k_0$ the first term on the right hand side is smaller then $\frac{1}{2}\varepsilon$. Since $\lim_{n\to\infty} \|f - f_n\|_{F_k} = 0$, by Theorem 7 the second term with $F_k = F_{k_0}$ is for sufficiently large $n = n_0$ smaller then $\frac{1}{2}\varepsilon$ as well. Since f_{n_0} is a simple integrable function with $\|f_{n_0}\|_E \leqq 1$, this proves the theorem.

As a consequence of this theorem and Theorem 3 we have the validity of Theorem 2 for general integrable functions f_n, $n = 1, 2, \ldots$. The following theorem shows that we obtain no further extension of the integral when we apply again the extension procedure to general integrable functions.

Theorem 15. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions converging almost everywhere $\boldsymbol{m}$ to a measurable function f and let the integrals $\int f_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$, be uniformly countably additive on $\mathfrak{S}(\mathscr{P})$. Then the function f is integrable, $\int_E f \, \mathrm{d}\boldsymbol{m} = \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\boldsymbol{m}$ for each set $E \in \mathfrak{S}(\mathscr{P})$ and this limit is uniform with respect to $E \in \mathfrak{S}(\mathscr{P})$.*

Proof. By Theorem 2 for each set $E \in \mathfrak{S}(\mathscr{P})$ there exists the limit $\lim_{n\to\infty} \int_E f_n \, \mathrm{d}\boldsymbol{m} = \boldsymbol{\nu}(E) \in Y$, and this limit is uniform with respect to $E \in \mathfrak{S}(\mathscr{P})$. Hence it is enough to prove that the function f is integrable and that $\boldsymbol{\nu}(E) = \int_E f \, \mathrm{d}\boldsymbol{m}$ for each set $E \in \mathfrak{S}(\mathscr{P})$.

Since f is a measurable function, by definition there is a sequence of simple integrable functions $\{\boldsymbol{h}_n\}_{n=1}^{\infty}$ converging on the whole T to the function f. Let us consider the sequence $\{f_1, \boldsymbol{h}_1, f_2, \boldsymbol{h}_2, \ldots, f_n, \boldsymbol{h}_n, \ldots\}$, put $F = \bigcup_{n=1}^{\infty} \{t \in T, |f_n(t)| + |\boldsymbol{h}_n(t)| > 0\}$ and apply Theorem 1. Then there is a set $N \in \mathfrak{S}(\mathscr{P})$, $N \subset F$ and a non decreasing sequence of sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ with $\bigcup_{k=1}^{\infty} F_k = F - N$, so that $\int_E f_n \cdot \chi_N \, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{h}_n \cdot \chi_N \, \mathrm{d}\boldsymbol{m} = 0$ for every $E \in \mathfrak{S}(\mathscr{P})$ and $n = 1, 2, \ldots$, and on each set F_k, $k = 1, 2, \ldots$ the sequence $\{f_1, \boldsymbol{h}_1, f_2, \boldsymbol{h}_2, \ldots, f_n, \boldsymbol{h}_n, \ldots\}$ converges uniformly to the function f. Let us choose a subsequence $\{n_k\}_{k=1}^{\infty}$ such that for each k it is $\|\boldsymbol{h}_{n_k} - f_{n_k}\|_{F_k} \cdot \hat{\boldsymbol{m}}(F_k) < 1/k$ and put $\boldsymbol{g}_k = \boldsymbol{h}_{n_k} \cdot \chi_N + \boldsymbol{h}_{n_k} \cdot \chi_{F_k}$ for each k. Then $\{\boldsymbol{g}_k\}_{k=1}^{\infty}$ is a sequence of simple integrable functions converging on the whole T to the function f and by Theorem 14 for each k and each set $E \in \mathfrak{S}(\mathscr{P})$

$$\left|\boldsymbol{\nu}(E) - \int_E \boldsymbol{g}_k \, \mathrm{d}\boldsymbol{m}\right| \leqq \left|\int_{E \cap F_k} (\boldsymbol{g}_k - f_{n_k}) \, \mathrm{d}\boldsymbol{m}\right| +$$

$$+ \left|\int_{E \cap (F - N - F_k)} f_{n_k} \, \mathrm{d}\boldsymbol{m}\right| + \left|\boldsymbol{\nu}(E) - \int_E f_{n_k} \, \mathrm{d}\boldsymbol{m}\right| \leqq 1/k + \ldots .$$

From here it is clear that $\lim_{k\to\infty} \int_E \boldsymbol{g}_k \, \mathrm{d}\boldsymbol{m} = \boldsymbol{\nu}(E)$ for each set $E \in \mathfrak{S}(\mathscr{P})$. Thus by Theorem 7 the function f is integrable and $\int_E f \, \mathrm{d}\boldsymbol{m} = \boldsymbol{\nu}(E)$ for each $E \in \mathfrak{S}(\mathscr{P})$. This proves the theorem.

From the definition of integrable functions and from the preceding theorem we immediately have the following characterization of the set of all integrable functions: *The set of all integrable functions is the smallest set of functions containing the set of all simple integrable functions for which Theorem 15 is valid.*

From Theorem 15 and from the theorem of Vitali-Hahn-Saks, see [10, IV.10.6], we obtain the following fundamental theorem:

Theorem 16. Theorem on interchange of limit and integral. *Let a sequence of integrable functions $\{f_n\}_{n=1}^{\infty}$ converge almost everywhere $\boldsymbol{m}$ to a measurable function f*

and let there exist the limit $\lim_{n\to\infty} \int_E f_n \,\mathrm{d}\boldsymbol{m} = \nu(E) \in Y$ *for each set* $E \in \mathfrak{S}(\mathscr{P})$. *Then the function* f *is integrable,* $\int_E f \,\mathrm{d}\boldsymbol{m} = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\boldsymbol{m}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$ *and this limit is uniform with respect to* $E \in \mathfrak{S}(\mathscr{P})$.

As a consequence of Theorems 14, 15 and 16 we obtain the validity of Theorems 9, 11 and 13 for general integrable functions.

From Theorems 7 and 16 we have the following characterization of the set of all integrable functions: *The set of all integrable functions is the smallest set of functions containing the set of all simple integrable functions for which Theorem* 16 *on interchange of limit and integral is valid.*

We say that a measurable function $f: T \to X$ is weakly integrable iff for each functional $y^* \in Y^*$ the function f is integrable with respect to the measure $y^*\boldsymbol{m} : \mathscr{P} \to X^*$. From the paragraph following Theorem 7 we immediately have that every integrable function is also weakly integrable. The following simple example shows that the converse is in general not true.

Example. Let T be the set of positive integers, $\mathscr{P}_0$ the δ-ring of all finite subsets of T and X the real Banach space c_0. Let further $\boldsymbol{m}(\{k\})\, x = x$ and $f(k) = \overbrace{[0, 0, \ldots, 0, 1}^{k}, 0, \ldots] \in c_0$ for each $k = 1, 2, \ldots$ and each $x \in c_0$. Since the dual of c_0 is the Banach space l_1, the function f is clearly measurable and weakly integrable, but it is not integrable.

Nevertheless, from the generalization of Orlicz's theorem, see Theorem 5 in [5], we have the following interesting result.

Theorem 17. *Let* Y *contain no subspace isomorphic to the space* c_0, *for example let* Y *be a weakly complete Banach space, see pp.* 160 *and* 161 *in* [5]. *Then every measurable and weakly integrable function* $f: T \to X$ *is integrable.*

Proof. According to Theorem 10 there is a set $N \in \mathfrak{S}(\mathscr{P})$, $N \subset F = \{t \in T, |f(t)| > 0\}$ and an increasing sequence of sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$, $\bigcup_{k=1}^{\infty} F_k = F - N$ such that $\int_{E\cap N} f \,\mathrm{d}\boldsymbol{m} = 0$ for each set $E \in \mathfrak{S}(\mathscr{P})$, and $f \cdot \chi_{F_k} \in \mathfrak{I}_s$ for each k. Since $\hat{\boldsymbol{m}}(F_k) < +\infty$ for each k, by Theorem 9 the function $f \cdot \chi_{F_k}$ is integrable for each k. We now prove that for each set $E \in \mathfrak{S}(\mathscr{P})$ the limit $\lim_{k\to\infty} \int_E f \cdot \chi_{F_k} \,\mathrm{d}\boldsymbol{m}$ exists, which owing to Theorem 16 will prove the integrability of the function f.

Suppose that for some set $E \in \mathfrak{S}(\mathscr{P})$ the limit $\lim_{k\to\infty} \int_E f \cdot \chi_{F_k} \,\mathrm{d}\boldsymbol{m}$ does not exists. Then there is an $\varepsilon > 0$ and a sequence of positive integers $k_1 < k_2 < k_3 < k_4 < \ldots$ such that

$$\left| \int_{E\cap(F_{k_{2i}} - F_{k_{2i-1}})} f \,\mathrm{d}\boldsymbol{m} \right| = \left| \int_E f \cdot \chi_{F_{k_{2i-1}}} \mathrm{d}\boldsymbol{m} - \int_E f \cdot \chi_{F_{k_{2i}}} \mathrm{d}\boldsymbol{m} \right| > \varepsilon$$

for every $i = 1, 2, \ldots$. But this is impossible, since owing to the weak integrability of the function f the series $\sum_{i=1}^{\infty} \int_{E \cap (F_{k_{2i}} - F_{k_{2i-1}})} f \, \mathrm{d}\boldsymbol{m}$ is weakly unconditionally convergent, and therefore, since Y contains no subspace isomorphic to the space c_0, this series is (strongly) unconditionally convergent, see Theorem 5 in [5]. Thus the theorem is proved.

§ 3. SOME SPECIAL CASES

In this section we compare the obtained integral with a few well known integrals. The examples treated below are continuations of the corresponding examples from section 1.1.

1. Let us show that the obtained integral in this case coincides with the classical Lebesgue integral on a general measure space, see for example [12].

By Theorem 6 each Lebesgue integrable function is integrable in our sense. Conversely, let f be an integrable function and according to Def. 2 let $\{f_n\}_{n=1}^{\infty}$ be a sequence of simple integrable functions converging almost everywhere $\boldsymbol{m}$ on T to f for which the integrals $\int f_n \, \mathrm{d}\boldsymbol{m}$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$. Then their scalar semivariations $\int |f_n| \, \mathrm{d}v(\boldsymbol{m}, .)$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$ as well, see the paragraph before Theorem 2. Let us use the notation of Theorem 1. Then for each $n, k = 1, 2, \ldots$ we have:

$$\int_{F_k} |f| \, \mathrm{d}v(\boldsymbol{m}, .) \leqq \int_{F_k} |f - f_n| \, \mathrm{d}v(\boldsymbol{m}, .) + \int_{F_k} |f_n| \, \mathrm{d}v(\boldsymbol{m}, .) + \int_{F - N - F_k} |f| \, \mathrm{d}v(\boldsymbol{m}, .) .$$

By Fatou's lemma

$$\int_{F - N - F_k} |f| \, \mathrm{d}v(\boldsymbol{m}, .) \leq \liminf_n \int_{F - N - F_k} |f_n| \, \mathrm{d}v(\boldsymbol{m}, .) ,$$

and this holds for sufficiently large k, owing to the uniform countable additivity of the integrals $\int |f_n| \, \mathrm{d}v(\boldsymbol{m}, .)$, $n = 1, 2, \ldots$, on $\mathfrak{S}(\mathscr{P})$ smaller then a given $\varepsilon > 0$. This proves the integrability of the function f in the classical sense. Further, it is not difficult to prove that $\left\| \int f \mathrm{d}\boldsymbol{m} \right\| (E) = \int_E |f| \, \mathrm{d}v(\boldsymbol{m}, .)$ for each set $E \in \mathfrak{S}(\mathscr{P})$.

Let us note that by the isomorphism of all n dimensional Banach spaces the result just proved can be easily extended to the case when X and Y are finite dimensional Banach spaces.

2. By Theorem 6 each Bochner integrable function is integrable in our sense. By Theorem 7 our integral in this case coincides with the so called second Dunford integral, see [13, pg 128]. Since each integrable function is by definition strongly measurable, each integrable function is integrable in the sense of Pettis, see the para-

graph after Theorem 7 and [14, 3.7.1]. The converse is in general not true since there are functions integrable in the sense of Pettis which are not strongly measurable. At this occasion see also Theorem 17.

Let us now introduce a simple example of an integrable function which is not integrable in the sense of Bochner. This example also shows that in general Theorem 16 on interchange of limit and integral is not valid for the Bochner integral.

Example. Let T be the set of all natural numbers, $\mathscr{P}_0$ the set of all finite subsets of T and let $X = Y = l_2$. Define $m(\{k\})\, x = x$ for each $k \in T$ and each $x \in l_2$ and $m(E) = \sum_{k \in E} m(\{k\})$ for $E \in \mathscr{P}_0$. Define further the function f by $f(k) = 1/k \,.\, e_k$ where $e_k = [\overbrace{0, 0, \ldots, 0, 1}^{k}, 0, \ldots] \in l_2$, $k = 1, 2, \ldots$. Then the function f is not integrable in the sense of Bochner since $\int_T |f| \, \mathrm{d}v(m, .) = \sum_{k=1}^{\infty} 1/k = +\infty$. On other hand, since $\sum_{k=1}^{\infty} 1/k^2 < +\infty$, by Riesz-Fischer Theorem the series $\sum_{k \in E} f(k)$ is unconditionally convergent for each subset $E \subset T$. Hence f is an integrable function. Let us note that if we define the function g by $q(k) = (1/k) \,.\, e_1$, $k = 1, 2, \ldots$, then $|g(k)| = |f(k)|$ for each k, and at the same time the function g is clearly not integrable.

3. Theorems 7 and 16 and Theorem 2.7 in [15] show that in this case our integral coincides with that in [15]. At the same time, if $\mathscr{P}$ is a σ-algebra, then this integral coinsides with that given in [2] or in [10, IV.10].

4. Let us compare the obtained integral with the general bilinear vector integral of R. G. Bartle [1] (of course, only for the countably additive case of [1]). First, what Bartle calls convergence in measure is in our terminology convergence in the semivariation $\hat{m}$. Since from each sequence of measurable functions converging in the semivariation $\hat{m}$ we can draw a subsequence converging almost everywhere m on the whole T, from Theorem 1 in [1] and from Theorem 7 we immediately have that each function integrable in the sense of Bartle is integrable in our sense. On the other hand, in Example 7″ we constructed a bounded integrable function which is not even measurable in the sense of Bartle. Hence Theorem 16 on interchange of limit and integral is in general not valid for the integral of Bartle. However, in the important particular case when the semivariation $\hat{m}$ is continuous on $\mathscr{P}$, see the $*$-Theorem in section 1.1, and when $\mathscr{P}$ is a σ-algebra, our integrals coincide, see Theorem 7 and Theorem 9 in [1].

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