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Bedřich Pondělíček

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ON A CERTAIN RELATION FOR CLOSURE OPERATION ON A SEMIGROUP

BEDŘICH PONDĚLÍČEK, Poděbrady

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Let S be a semigroup. It is well known that S is regular if and only if the relation

$$(*) \qquad A \cap B = AB$$

holds for every right ideal A and for every left ideal B of S. (See [1].)

If the relation (*) holds for every left ideals A, B of S, then every left ideal of S is a two-sided ideal of S and S is a left regular semigroup. Analogously for right ideals of S. (See [2].) Finally, if the relation (*) holds for any left ideals A, B of S and for any right ideals A, B of S, then S is a semilattice of groups. (See [3].)

In this paper we consider semigroups satisfying the relation (*) for every $\mathbf{U}$-closed non-empty subset A of S and for every $\mathbf{V}$-closed non-empty subset B of S where $\mathbf{U}$, $\mathbf{V}$ are arbitrary closure operations on S.

I

In this section, S will be a fixed non-empty set.

Definition 1. The mapping $\mathbf{U} : \exp S \to \exp S$ is said to be a *topological Čech's closure operation* (or simply a *$\mathscr{C}$-closure operation*) if the mapping $\mathbf{U}$ satisfies the following conditions:

1. $\mathbf{U}(\emptyset) = \emptyset$;
2. if $A \subset B \subset S$, then $\mathbf{U}(A) \subset \mathbf{U}(B)$;
3. $A \subset \mathbf{U}(A)$ for each $A \subset S$;
4. $\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A)$ for each $A \subset S$.

For $x \in S$ we write simply $\mathbf{U}(x)$ instead of $\mathbf{U}(\{x\})$. The set of all $\mathscr{C}$-closure operations for the set S will be denoted by $\mathscr{C}(S)$. (See [4] and [5].)

Lemma 1. *Let* $\mathbf{U} \in \mathscr{C}(S)$ *and* $A_i \subset S$ $(i \in I \neq \emptyset)$. *Then*

a) $\bigcup_{i\in I} \mathbf{U}(A_i) \subset \mathbf{U}(\bigcup_{i\in I} A_i)$;

b) $\mathbf{U}(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} \mathbf{U}(A_i)$.

Proof follows from Definition 1.

Definition 2. A $\mathscr{C}$-closure operation $\mathbf{U}$ is said to be a *quasi-discrete closure operation* (or simply a *$\mathscr{Q}$-closure operation*) if there holds

5. $\mathbf{U}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathbf{U}(A_i)$ for $A_i \subset S$ $(i \in I \neq \emptyset)$.

Let $\mathscr{Q}(S)$ be the set of all $\mathscr{Q}$-closure operations for the set S. (See p. 479 [6].)

Definition 3. Let $\mathbf{U} \in \mathscr{C}(S)$. A subset A of S will be called $\mathbf{U}$-*closed* if $\mathbf{U}(A) = A$, $\mathbf{U}$-*open* if $\mathbf{U}(S - A) = S - A$. The set of all $\mathbf{U}$-closed ($\mathbf{U}$-open) subsets of S will be denoted by $\mathscr{F}(\mathbf{U})$ $(\mathscr{O}(\mathbf{U}))$.

Theorem 1. *Let* $\mathbf{U} \in \mathscr{C}(S)$. *Then*:

1. $\emptyset, S \in \mathscr{F}(\mathbf{U})$;
2. *if* $A_i \in \mathscr{F}(\mathbf{U})$ $(i \in I \neq \emptyset)$, *then* $\bigcap_{i\in I} A_i \in \mathscr{F}(\mathbf{U})$;
3. *if* $A \subset S$, *then* $\mathbf{U}(A) = \bigcap_{i\in I} A_i$ *where* A_i $(i \in I)$ *are all* $\mathbf{U}$-*closed subsets of* S *such that* $A \subset A_i$.

Proof. 1. Evident. 2. If $A_i \in \mathscr{F}(\mathbf{U})$ $(i \in I \neq \emptyset)$, then $\mathbf{U}(A_i) = A_i$. From Definition 1 and Lemma 1 it follows that $\bigcap_{i\in I} A_i \subset \mathbf{U}(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} \mathbf{U}(A_i) = \bigcap_{i\in I} A_i$. Thus $\mathbf{U}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} A_i \in \mathscr{F}(\mathbf{U})$.

3. Clearly $\mathbf{U}(A)$ is a $\mathbf{U}$-closed subset of S. If A_i is an arbitrary $\mathbf{U}$-closed subset of S such that $A \subset A_i$, then by Definition 1 we have $\mathbf{U}(A) \subset \mathbf{U}(A_i) = A_i$. Therefore, we have $\mathbf{U}(A) \subset \bigcap_{i\in I} A_i$ and since $\bigcap_{i\in I} A_i \subset \mathbf{U}(A)$ we get the required result.

Remark 1. If $\mathbf{U}$ is a $\mathscr{Q}$-closure operation, then we also have:

4. if $A_i \in \mathscr{F}(\mathbf{U})$ $(i \in I \neq \emptyset)$, then $\bigcup_{i\in I} A_i \in \mathscr{F}(\mathbf{U})$.

Proof follows from Definition 2.

Now we shall introduce an order relation $\leqq$ in the set $\mathscr{C}(S)$.

Definition 4. If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then $\mathbf{U} \leqq \mathbf{V}$ if and only if $\mathbf{U}(A) \subset \mathbf{V}(A)$ for each $A \subset S$. (See [4].)

Put $\mathbf{O}(A) = A$ for each $A \subset S$ and $\mathbf{I}(A) = S$ for each $A \subset S$, $A \neq \emptyset$; $\mathbf{I}(\emptyset) = \emptyset$. Then $\mathbf{O}, \mathbf{I} \in \mathscr{Q}(S)$ and for each $\mathbf{U} \in \mathscr{C}(S)$,

$$\mathbf{O} \leqq \mathbf{U} \leqq \mathbf{I}$$

holds.

Remark 2. If $\mathbf{U}$, $\mathbf{V}$ are $\mathscr{Q}$-closure operations, then $\mathbf{U} \leqq \mathbf{V}$ if and only if $\mathbf{U}(x) \subset \mathbf{V}(x)$ for every $x \in S$.

Proof. If $\mathbf{U} \leqq \mathbf{V}$, then by Definition 4 we have $\mathbf{U}(x) \subset \mathbf{V}(x)$ for every $x \in S$. Conversely, let $\mathbf{U}(x) \subset \mathbf{V}(x)$ for every $x \in S$. It follows from Definition 2 that $\mathbf{U}(A) = \bigcup_{x \in A} \mathbf{U}(x) \subset \bigcup_{x \in A} \mathbf{V}(x) = \mathbf{V}(A)$ for each $A \subset S$, $A \neq \emptyset$.

Theorem 2. *If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then $\mathbf{U} \leqq \mathbf{V}$ if and only if $\mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U})$.*

Proof. Let $\mathbf{U} \leqq \mathbf{V}$. If $A \in \mathscr{F}(\mathbf{V})$, then $A = \mathbf{V}(A)$. By Definition 1 we have $A \subset \mathbf{U}(A) \subset \mathbf{V}(A) = A$. Hence $A = \mathbf{U}(A) \in \mathscr{F}(\mathbf{U})$. This implies $\mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U})$.

Let $\mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U})$. If $A \subset S$, then it follows from Theorem 1 that $\mathbf{U}(A) = \bigcap_{i \in I} A_i$ where A_i $(i \in I)$ are all $\mathbf{U}$-closed subsets of S such that $A \subset A_i$. Since $\mathscr{F}(\mathbf{V})$ is non-empty (it contains S) there exists a subset of indices $K \subset I$ such that A_k $(k \in K)$ are all $\mathbf{V}$-closed subsets of S containing A. Hence it follows that $\mathbf{U}(A) = \bigcap_{i \in I} A_i \subset \bigcap_{k \in K} A_k = \mathbf{V}(A)$. Therefore $\mathbf{U} \leqq \mathbf{V}$.

Corollary. *If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then $\mathbf{U} = \mathbf{V}$ if and only if $\mathscr{F}(\mathbf{U}) = \mathscr{F}(\mathbf{V})$.*

Theorem 3. *Let $\mathscr{F} \subset \exp S$ and*

1. *$\emptyset, S \in \mathscr{F}$;*
2. *if $A_i \in \mathscr{F}$ $(i \in I \neq \emptyset)$, then $\bigcap_{i \in I} A_i \in \mathscr{F}$.*

Then there exists a unique $\mathscr{C}$-closure operation $\mathbf{U}$ such that $\mathscr{F} = \mathscr{F}(\mathbf{U})$.

Proof. If $A \subset S$, then we put $\mathbf{U}(A) = \bigcap_{i \in I} A_i$ where A_i $(i \in I)$ are all sets from $\mathscr{F}$ such that $A \subset A_i$. Evidently $\mathbf{U}$ is a $\mathscr{C}$-closure operation. The unicity of $\mathbf{U}$ follows from Corollary to Theorem 2.

Remark 3. Let $\mathscr{F}$ satisfy the conditions of Theorem 3 and the following condition:

3. *if $A_i \in \mathscr{F}$ $(i \in I \neq \emptyset)$, then $\bigcup_{i \in I} A_i \in \mathscr{F}$.*

Then there exists a unique $\mathscr{Q}$-closure operation $\mathbf{U}$ such that $\mathscr{F} = \mathscr{F}(\mathbf{U})$.

Proof. It follows from Theorem 3 that there exists a unique $\mathbf{U} \in \mathscr{C}(S)$ such that $\mathscr{F} = \mathscr{F}(\mathbf{U})$. We shall prove that $\mathbf{U} \in \mathscr{Q}(S)$. Let $A_i \subset S$ $(i \in I \neq \emptyset)$. It follows from Lemma 1 and Definition 1 that $\bigcup_{i \in I} A_i \subset \bigcup_{i \in I} \mathbf{U}(A_i) \subset \mathbf{U}(\bigcup_{i \in I} A_i)$. Thus $\mathbf{U}(\bigcup_{i \in I} A_i) \subset$

$\subset \mathbf{U}(\bigcup_{i\in I} \mathbf{U}(A_i)) = \bigcup_{i\in I} \mathbf{U}(A_i) \subset \mathbf{U}(\bigcup_{i\in I} A_i)$. Hence $\mathbf{U}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathbf{U}(A_i)$. This implies $\mathbf{U} \in \mathscr{Q}(S)$.

If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then their greatest lower bound in $\mathscr{C}(S)$ will be denoted by $\mathbf{U} \wedge \mathbf{V}$ and their least upper bound in $\mathscr{C}(S)$ will be denoted by $\mathbf{U} \vee \mathbf{V}$.

Theorem 4. *If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then there exist $\mathbf{U} \vee \mathbf{V}$, $\mathbf{U} \wedge \mathbf{V}$ and*

1. $\mathscr{F}(\mathbf{U} \vee \mathbf{V}) = \mathscr{F}(\mathbf{U}) \cap \mathscr{F}(\mathbf{V})$;
2. $\mathscr{F}(\mathbf{U} \wedge \mathbf{V}) = \{A \cap B / A \in \mathscr{F}(\mathbf{U}), B \in \mathscr{F}(\mathbf{V})\}$.

The ordered set $\mathscr{C}(S)$ is a lattice.

Proof follows from Theorem 2 and Theorem 3.

Remark 4. Let for $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ be $\mathbf{W} = \mathbf{U} \wedge \mathbf{V}$. Evidently, if $A \subset S$, then we have $\mathbf{W}(A) = \mathbf{U}(A) \cap \mathbf{V}(A)$.

Remark 5. If $\mathbf{U}, \mathbf{V} \in \mathscr{Q}(S)$, then $\mathbf{U} \vee \mathbf{V} \in \mathscr{Q}(S)$. The following example shows that $\mathbf{U} \wedge \mathbf{V} \in \mathscr{Q}(S)$ does not hold in general.

Let $S = \{a, b, c, d\}$, $\mathscr{F}(\mathbf{U}) = \{\emptyset, \{a, b\}, \{c, d\}, S\}$ and $\mathscr{F}(\mathbf{V}) = \{\emptyset, \{a, c\}, \{b, d\}, S\}$. Evidently $\mathbf{U}, \mathbf{V} \in \mathscr{Q}(S)$. Further we have $\mathscr{F}(\mathbf{U} \wedge \mathbf{V}) = \mathscr{F}(\mathbf{U}) \cup \mathscr{F}(\mathbf{V}) \cup \cup \{a\} \cup \{b\} \cup \{c\} \cup \{d\}$. This implies $\mathbf{U} \wedge \mathbf{V} \in \mathscr{C}(S) - \mathscr{Q}(S)$.

Definition 5. Let $\mathbf{U} \in \mathscr{C}(S)$. We define $\mathbf{U}^* : \exp S \to \exp S$. If $A \subset S$, then $x \in \mathbf{U}^*(A)$ if and only if $\mathbf{U}(x) \cap A \neq \emptyset$.

Theorem 5. *If $\mathbf{U} \in \mathscr{C}(S)$, then $\mathbf{U}^* \in \mathscr{Q}(S)$.*

Proof. We shall show that $\mathbf{U}^*$ satisfies conditions 1, 2, 3 and 4 of Definition 1 and condition 5 of Definition 2.

1. Evident. 2. Let $A \subset B \subset S$. If $x \in \mathbf{U}^*(A)$, then $\mathbf{U}(x) \cap A \neq \emptyset$. Thus $\mathbf{U}(x) \cap B \neq \neq \emptyset$. Hence $x \in \mathbf{U}^*(B)$. Therefore $\mathbf{U}^*(A) \subset \mathbf{U}^*(B)$.

3. Let $A \subset S$. If $x \in A$, then it follows from Definition 1 that $x \in \mathbf{U}(x) \cap A$. Thus $x \in \mathbf{U}^*(A)$. This implies $A \subset \mathbf{U}^*(A)$.

4. Let $A \subset S$. From 3 and 2 it follows that $\mathbf{U}^*(A) \subset \mathbf{U}^*(\mathbf{U}^*(A))$. If $x \in \mathbf{U}^*(\mathbf{U}^*(A))$, then $\mathbf{U}(x) \cap \mathbf{U}^*(A) \neq \emptyset$. This implies that there exists some $z \in \mathbf{U}(x) \cap \mathbf{U}^*(A)$. Since $z \in \mathbf{U}^*(A)$, we have $\mathbf{U}(z) \cap A \neq \emptyset$. Definition 1 implies $\mathbf{U}(z) \subset \mathbf{U}(x)$, hence $\mathbf{U}(x) \cap A \neq \emptyset$. Therefore we have $x \in \mathbf{U}^*(A)$. Hence $\mathbf{U}^*(\mathbf{U}^*(A)) \subset \mathbf{U}^*(A)$. Therefore $\mathbf{U}^*(A) = \mathbf{U}^*(\mathbf{U}^*(A))$. By Definition 1 we have $\mathbf{U}^* \in \mathscr{C}(S)$.

5. Let $A_i \subset S$ $(i \in I \neq \emptyset)$. It follows from Lemma 1 that $\bigcup_{i\in I} \mathbf{U}^*(A_i) \subset \mathbf{U}^*(\bigcup_{i\in I} A_i)$. If $x \in \mathbf{U}^*(\bigcup_{i\in I} A_i)$, then $\mathbf{U}(x) \cap (\bigcup_{i\in I} A_i) \neq \emptyset$. There exists therefore some $k \in I$ such that $\mathbf{U}(x) \cap A_k \neq \emptyset$. Thus $x \in \mathbf{U}^*(A_k)$, hence $x \in \bigcup_{i\in I} \mathbf{U}^*(A_i)$. Therefore $\bigcup_{i\in I} \mathbf{U}^*(A_i) = = \mathbf{U}^*(\bigcup_{i\in I} A_i)$ and $\mathbf{U}^* \in \mathscr{Q}(S)$.

Theorem 6. *Let* $\mathbf{U} \in \mathscr{C}(S)$. *Then*:

1. $\mathbf{U}^{**}(x) = \mathbf{U}(x)$ *for every* $x \in S$;
2. $\mathbf{U}^{**} \leqq \mathbf{U}$;
3. $\mathscr{F}(\mathbf{U}) \subset \mathscr{O}(\mathbf{U}^*)$;
4. $\mathscr{O}(\mathbf{U}) \subset \mathscr{F}(\mathbf{U}^*)$.

Proof. 1. The proof follows from $z \in \mathbf{U}(x) \Leftrightarrow x \in \mathbf{U}^*(z) \Leftrightarrow z \in \mathbf{U}^{**}(x)$.

2. Let $A \subset S$. By Theorem 5 and Lemma 1 we have $\mathbf{U}^{**}(A) = \bigcup_{x \in A} \mathbf{U}^{**}(x) = \bigcup_{x \in A} \mathbf{U}(x) \subset \mathbf{U}(A)$. Hence $\mathbf{U}^{**} \leqq \mathbf{U}$.

3. Let $A \in \mathscr{F}(\mathbf{U})$. Suppose that $A \notin \mathscr{O}(\mathbf{U}^*)$. Then $S - A \neq \mathbf{U}^*(S - A)$. There exists therefore some x such that $x \in \mathbf{U}^*(S - A)$ and $x \notin S - A$. Thus $\mathbf{U}(x) \cap (S - A) \neq \emptyset$ and $x \in A$. Consequently, there exists some z such that $z \in \mathbf{U}(x)$, $z \notin A$. On the other hand, $z \in \mathbf{U}(x) \subset \mathbf{U}(A) = A$. This is a contradiction. Hence $A \in \mathscr{O}(\mathbf{U}^*)$ and $\mathscr{F}(\mathbf{U}) \subset \mathscr{O}(\mathbf{U}^*)$.

4. Let $A \in \mathscr{O}(\mathbf{U})$. Then $S - A \in \mathscr{F}(\mathbf{U})$. By 3 it follows that $S - A \in \mathscr{O}(\mathbf{U}^*)$. Hence $A \in \mathscr{F}(\mathbf{U}^*)$. Consequently $\mathscr{O}(\mathbf{U}) \subset \mathscr{F}(\mathbf{U}^*)$.

Theorem 7. *Let* $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. *If* $\mathbf{U} \leqq \mathbf{V}$, *then* $\mathbf{U}^* \leqq \mathbf{V}^*$.

Proof. Let $\mathbf{U} \leqq \mathbf{V}$ and $A \subset S$. If $x \in \mathbf{U}^*(A)$, then $\mathbf{U}(x) \cap A \neq \emptyset$. Since $\mathbf{U}(x) \subset \mathbf{V}(x)$, we have $\mathbf{V}(x) \cap A \neq \emptyset$. Hence $x \in \mathbf{V}^*(A)$. Therefore $\mathbf{U}^*(A) \subset \mathbf{V}^*(A)$. This implies $\mathbf{U}^* \leqq \mathbf{V}^*$.

Theorem 8. *Let* $\mathbf{U} \in \mathscr{C}(S)$. *Then the following conditions are equivalent*:

1. $\mathbf{U} \in \mathscr{Q}(S)$;
2. $\mathbf{U} = \mathbf{U}^{**}$;
3. $\mathscr{F}(\mathbf{U}) = \mathscr{O}(\mathbf{U}^*)$;
4. $\mathscr{O}(\mathbf{U}) = \mathscr{F}(\mathbf{U}^*)$.

Proof. $1 \Rightarrow 2$. This follows from Theorem 6 and Definition 2.

$2 \Rightarrow 3$. It follows from Theorem 6 that $\mathscr{F}(\mathbf{U}) \subset \mathscr{O}(\mathbf{U}^*) \subset \mathscr{F}(\mathbf{U}^{**}) = \mathscr{F}(\mathbf{U})$. This implies $\mathscr{F}(\mathbf{U}) = \mathscr{O}(\mathbf{U}^*)$.

$3 \Rightarrow 4$. Evident.

$4 \Rightarrow 1$. Let $A_i \in \mathscr{F}(\mathbf{U})$ $(i \in I \neq \emptyset)$. Then $S - A_i \in \mathscr{O}(\mathbf{U}) = \mathscr{F}(\mathbf{U}^*)$. According to Theorem 1, $S - \bigcup_{i \in I} A_i = \bigcap_{i \in I} (S - A_i) \in \mathscr{F}(\mathbf{U}^*) = \mathscr{O}(\mathbf{U})$. Thus $\bigcup_{i \in I} A_i \in \mathscr{F}(\mathbf{U})$. From Remark 3 it follows that $\mathbf{U} \in \mathscr{Q}(S)$.

Corollary. *If* $\mathbf{U} \in \mathscr{Q}(S)$, *then* $\mathbf{U} = \mathbf{U}^*$ *or* $\mathbf{U} \parallel \mathbf{U}^*$.

Proof. If $\mathbf{U} \leqq \mathbf{U}^*$, then by Theorem 7 and Theorem 8 $\mathbf{U}^* \leqq \mathbf{U}^{**} = \mathbf{U}$ holds. Hence $\mathbf{U} = \mathbf{U}^*$. Similarly, if $\mathbf{U}^* \leqq \mathbf{U}$, then $\mathbf{U} = \mathbf{U}^*$.

Remark 6. Evidently $\mathbf{O} = \mathbf{O}^*$ and $\mathbf{I} = \mathbf{I}^*$.

II

Let now S be an arbitrary semigroup.

Definition 6. Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. We shall say that $\mathbf{U} \varrho \mathbf{V}$ if there holds

(*) $$A \cap B = AB$$

for every $\mathbf{U}$-closed non-empty subset A of S and for every $\mathbf{V}$-closed non-empty subset B of S.

From Definition 6 and Theorem 2 there follows

Lemma 2. *Let $\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}_1, \mathbf{V}_2 \in \mathscr{C}(S)$ and $\mathbf{U}_1 \leqq \mathbf{U}_2$, $\mathbf{V}_1 \leqq \mathbf{V}_2$. If $\mathbf{U}_1 \varrho \mathbf{V}_1$, then $\mathbf{U}_2 \varrho \mathbf{V}_2$.*

Let $A \subset S$, $A \neq \emptyset$. Put $\mathbf{L}(A) = S^1A = SA \cup A$ and $\mathbf{R}(A) = AS^1 = AS \cup A$. Finally $\mathbf{L}(\emptyset) = \emptyset = \mathbf{R}(\emptyset)$. Clearly $\mathbf{L}, \mathbf{R} \in \mathscr{Q}(S)$ and $\mathscr{F}(\mathbf{L})$ is the set of all left ideals of S (including $\emptyset$), $\mathscr{F}(\mathbf{R})$ is the set of all right ideals of S (including $\emptyset$).

Theorem 9. *Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. Then $\mathbf{U} \varrho \mathbf{V}$ if and only if $\mathbf{R} \leqq \mathbf{U}$, $\mathbf{L} \leqq \mathbf{V}$ and $x \in \mathbf{U}(x)\, \mathbf{V}(x)$ for every $x \in S$.*

Proof. 1. Let $\mathbf{U} \varrho \mathbf{V}$. Clearly $S \in \mathscr{F}(\mathbf{V})$. If $A \in \mathscr{F}(\mathbf{U})$, then $A = A \cap S = AS$. Thus $A \in \mathscr{F}(\mathbf{R})$, hence $\mathscr{F}(\mathbf{U}) \subset \mathscr{F}(\mathbf{R})$. From Theorem 3 it follows that $\mathbf{R} \leqq \mathbf{U}$. Similarly we can show that $\mathbf{L} \leqq \mathbf{V}$. Finally, by Definition 1 we have $\mathbf{U}(x) \in \mathscr{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathscr{F}(\mathbf{V})$. Thus $x \in \mathbf{U}(x) \cap \mathbf{V}(x) = \mathbf{U}(x)\, \mathbf{V}(x)$.

2. Let now $\mathbf{R} \leqq \mathbf{U}$, $\mathbf{L} \leqq \mathbf{V}$ and $x \in \mathbf{U}(x)\, \mathbf{V}(x)$ for every $x \in S$. If $A \in \mathscr{F}(\mathbf{U})$, $A \neq \emptyset$ and $B \in \mathscr{F}(\mathbf{V})$, $B \neq \emptyset$, then according to Theorem 2 $A \in \mathscr{F}(\mathbf{R})$ and $B \in \mathscr{F}(\mathbf{L})$. Thus $AB \subset AS \subset A$ and $AB \subset SB \subset B$. Hence $AB \subset A \cap B$.

Let $x \in A \cap B$. Since $x \in A$, there holds $\mathbf{U}(x) \subset A$. Similarly we obtain that $\mathbf{V}(x) \subset B$. Thus $x \in \mathbf{U}(x)\, \mathbf{V}(x) \subset AB$. Hence $A \cap B \subset AB$. This implies (*).

Remark 7. It is clear that $\mathbf{I} \varrho \mathbf{I}$ if and only if $S^2 = S$.

Put $\mathbf{M} = \mathbf{L} \vee \mathbf{R}$, $\mathbf{H} = \mathbf{L} \wedge \mathbf{R}$. Evidently $\mathbf{M} \in \mathscr{Q}(S)$ and $\mathbf{H} \in \mathscr{C}(S)$. By Theorem 4 it follows that $\mathscr{F}(\mathbf{M})$ is the set of all two-sided ideals of S (including $\emptyset$) and $\mathscr{F}(\mathbf{H})$ is the set of all quasi-ideals of S (including $\emptyset$).

A semigroup S is called *left regular* (*right regular*, *regular*) if $x \in Sx^2$ ($x \in x^2S$, $x \in xSx$) for every $x \in S$.

Lemma 3. *A semigroup S is left regular (right regular, regular) if and only if $x \in S^1x^2$ ($x \in x^2S^1$, $x \in xS^1x$) for every $x \in S$.*

Proof is obvious.

Theorem 10. $\mathbf{R} \varrho \mathbf{L}$ *if and only if the semigroup S is regular.*

Proof. Let $\mathbf{R} \varrho \mathbf{L}$. Then by Theorem 9 $x \in \mathbf{R}(x)\, \mathbf{L}(x) = xS^1S^1x \subset xS^1x$. It follows from Lemma 3 that S is regular.

Let S be a regular semigroup. Then $x \in xSx \subset \mathbf{R}(x)\, \mathbf{L}(x)$. Theorem 9 implies that $\mathbf{R} \varrho \mathbf{L}$.

(See [1].)

Theorem 11. *The following conditions on S are equivalent:*

1. $\mathbf{L} \varrho \mathbf{L}$;
2. $\mathbf{L} \varrho \mathbf{M}$;
3. *S is left regular and* $\mathbf{R} \leqq \mathbf{L}$.

Proof. $1 \Rightarrow 2$. This follows from Lemma 2.

$2 \Rightarrow 3$. By Theorem 9 we have $\mathbf{R} \leqq \mathbf{L}$ and $x \in \mathbf{L}(x)\, \mathbf{M}(x) = \mathbf{L}(x)\, \mathbf{L}(x) = S^1xS^1x = S^1\, \mathbf{R}(x)\, x \subset S^1\, \mathbf{L}(x)\, x \subset S^1x^2$. It follows from Lemma 3 that S is left regular.

$3 \Rightarrow 1$. If S is left regular and $\mathbf{R} \leqq \mathbf{L}$, then $x \in Sx^2$. Hence $x \in \mathbf{L}(x)\, \mathbf{L}(x)$. Theorem 9 implies that $\mathbf{L} \varrho \mathbf{L}$.

(See [2].)

The following left-right dual of Theorem 11 holds:

Theorem 12. *The following conditions on S are equivalent:*

1. $\mathbf{R} \varrho \mathbf{R}$;
2. $\mathbf{M} \varrho \mathbf{R}$;
3. *S is right regular and* $\mathbf{L} \leqq \mathbf{R}$.

Theorem 13. $\mathbf{H} \varrho \mathbf{M}$ *if and only if the semigroup S is regular and* $\mathbf{R} \leqq \mathbf{L}$.

Proof. 1. Let $\mathbf{H} \varrho \mathbf{M}$. Then by Theorem 9 $\mathbf{R} \leqq \mathbf{H}$ and $x \in \mathbf{H}(x)\, \mathbf{M}(x)$. Thus $\mathbf{R} \leqq \mathbf{L}$ and $x \in \mathbf{R}(x)\, \mathbf{L}(x) = xS^1S^1x$. Lemma 3 implies that S is regular.

2. If S is regular and $\mathbf{R} \leqq \mathbf{L}$, then by Theorem 10 $\mathbf{R} \varrho \mathbf{L}$. Hence $\mathbf{H} \varrho \mathbf{M}$.

Theorem 14. $\mathbf{M} \varrho \mathbf{H}$ *if and only if the semigroup S is regular and* $\mathbf{L} \leqq \mathbf{R}$.

Proof. The proof is dual to the proof of Theorem 13.

Lemma 4. *Let* $\mathbf{L} = \mathbf{R}$. *A semigroup S is regular if and only if S is left regular (right regular).*

Proof. This follows from Lemma 3 and from $x^2S^1 = x\, \mathbf{R}(x) = x\, \mathbf{L}(x) = xS^1x = \mathbf{R}(x)\, x = \mathbf{L}(x)\, x = S^1x^2$.

Lemma 5. *Let* $\mathbf{L} = \mathbf{R}$. *Then $ef = fe$ for any couple of idempotents $e, f \in S$.*

Proof. The proof is an easy modification of the proof of Lemma 1 [7]. Evidently

$\mathbf{L}(e) = \mathbf{R}(e)$. Thus $ef \in eS^1 = S^1e = Se$ and $fe \in S^1e = eS^1 = eS$. This implies that $ef = ue$ for some $u \in S$ and $fe = ev$ for some $v \in S$. Hence $ef = (ue)\,e = efe = e(ev) = fe$.

Lemma 6. *S is a semilattice of groups if and only if S is regular and $\mathbf{L} = \mathbf{R}$.*

Proof. Let be $\mathbf{L} = \mathbf{R}$ and S regular. It follows from Lemma 4, Lemma 5 and Theorem 8 [8] that S is a semilattice of groups.

If S is a semilattice of groups, then clearly S is regular. From Remark to Theorem 2 [9] it follows that $\mathbf{L} = \mathbf{R}$.

Theorem 15. *The following conditions on S are equivalent:*

1. $\mathbf{L} \varrho \mathbf{R}$;
2. $\mathbf{L} \varrho \mathbf{L}$ *and* $\mathbf{R} \varrho \mathbf{R}$;
3. $\mathbf{L} \varrho \mathbf{M}$ *and* $\mathbf{M} \varrho \mathbf{R}$;
4. *S is a semilattice of groups.*

Proof. $1 \Rightarrow 2$. From Theorem 9 we have $\mathbf{L} \leqq \mathbf{R}$ and $\mathbf{R} \leqq \mathbf{L}$. Hence $\mathbf{L} = \mathbf{R}$ and thus $\mathbf{L} \varrho \mathbf{L}$, $\mathbf{R} \varrho \mathbf{R}$.

$2 \Rightarrow 3$. This follows from Lemma 2.

$3 \Rightarrow 4$. This follows from Theorem 11, Theorem 12, Lemma 4 and Lemma 6.

$4 \Rightarrow 1$. By Lemma 6 it follows that S is regular and $\mathbf{L} = \mathbf{R}$. Theorem 10 implies that $\mathbf{R} \varrho \mathbf{L}$. Hence $\mathbf{L} \varrho \mathbf{R}$.

(See [3].)

Lemma 7. $\mathbf{L} \vee \mathbf{R}^* = \mathbf{I} = \mathbf{L}^* \vee \mathbf{R}$.

Proof. Let $A \in \mathscr{F}(\mathbf{L} \vee \mathbf{R}^*)$, $A \neq \emptyset$. Then by Theorem 4 and Theorem 8 it follows that $A \in \mathscr{F}(\mathbf{L})$ and $S - A \in \mathscr{F}(\mathbf{R})$. Suppose that $A \neq S$. Then we obtain $(S - A)\,A \subset A \cap (S - A)$ which is a contradiction. Hence $A = S$ and $\mathbf{L} \vee \mathbf{R}^* = \mathbf{I}$. Similarly we obtain that $\mathbf{L}^* \vee \mathbf{R} = \mathbf{I}$.

A semigroup S is called *simple* (*left simple, right simple*) if $\mathbf{M} = \mathbf{I}$ ($\mathbf{L} = \mathbf{I}$, $\mathbf{R} = \mathbf{I}$).

Lemma 8. *A semigroup S is simple if and only if* $\mathbf{L} \leqq \mathbf{M}^*$ ($\mathbf{R} \leqq \mathbf{M}^*$).

Proof. 1. If $\mathbf{M} = \mathbf{I}$, then $\mathbf{L} \leqq \mathbf{I} = \mathbf{I}^* = \mathbf{M}^*$ and $\mathbf{R} \leqq \mathbf{I} = \mathbf{M}^*$.

2. Let $\mathbf{L} \leqq \mathbf{M}^*$. Since $\mathbf{R} \leqq \mathbf{M}$ we have by Theorem 7 $\mathbf{R}^* \leqq \mathbf{M}^*$. Now from Lemma 7 it follows that $\mathbf{I} = \mathbf{L} \vee \mathbf{R}^* \leqq \mathbf{M}^*$. Thus $\mathbf{M}^* = \mathbf{I}$. By Corollary to Theorem 8 we have $\mathbf{M} = \mathbf{I}$.

Theorem 16. *The following conditions on S are equivalent:*

1. $\mathbf{M}^* \varrho \mathbf{I}$;
2. $\mathbf{I} \varrho \mathbf{M}^*$;
3. *S is simple.*

Proof. $1 \Rightarrow 3$ and $2 \Rightarrow 3$ follow from Theorem 9 and Lemma 8.

$3 \Rightarrow 1$ and $3 \Rightarrow 2$. If S is simple, then clearly $S^2 = S$ and $\mathbf{M} = \mathbf{I} \approx \mathbf{M}^*$. Remark 7 implies that $\mathbf{M}^* \varrho \mathbf{I}$ and $\mathbf{I} \varrho \mathbf{M}^*$.

Lemma 9. *A semigroup S is left simple if and only if $\mathbf{R} \leqq \mathbf{L}^*$.*

Proof. 1. If $\mathbf{L} = \mathbf{I}$, then $\mathbf{R} \leqq \mathbf{I} = \mathbf{I}^* = \mathbf{L}^*$.

2. Let $\mathbf{R} \leqq \mathbf{L}^*$. Lemma 7 implies that $\mathbf{I} = \mathbf{L}^* \vee \mathbf{R} \leqq \mathbf{L}^*$. Thus $\mathbf{L}^* = \mathbf{I}$ and $\mathbf{L} = \mathbf{I}$.

Theorem 17. *The following conditions on S are equivalent:*

1. $\mathbf{L}^* \varrho \mathbf{I}$;
2. $\mathbf{L} \varrho \mathbf{L}^*$;
3. *S is left simple.*

Proof. The proof is analogous to the proof of Theorem 16.

Lemma 10. *A semigroup S is right simple if and only if $\mathbf{L} \leqq \mathbf{R}^*$.*

Theorem 18. *The following conditions on S are equivalent:*

1. $\mathbf{I} \varrho \mathbf{R}^*$;
2. $\mathbf{R}^* \varrho \mathbf{R}$;
3. *S is right simple.*

Evidently, a semigroup S is a *group* if and only if S is left simple and right simple, i.e. $\mathbf{H} = \mathbf{I}$.

Theorem 19. *The following conditions on S are equivalent:*

1. $\mathbf{L}^* \varrho \mathbf{R}^*$;
2. $\mathbf{L}^* \varrho \mathbf{R}$;
3. $\mathbf{L} \varrho \mathbf{R}^*$;
4. $\mathbf{L}^* \varrho \mathbf{I}$ *and* $\mathbf{I} \varrho \mathbf{R}^*$;
5. $\mathbf{L} \varrho \mathbf{L}^*$ *and* $\mathbf{R}^* \varrho \mathbf{R}$;
6. *S is a group.*

Proof. $1 \Rightarrow 2$. From Theorem 9 we have $\mathbf{L} \leqq \mathbf{R}^*$. By Lemma 10 it follows that $\mathbf{R} = \mathbf{I} = \mathbf{R}^*$. Thus $\mathbf{L}^* \varrho \mathbf{R}$.

$2 \Rightarrow 3$. From Theorem 9 and Lemma 9 we obtain $\mathbf{L} = \mathbf{I} = \mathbf{L}^*$ and $\mathbf{L} \leqq \mathbf{R}$. Thus $\mathbf{R} = \mathbf{I} = \mathbf{R}^*$. Hence $\mathbf{L} \varrho \mathbf{R}^*$.

$3 \Rightarrow 4$. It follows from Theorem 9 and Lemma 10 that $\mathbf{R} = \mathbf{I} = \mathbf{R}^*$ and $\mathbf{L} = \mathbf{I} = \mathbf{L}^*$. Thus $\mathbf{L}^* \varrho \mathbf{I}$ and $\mathbf{I} \varrho \mathbf{R}^*$.

$4 \Rightarrow 5 \Rightarrow 6$. This follows from Theorem 17 and Theorem 18.

$6 \Rightarrow 1$. If S is a group, then $\mathbf{L} = \mathbf{I} = \mathbf{L}^*$, $\mathbf{R} = \mathbf{I} = \mathbf{R}^*$ and $S^2 = S$. By Remark 7 we have $\mathbf{L}^* \varrho \mathbf{R}^*$.

A simple semigroup S is called *completely simple* if it contains at least one minimal left and at least one minimal right ideal of S.

Lemma 11. *A semigroup S is completely simple if and only if $\mathbf{L} = \mathbf{L}^*$ and $\mathbf{R} = \mathbf{R}^*$.*

Proof. 1. If S is a completely simple semigroup, then by [10] every left ideal of S is a union of disjoint minimal left ideals and every right ideal of S is a union of disjoint minimal right ideals. Clearly $\mathbf{L} = \mathbf{L}^*$ and $\mathbf{R} = \mathbf{R}^*$.

2. Let $\mathbf{L} = \mathbf{L}^*$ and $\mathbf{R} = \mathbf{R}^*$. Then $\mathbf{M} = \mathbf{L} \vee \mathbf{R} = \mathbf{L} \vee \mathbf{R}^* = \mathbf{I}$ (Lemma 7), and S is simple. Let $a \in S$. Evidently $\mathbf{L}(a)$ is a left ideal of S. We shall show that $\mathbf{L}(a)$ is a minimal left ideal of S. Let A be a left ideal of S such that $A \subset \mathbf{L}(a)$. If $x \in A$, then $x \in \mathbf{L}(x) \subset A \subset \mathbf{L}(a)$. It follows from Definition 5 that $a \in \mathbf{L}^*(x) = \mathbf{L}(x)$. This implies that $\mathbf{L}(a) \subset \mathbf{L}(x)$ and therefore $\mathbf{L}(x) = A = \mathbf{L}(a)$. Hence $\mathbf{L}(a)$ is a minimal left ideal. Similarly we obtain that $\mathbf{R}(a)$ is a minimal right ideal. Consequently, S is completely simple.

Theorem 20. *The following conditions on S are equivalent:*

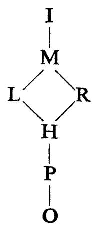
1. $\mathbf{R}^* \varrho \mathbf{L}^*$;
2. $\mathbf{R} \varrho \mathbf{L}^*$ *and* $\mathbf{R}^* \varrho \mathbf{L}$;
3. $\mathbf{I} \varrho \mathbf{L}^*$ *and* $\mathbf{R}^* \varrho \mathbf{I}$;
4. *S is completely simple.*

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. This follows from Theorem 9, Corollary to Theorem 8, Lemma 2 and Lemma 11.

$4 \Rightarrow 1$. Let S be completely simple. Then by Lemma 11 $\mathbf{R} = \mathbf{R}^*$ and $\mathbf{L} = \mathbf{L}^*$. Obviously S is regular and Theorem 10 implies that $\mathbf{R} \varrho \mathbf{L}$. Thus $\mathbf{R}^* \varrho \mathbf{L}^*$.

Put $\mathbf{P}(\emptyset) = \emptyset$. If $A \subset S$, $A \neq \emptyset$, then we denote by $\mathbf{P}(A)$ the subsemigroup generated by all elements of A. It is clear that $\mathbf{P} \in \mathscr{C}(S)$ and $\mathscr{F}(\mathbf{P})$ is the set of all subsemigroups of S (including $\emptyset$). Further $\mathbf{P} \leqq \mathbf{H}$.

Evidently the set $\{\mathbf{O}, \mathbf{P}, \mathbf{H}, \mathbf{L}, \mathbf{R}, \mathbf{M}, \mathbf{I}\}$ is ordered according to the following diagram:



Remark 8. Let $A \subset S$, $A \neq \emptyset$. It follows from Definition 5 that $\mathbf{P}^*(A)$ is the set of all almost nilpotent elements with respect to A in the sense of paper [11].

Lemma 12. $\mathbf{P} \leqq \mathbf{P}^*$ *if and only if* $\mathbf{P} = \mathbf{O}$.

Proof. 1. Let $\mathbf{P} \leqq \mathbf{P}^*$. According to Theorem 7 and Theorem 6 it follows that $\mathbf{P}^* \leqq \mathbf{P}^{**} \leqq \mathbf{P}$. Hence $\mathbf{P} = \mathbf{P}^*$. If $x \in S$, then $\mathbf{P}(x) = \mathbf{P}^*(x)$. Thus $x = x^n$ for some integer $n > 1$. Evidently $\mathbf{P}(x)$ is a cyclic subgroup of S. Let e be an identity of $\mathbf{P}(x)$. Since $e \in \mathbf{P}^*(x)$, there exists some positive integer m such that $x = e^m = e$. Consequently $\mathbf{P}(x) = \{x\} = \mathbf{O}(x)$ for every $x \in S$. It follows from Remark 2 and Theorem 5 that $\mathbf{P} = \mathbf{O}$.

2. If $\mathbf{P} = \mathbf{O}$, then it is clear that $\mathbf{P} \leqq \mathbf{P}^* = \mathbf{O}$.

A semigroup S is called a *left zero* (*right zero*) *semigroup* if $xy = x$ ($xy = y$) for every $x, y \in S$. Evidently, each left zero semigroup (right zero semigroup) is left simple (right simple).

Clearly:

Lemma 13. *A semigroup S is a left zero semigroup* (*right zero semigroup*) *if and only if* $\mathbf{R} = \mathbf{O}$ ($\mathbf{L} = \mathbf{O}$).

Theorem 21. *The following conditions on S are equivalent*:

1. $\mathbf{P} \, \varrho \, \mathbf{M}$;
2. $\mathbf{O} \, \varrho \, \mathbf{I}$;
3. $\mathbf{P}^* \, \varrho \, \mathbf{I}$;
4. *S is a left zero semigroup.*

Proof. $1 \Rightarrow 2$. It follows from Theorem 9 that $\mathbf{R} \leqq \mathbf{P}$ and $x \in \mathbf{P}(x)\,\mathbf{M}(x)$ for every $x \in S$. Thus $\mathbf{P} = \mathbf{R} \leqq \mathbf{L} = \mathbf{M}$. If $x \in S$, then $x \in \mathbf{P}(x)\,\mathbf{M}(x) = \mathbf{R}(x)\,\mathbf{L}(x) = xS^1S^1x \subset \subset xS^1x = \mathbf{R}(x)\,x = \mathbf{P}(x)\,x$. Hence there exists some integer $n > 1$ such that $x = x^n$. Evidently $\mathbf{P}(x)$ is a cyclic subgroup of S. Let e be an identity of $\mathbf{P}(x)$. Then $ex \in \mathbf{R}(e) = \mathbf{P}(e) = \{e\}$ and $x = ex = e$. Every element x of S is an idempotent. Consequently $\mathbf{P}(x) = \{x\} = \mathbf{O}(x)$ for every $x \in S$. Thus $\mathbf{P} = \mathbf{O}$ and $\mathbf{R} = \mathbf{O}$, $\mathbf{L} = \mathbf{I}$. Hence $\mathbf{O} \, \varrho \, \mathbf{I}$.

$2 \Rightarrow 3$. This follows from Lemma 2.

$3 \Rightarrow 4$. It follows from Theorem 9 that $\mathbf{R} \leqq \mathbf{P}^*$ and thus $\mathbf{P} \leqq \mathbf{P}^*$. By Lemma 12 we have $\mathbf{P} = \mathbf{O}$. Hence $\mathbf{R} = \mathbf{O}$. According to Lemma 13 S is a left zero semigroup.

$4 \Rightarrow 1$. If S is a left zero semigroup, then it follows from Lemma 13 that $\mathbf{R} = \mathbf{O}$. Thus $\mathbf{L} = \mathbf{I}$. Since $x \in \mathbf{O}(x) = \mathbf{O}(x)\,\mathbf{I}(x)$, we get by Theorem 9 that $\mathbf{O} \, \varrho \, \mathbf{I}$. Thus $\mathbf{P} \, \varrho \, \mathbf{M}$.

Dually we have the following:

Theorem 22. *The following conditions on S are equivalent*:

1. $\mathbf{M} \, \varrho \, \mathbf{P}$;
2. $\mathbf{I} \, \varrho \, \mathbf{O}$;
3. $\mathbf{I} \, \varrho \, \mathbf{P}^*$;
4. *S is a right zero semigroup.*

Theorem 23. *The following conditions on S are equivalent*:

1. $\mathbf{L} \varrho \mathbf{P}$;
2. $\mathbf{P} \varrho \mathbf{R}$;
3. $\mathbf{L}^* \varrho \mathbf{P}$;
4. $\mathbf{P} \varrho \mathbf{R}^*$;
5. $\mathbf{L}^* \varrho \mathbf{P}^*$;
6. $\mathbf{P}^* \varrho \mathbf{R}^*$;
7. $\mathbf{L} \varrho \mathbf{P}^*$;
8. $\mathbf{P}^* \varrho \mathbf{R}$;
9. $\mathbf{O} \varrho \mathbf{I}$ *and* $\mathbf{I} \varrho \mathbf{O}$;
10. $S = \{e\}$ *where* $e^2 = e$.

Proof follows from Theorem 21 and Theorem 22.

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Author's address: Poděbrady, zámek, ČSSR (České vysoké učení technické).