Czechoslovak Mathematical Journal

Ján Jakubík

The mixed product decompositions of partially ordered groups

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 2, 184–206

Persistent URL: http://dml.cz/dmlcz/100960

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

THE MIXED PRODUCT DECOMPOSITIONS OF PARTIALLY ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received July 3, 1968)

The concept of the mixed product of partially ordered groups is a common generalization of the concepts of the complete direct product and the lexicographic product. The mixed products were used by CONRAD, HARVEY and HOLLAND [3] and by CONRAD [2] to the study of the structure of the abelian l-groups and abelian partially ordered groups.

The main result of this paper consists in constructing the isomorphic refinements of any two mixed product decompositions

$$G = \Omega_{i \in I} A_i \,, \quad G = \Omega_{j \in J} B_j$$

where G is a partially ordered group and all factors A_i, B_j are directed, $A_i \neq \{0\} \neq B_j$. An analogous result was proved by MALCEV [6] for the lexicographic σ-products of linearly ordered groups. FUCHS ([4], Chap. II, Theorem 9) generalized Mal'cev's theorem for lexicographic σ-products with directed factors. Lexicographic products and lexicographic σ-products of a certain type of partially ordered grup-poids were considered in [5].

1. DEFINITIONS AND NOTATION

For partially ordered groups we shall use the concepts and the notation from [1]. The group operation will be denoted additively (the commutativity not being assumed). $\bigcap$, $\bigcup$, $\subset$ and $\wedge$, $\vee$ are the usual set-theoretical and lattice-theoretical symbols, respectively. If X, Y are any sets, then $X \setminus Y$ is the set of all elements of X that do not belong to Y. Let A and B be partially ordered sets; $A \circ B$ is the lexicographic product of A and B (cf. [1]). For $a_1, a_2 \in A$ the symbol $a_1 \parallel a_2$ denotes that a_1 and a_2 are incomparable.

1.1. Let $I \neq \emptyset$ be a partially ordered set and for any $i \in I$ let A_i be a partially ordered group. Let us denote by G_1 the Cartesian product of all sets A_i, i.e., G_1 is the system of all maps $f : I \to \bigcup A_i$ such that $f(i) \in A_i$ for each $i \in I$. The element $f(i)$ is said to be the component of f in A_i. For $f(i)$ and f we shall use also the symbols f_i and $(\ldots, f_i, \ldots)_{i \in I}$, respectively. If $f \in G_1$, let us put

$$I(f) = \{i \in I : f(i) \neq 0\}.$$

Further we denote by

$$G_2 = [\Omega_{i \in I} A_i]$$

the system of all $f \in G_1$ such that $I(f)$ satisfies the descending chain condition. Now we define in G_2 the operation $+$ componentwise. If $f, g \in G_2, f \neq g$, we denote

$$I(f, g) = \{i \in I : f(i) \neq g(i)\}.$$

Let $\min I(f, g)$ be the set of all minimal elements of the set $I(f, g)$. Let us put $f < g$ if $f(i) < g(i)$ for each $i \in \min I(f, g)$. Then $(G_2; +, \leqq)$ is a partially ordered group; G_2 is the *mixed product of partially ordered groups* A_i.

1.1.1. Analogously as in the case of direct products it is sometimes convenient to replace the partially ordered groups A_i by some subgroups $\bar{A}_i$ of G_2 that are isomorphic to A_i. Let $i \in I$ be fixed and let us put

$$\bar{A}_i = \{f \in G_2 : f(i_1) = 0 \text{ for each } i_1 \in I,\ i_1 \neq i\}.$$

Then $\bar{A}_i$ is a partially ordered group isomorphic to A_i. For each $g \in G_2$ we put $\varphi_i(g) = f$ where $f \in \bar{A}_i$, $g(i) = f(i)$. The mappings φ_i have the following properties: if i_1, $i_2 \in I$, $i_1 \neq i_2$, $g \in \bar{A}_{i_1}$, then $\varphi_{i_1}(g) = g$, $\varphi_{i_2}(g) = 0$. Moreover, the map $g \to \varphi(g) = (\ldots, \varphi_i(g), \ldots)_{i \in I}$ is an isomorphism of the partially ordered group G_2 onto the partially ordered group $[\Omega_{i \in I} \bar{A}_i]$.

We can formulate now the definition of the mixed product decomposition of G.

1.2. Let G be a partially ordered group and let $I \neq \emptyset$ be an ordered set. For any $i \in I$ let A_i be a subgroup of G (with the induced partial order). Assume that for each $i \in I$ there exists a mapping φ_i of G onto A_i such that

(a) $x \in A_i \Rightarrow \varphi_i(x) = x$, $\varphi_{i_1}(x) = 0$ for any $i_i \in I$, $i_1 \neq i$;
(b) the mapping $\varphi(x) = (\ldots, \varphi_i(x), \ldots)$ is an isomorphism of G onto $[\Omega_{i \in I} A_i]$.

In such case we will write

$$G = \Omega_{i \in I} A_i\,; \tag{1.1}$$

this equation represents a *mixed product decomposition* of the partially ordered group G.

1.3. Assume that (1.1) holds. If the mappings φ_i are fixed, then we write also x_i,

$x(A_i)$ instead of $\varphi_i(x)$. For $X \subset G$ we denote $X(A_i) = \{x(A_i)\}_{x \in X}$. If $I_1 \subset I$ and if B_i $(i \in I_1)$ is a subgroup of A_i (with the induced partial order) we define the partially ordered group

$$H = \Omega_{i \in I_1} B_i \,.$$

as follows: we put $B_i = \{0\}$ for each $i \in I \setminus I_1$ and denote

$$H = \varphi^{-1}([\Omega_{i \in I} B_i]) \,.$$

1.4. Assume that (1.1) is valid and let another mixed product decomposition

$$G = \Omega_{j \in J} B_j \tag{1.2}$$

be given. The decompositions (1.1) and (1.2) are isomorphic, if there exists an isomorphism ψ of the partially ordered set I onto J such that the partially ordered groups A_i and $B_{\psi(i)}$ are isomorphic for each $i \in I$. The decomposition (1.2) is a refinement of (1.1), if for each $i \in I$ there exists a subset $J_i \subset J$ such that $A_i = \Omega_{j \in J_i} B_j$.

1.5. Let X be a subgroup of the partially ordered group G. X is a factor in G if there exists a decomposition (1.1) and an element $i_0 \in I$ such that $X = A_{i_0}$. A factor X is nontrivial if $X \neq \{0\}$. An immediate consequence of the definition 1.2 is the following "substitution rule": Let X be a factor in G (under the notation just used) and let

$$X = \Omega_{k \in K} C_k \,.$$

Denote $M = (I \setminus \{i_0\}) \cup K$. On the set M we introduce a partial order $\leqq$ in such a way that on the set $I \setminus \{i_0\}$ we take the original partial order (induced by I) and for any $m_1 \in I \setminus \{i_0\}$, $m_2 \in K$ we put $m_1 < m_2$ $(m_2 < m_1)$ if and only if $m_1 < i_0$ $(i_0 < m_1)$. Now we put $D_m = A_m$ if $m \in I \setminus \{i_0\}$ and $D_m = C_m$ if $m \in K$. Then $G = \Omega_{m \in M} D_m$ holds.

1.6. Throughout the paper we shall suppose that $G \neq \{0\}$ (our considerations being trivial for $G = \{0\}$). Let us consider the decomposition (1.1) and denote

$$I' = \{i \in I : A_i \neq \{0\}\} \,.$$

Then $I' \neq \emptyset$ and from the definition 1.2 it follows

$$G = \Omega_{i \in I'} A_i \,.$$

If $M \supset I$ is a partially ordered set and if we put $A_i = \{0\}$ for each $i \in M \setminus I$, then from 1.1 we get $G = \Omega_{m \in M} A_m$. Hence any number of trivial factors can be removed from or added to a decomposition. We shall often restrict ourselves to decompositions with non-trivial factors only.

1.7. Assume that (1.1) holds and $I = \{1, 2\}$ (with the natural order). In such a case we write

$$G = A_1 \circ A_2$$

instead of (1.1). From the conditions (a) and (b) in 1.2 it follows that each element $x \in G$ can be uniquely written in the form $x = x_1 + x_2$, $x_1 \in A_1$, $x_2 \in A_2$; if, at the same time, $y = y_1 + y_2$, $y_i \in A_i$, then $x < y$ if and only if either $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$. Moreover, if $A_2 = A_3 \circ A_4$, then by 1.5 $G = A_1 \circ (A_3 \circ A_4)$. It is easy to prove that this is equivalent with $G = (A_1 \circ A_3) \circ A_4$ and therefore we write simply $G = A_1 \circ A_3 \circ A_4$ (cf. also [5], section 6).

2. THE SUBGROUP $A(i)$

In this section we suppose that there are given two decompositions

$$(\alpha)\ G = \Omega_{i \in I} A_i\,,$$
$$(\beta)\ G = \Omega_{j \in J} B_j$$

of the partially ordered group G and that all factors A_i, B_j are directed. For any $i_0 \in I$ denote

$$A(i_0) = \Omega A_i\ (i \in I,\ i \geqq i_0)\,,$$
$$A'(i_0) = \Omega A_i\ (i \in I,\ i > i_0)\,.$$

The symbols $B(j_0)$, $B'(j_0)$ ($j_0 \in J$) have the analogous meaning. Since all factors A_i are directed, $A(i_0)$ and $A'(i_0)$ are directed as well.

2.1. *$A(i_0)$ is a convex subset of G.*

Proof. Let $x, z \in A(i_0)$, $y \in G$, $x < y < z$. Assume that $y \notin A(i_0)$. Hence there exists $i \in I$ such that $i \ngeqq i_0$, $y_i \neq 0$. Then $x_i \neq y_i$ and therefore there exists $i_1 \in \min I(x, y)$, $i_1 \leqq i$. Since $x < y$ and $x_{i_1} = 0$, we get $y_{i_1} > 0$. At the same time $z_{i_2} = y_{i_2} = 0$ for each $i_2 < i_1$ and $z_{i_1} = 0$. Hence $i_1 \in \min I(y, z)$, $y_{i_1} > z_{i_1}$, a contradiction.

2.2. *Let $x \in G$, $x > 0$, $i_1 \in \min I(x, 0)$. Then (a) $2x > x_{i_1}$, (b) $2x \ngtr 3x_{i_1}$.*

Proof. Clearly $\min I(x, 0) = \min I(2x, x_{i_1}) = \min I(2x, 3x_{i_1})$. For any $i \in \min I(x, 0)$, $(2x)_i > (x_{i_1})_i$ holds. Moreover, $i_1 \in \min I(2x, 3x_{i_1})$, $(2x)_{i_1} < (3x_{i_1})_{i_1}$.

2.3. *Let $x \in A(i_0)$, $x > 0$. Then $x_j \in A(i_0)$ for any $j \in J$.*

Proof. For $x_j = 0$ the assertion is trivial; let $x_j \neq 0$. There exists $j_1 \in \min J(x, 0)$, $j_1 \leqq j$ with $0 < x_{j_1}$. By 2.2 $x_{j_1} < 2x$. Since $2x \in A(i_0)$, we get $x_{j_1} \in A(i_0)$ by 2.1. If

$j = j_1$, we have $x_j \in A(i_0)$. Let $j_1 < j$. Then for each $z \in B_j^+$ $0 \leqq z < x_{j_1}$ holds, and thus by 2.1 $B_j^+ \subset A(i_0)^+$. Each element of B_j is a diference of positive elements of B_j (since B_j is directed) and this implies $B_j \subset A(i_0)$, whence $x_j \in A(i_0)$.

By a dual argument the analogous proposition for $x < 0$ can be proved. Since $A(i_0)$ is directed, we have:

2.4. *If $x \in A(i_0)$, then $x_j \in A(i_0)$ for each $j \in J$.*

Now we will prove that from $x \in G$, $x_j \in A(i_0) \neq \{0\}$ for each $j \in J$ it follows $x \in A(i_0)$. We need some auxiliary lemmas.

2.5. *Let $x, v \in G$, $x > 0$, $v > 0$ and let $x_i \leqq v$ for each $i \in I$. Then $x < 2v$.*

Proof. Assume, at first, that $x = 2v$ and let $i_1 \in \min I(v, 0)$. Then $0 < v_{i_1} < 2v_{i_1}$, $i_1 \in \min I(v, 2v_{i_1})$, hence $2v_{i_1} \nleqq v$. But $2v_{i_1} = x_{i_1} \leqq v$, a contradiction. Therefore, $x \neq 2v$. If $x = v$, then $x < 2v$. Let $x \neq v$ and let $i_1 \in \min I(x, 2v)$. Suppose that there exists $i_2 < i_1$ such that $x_{i_2} \neq 0$. Then there exists $i_3 \leqq i_2$, $i_3 \in \min I(x, 0)$. Since $i_3 < i_1$, we have $i_3 \in \min I(2v, 0) = \min I(v, 0)$, hence $0 < x_{i_3} = 2v_{i_3} > v_{i_3}$. But, at the same time, $i_3 \in \min I(x_{i_3}, v)$, thus $x_{i_3} < v_{i_3}$, a contradiction. Therefore $x_i = 2v_i = v_i = 0$ for each $i < i_1$. If $x_{i_1} = 0$, then $2v_{i_1} \neq 0$, whence $v_{i_1} \neq 0$ and $i_1 \in \min I(v, 0)$, thus $v_{i_1} > 0$, $2v_{i_1} > 0 = x_{i_1}$. If $x_{i_1} \neq 0$, then $i_1 \in \min I(x, 0)$ and $x_{i_1} > 0$. Now we have etiher $x_{i_1} = v_{i_1}$ and $x_{i_1} < 2v_{i_1}$, or $x_{i_1} \neq v_{i_1}$ and $i_1 \in \in \min I(x, v)$, whence $x_{i_1} < v_{i_1}$, $x_{i_1} < 2v_{i_1}$. The proof is complete.

2.6. *Let $x \in G$, $x > 0$, $x_j \in A(i_0)$ for each $j \in \min J(x, 0)$. If $i < i_0$, then $x_i = 0$.*

Proof. Assume that $i < i_0$, $x_i \neq 0$. Then there exists $i_1 \in \min I(x, 0)$, $i_1 \leqq i$. According to 2.2 $2x > x_{i_1}$; clearly $x_{i_1} > t$ for each $t \in A(i_0)$, hence $2x > t$ for each $t \in A(i_0)$. Let $j_1 \in \min J(x, 0)$. We have $x_{j_1} \in A(i_0)$, hence $3x_{j_1} \in A(i_0)$ and thus $2x > 3x_{j_1}$. By 2.2 (b) $2x \ngtr 3x_{j_1}$, a contradiction.

2.7. *Let $z \in G$, $z > 0$, $i_0 \in I$. Suppose that $z_{j_1} \in A(i_0)$ for each $j_1 \in \min J(z, 0)$. Then $z_j \in A(i_0)$ for each $j \in J$.*

Proof. Let $j \in J$, $j \notin \min J(z, 0)$. The case $z_j = 0$ is trivial. Let $z_j \neq 0$; then there exists $j_1 < j$, $j_1 \in \min J(z, 0)$. For each $b_j \in B_j^+$ we have $0 \leqq b_j < z_{j_1}$, thus, by the convexity of $A(i_0)$, $b_j \in A(i_0)$. Therefore, since B_j is directed, $B_j \subset A(i_0)$ and so $z_j \in A(i_0)$.

2.8. *Let $y, z \in G$, $0 < y < z$, $i_0 \in I$ and let $z_j \in A(i_0)$ for each $j \in J$. Then $y_j \in \in A(i_0)$ for each $j \in J$.*

Proof. Let $j_1 \in \min J(y, 0)$. Then $y_{j_1} > 0$. If $j_1 \in \min J(y, z)$, we get $0 < y_{j_1} < < z_{j_1}$, hence from the convexity of $A(i_0)$ it follows $y_{j_1} \in A(i_0)$. If $j_1 \notin \min J(y, z)$, then there exists $j_2 < j_1$, $j_2 \in \min J(y, z)$ and $0 < y_{j_1} < z_{j_2}$; therefore $y_{j_1} \in A(i_0)$. According to 2.7 this implies that $y_j \in A(i_0)$ for each $j \in J$.

2.9. *Let $x \in G$, $x > 0$, $i_0 \in I$, $x_j \in A(i_0)$ for each $j \in J$. Let $A_{i_0} \neq \{0\}$. If $i \in I$, $i \mid i_0$, then $x_i = 0$.*

Proof. Let $i \in I$, $i \mid i_0$. Assume that $x_i \neq 0$. Then there exists $i_1 \in \min I(x, 0)$, $i_1 \leqq i$. According to 2.6 $i_1 \mid i_0$. By 2.2 $0 < x_{i_1} < 2x$. Let $j \in J$. Since $(2x)_j = 2x_j \in A(i_0)$, by 2.8 we have

$$(2.1) \qquad (x_{i_1})_j \in A(i_0)\,.$$

At the same time $x_{i_1} \in A(i_1)$, hence by 2.4 $(x_{i_1})_j \in A(i_1)$ and therefore from $i_1 \mid i_0$ we get

$$(2.2) \qquad ((x_{i_1})_j)_{i_0} = 0\,.$$

It follows from (2.1) and (2.2) that $i_2 > i_0$ for each $i_2 \in \min I((x_{i_1})_j, 0)$. Since $A_{i_0} \neq \{0\}$, there exists $a \in A_{i_0}$, $a > 0$. We have $(x_{i_1})_j < a_{i_0}$ for each $j \in J$, hence by 2.5 $x_{i_1} < 2a$. From the relations $2a \in A_{i_0}$, $x_{i_1} \in A_{i_1}$, $i_0 \mid i_1$ we obtain $2a \mid x_{i_1}$, a contradiction.

2.10. *Let $x \in G$, $x > 0$, $i_0 \in I$, $x_j \in A(i_0)$ for each $j \in J$, $A_{i_0} \neq \{0\}$. Then $x \in A(i_0)$.*

This follows from 2.6 and 2.9.

2.11. *Let $x \in G$, $x < 0$, $i_0 \in I$, $x_j \in A(i_0)$ for each $j \in J$, $A_{i_0} \neq \{0\}$. Then $x \in A(i_0)$.*

Proof. Put $y = -x$. Then $y_j = -x_j \in A(i_0)$, hence by 2.10 $y \in A(i_0)$ and therefore $x \in A(i_0)$.

2.12. *The set $A(i_0) \cap B_j$ is directed.*

Proof. Let $d \in A(i_0) \cap B_j$, $d \mid 0$. Since $A(i_0)$ is directed, there exists $d' \in A(i_0)$ such that $d' > 0$, $d' > d$. Therefore from $d = d_j \mid 0$ it follows that there exists $j_1 \in \min J(d', 0)$, $j_1 \leqq j$. If $j_1 = j$, then $d'_{j_1} > 0$ and $j_1 \in \min J(d, d')$, whence $d'_{j_1} > d_j = d$, $d'_{j_1} \in B_j$. According to 2.4 $d'_{j_1} \in A(i_0)$, hence $d'_{j_1} \in A(i_0) \cap B_j$. If $j_1 < j$, then $0 \leqq b_j < d'_{j_1}$ for each $b_j \in B_j^+$, thus with respect to the convexity of $A(i_0)$ and from $d'_{j_1} \in A(i_0)$ it follows $b_j \in A(i_0)$. Therefore $B_j^+ \subset A(i_0)$ and $B_j \subset A(i_0)$. Since B_j is directed, there exists $b_j \in B_j$ such that $0 \leqq b_j$, $d \leqq b_j$. This proves that $A(i_0) \cap B_j$ is up-directed; by a dual argument we can show that it is down-directed.

2.13. *Let $x \in G$, $i_0 \in I$, $x_j \in A(i_0)$ for each $j \in J$, $A_{i_0} \neq \{0\}$. Then $x \in A(i_0)$.*

Proof. The assertion is trivial for $x = 0$; let $x \neq 0$, $j_1 \in \min J(x, 0)$. Then $0 \neq$

$\neq x_{j_1} \in A(i_0) \cap B_{j_1}$. By 2.12 there exist elements u^{j_1}, $v^{j_1} \in A(i_0) \cap B_{j_1}$ such that $u^{j_1} < 0$, $u^{j_1} < x_{j_1}$, $v^{j_1} > 0$, $v^{j_1} > x_{j_1}$. Assume that we have chosen such elements u^{j_1}, v^{j_1} for each $j_1 \in \min J(x, 0)$. There exist $u, v \in G$ satisfying $u_j = u^j$, $v_j = v^j$ for $j \in \min J(x, 0)$ and $u_j = v_j = 0$ for $j \notin \min J(x, 0)$. Then $u < 0 < v$ and according to 2.10 and 2.11 u and v belong to $A(i_0)$. Obviously $u < x < v$, and therefore $x \in A(i_0)$.

2.14. Theorem. *If $i_0 \in I$, $A_{i_0} \neq \{0\}$, then*

$$A(i_0) = \Omega_{j\in J}(A(i_0) \cap B_j)\,.$$

The proof follows from 2.4 and 2.13.

2.15. $A(i_0) \cap B_j = A(i_0)(B_j)$ *for any $i_0 \in I, j \in J$.*

Proof. Obviously $A(i_0)(B_j) \subset B_j$, hence by 2.4 $A(i_0)(B_j) \subset A(i_0) \cap B_j$. Let $t \in A(i_0) \cap B_j$. Then $t \in B_j$, hence $t(B_j) = t$. From $t \in A(i_0)$ we obtain $t(B_j) \in A(i_0)(B_j)$, whence $A(i_0) \cap B_j \subset A(i_0)(B_j)$.

From 2.14 and 2.15 it follows:

2.16. *If $i_0 \in I$, $A_{i_0} \neq \{0\}$, then*

$$A(i_0) = \Omega_{j\in J}\, A(i_0)(B_j)\,.$$

2.17. *If $A_{i_0} \neq \{0\}$, then*

$A(i_0) = \{x \in G : \textit{there exist } u, v \in A_{i_0} \textit{ such that } u \leqq x \leqq v\}$,

$A'(i_0)^+ = \{x \in G^+ : nx < a \textit{ for any } a \in A_{i_0}, a > 0 \textit{ and any positive integer } n\}$.

Proof. If $u, v \in A_{i_0}$, $x \in G$, $u \leqq x \leqq v$, then by the convexity of $A(i_0)$ we have $x \in A(i_0)$. Let $t \in A(i_0)$. Since $A_{i_0} \neq \{0\}$ is directed, there exist $u, v \in A_{i_0}$ such that $u < 0 < v$, $u < t_{i_0} < v$. This implies $u < t < v$.

Denote $Z = \{x \in G^+, nx < a$ for any $a \in A_{i_0}$, $a > 0$ and any positive integer $n\}$. Obviously $A'(i_0)^+ \subset Z$. Let $x \in Z$, $a \in A_{i_0}$, $a > 0$. Since $x < a$, we have $x \in A(i_0)$. Assume that $x_{i_0} \neq 0$. Then $x_{i_0} > 0$, $x_{i_0} \in A_{i_0}$, hence from $x \in Z$ we obtain $2x < x_{i_0}$. According to 2.2 $x_{i_0} < 2x$, a contradiction. Therefore $x_{i_0} = 0$ and $x \in A'(i_0)$.

As an immediate consequence it follows from 2.17:

2.17.1. *If $A_{i_0} \neq \{0\}$, then $A'(i_0) = \{x - y : x \in Z, y \in Z\}$ where Z has the same meaning as in the proof of* 2.17.

2.17.2. *Let $i_0 \in I$, $j_0 \in J$. If $B_{j_0} \subset A_{i_0}$, then $B(j_0) \subset A(i_0)$, $A'(i_0) \subset B'(j_0)$.*

2.18. *Let $i_1, i_2 \in I$, $i_1 \neq i_2$. Then $[A(i_1) \setminus A'(i_1)] \cap [A(i_2) \setminus A'(i_2)] = \emptyset$.*

Proof. Let $x \in [A(i_1) \setminus A'(i_1)] \cap [A(i_2) \setminus A'(i_2)]$. If i_1, i_2 are comparable, we may assume $i_1 < i_2$. Then $A(i_2) \subset A'(i_1)$. Since $x \in A(i_2)$, we have $x \notin A(i_1) \setminus A'(i_1)$, a contradiction. Let $i_1 \mid i_2$. Since $x \in A(i_1)$ and $i_1 \nleqq i_2$, we obtain $x_{i_2} = 0$. From this and from $x \in A(i_2)$ it follows $x \in A'(i_2)$, hence $x \notin A(i_2) \setminus A'(i_2)$, a contradiction.

3. THE DECOMPOSITION $G = C \circ D$

In this section we shall consider the decompositions

$$G = C \circ D\,, \tag{3.1}$$

$$G = \Omega_{i \in I} A_i \tag{3.2}$$

under the assumption that C, D, A_i $(i \in I)$ are directed.

3.1. $D = \Omega_{i \in I}(D \cap A_i) = \Omega_{i \in I} D(A_i)$.

Proof. For $D = \{0\}$ the assertion is trivial. Let $D \neq \{0\}$. Then by 2.14 $D = \Omega_{i \in I}(D \cap A_i)$. According to 2.15 $D \cap A_i = D(A_i)$.

3.2. *Let $\varphi : G \to [\Omega_{i \in I} A_i(C)]$ be a mapping defined by*

$$\varphi(x) = (\ldots, x_i(C), \ldots)_{i \in I}$$

for any $x \in G$. Then the partial map $\varphi_C : C \to [\Omega_{i \in I} A_i(C)]$ is an isomorphism with respect to the group operation.

Proof. Obviously φ is a homomorphism (into) with respect to the group operation. Let $c, c' \in C$, $\varphi(c) = \varphi(c')$. Then $\varphi(c - c') = 0$, hence $(c_i - c'_i)(C) = 0$ and therefore $c_i - c'_i \in D$ for any $i \in I$. Thus by 3.1 $c - c' \in D$. Since $c - c' \in C$, we get $c - c' = 0$. This shows that φ_C is a monomorphism. Let $y \in [\Omega_{i \in I} A_i(C)]$. Then there exist elements $a^i \in A_i$ such that

$$y = (\ldots, a^i(C), \ldots)\,.$$

For $a^i(C) = 0$ we can put $a^i = 0$. If we do so, then each non-empty subset of the set $I_1 = \{i \in I : a^i \neq 0\} = \{i \in I : a^i(C) \neq 0\}$ satisfies the descending chain condition (cf. 1.1). Thus there exists $a \in G$ such that $a_i = a^i$ for each $i \in I$. According to (3.1) $a = c + d$, $c \in C$, $d \in D$. Then we have $a_i = c_i + d_i$, $a_i(C) = c_i(C) + d_i(C)$. By 3.1 $d_i \in D$, hence $d_i(C) = 0$, $a_i(C) = c_i(C)$. Therefore we have $c_i(C) = a^i(C)$ for each $i \in I$ and hence $\varphi(c) = y$.

3.3. *If $x \in C$, $x > 0$, then $\varphi(x) > 0$.*

Proof. Let $x \in C$, $x > 0$. By 3.2 $\varphi(x) \neq 0$. Let $i_1 \in \min I(\varphi(x), 0)$. Hence $x_{i_1}(C) \neq 0$. Assume that there exists $i \in I$ such that $i < i_1$, $x_i \neq 0$. Then there exists $i_2 \leqq i$,

$i_2 \in \min I(x, 0)$. Since $i_2 < i_1$, we have $x_{i_2}(C) = 0$, hence $x_{i_2} \in D \cap A_{i_2}$. From $x_{i_2} \neq 0$ we get $D \cap A_{i_2} \neq \{0\}$. According to 2.12 $D \cap A_{i_2}$ is directed, thus there exists $a \in D \cap A_{i_2}$, $a > 0$. Then $0 \leqq t < a$ for each $t \in (A_{i_1})^+$ and by the convexity of D (cf. 2.1) $(A_{i_1})^+ \subset D$, whence $A_{i_1} \subset D$ and $x_{i_1} \in D$. This implies $x_{i_1}(C) = 0$, a contradiction. Therefore $i_1 \in \min I(x, 0)$. From $x > 0$ we get now $x_{i_1} > 0$. From $x_{i_1} = x_{i_1}(C) + x_{i_1}(D)$, $x_{i_1}(C) \neq 0$ it follows $x_{i_1}(C) > 0$. Hence $\varphi(x) > 0$.

3.4. *Let $c \in C$, $\varphi(c) > 0$. Then $c > 0$.*

Proof. For $i \in I$ we put $d^i = c_i$ or $d^i = 0$ if $c_i(C) = 0$ or $c_i(C) \neq 0$, respectively. There exists $d \in G$ such that $d_i = d^i$ for each $i \in I$. All d_i belong to D, hence by 3.1 $d \in D$. Denote $c - d = c'$. Thus $c'_i = 0$ if and only if $c_i(C) = (\varphi(c))_i = 0$. This implies $\min I(c', 0) = \min I(\varphi(c), 0)$. Let $i \in \min I(c', 0)$. Then $i \in \min I(\varphi(c), 0)$, hence $(\varphi(c))_i > 0$, i.e., $c_i(C) > 0$. Since $c_i = c_i(C) + c_i(D)$, we get $c_i > 0$. From this it follows $d^i = 0$, whence $c'_i = c_i$, $c'_i > 0$. This shows that $c' > 0$. From $c' = c - d$, $c \neq 0$ (this follows from $\varphi(c) \neq 0$) we conclude by (3.1) that $c > 0$ holds.

From 3.2, 3.3 and 3.4 it follows:

3.5. *φ_C is an isomorphism of the partially ordered group C onto $[\Omega_{i \in I} A_i(C)]$.*

For $c \in C$, $i \in I$ denote $c_i(C) = \varphi_i(c)$.

3.6. *Let $i, j \in I$, $i \neq j$, $c \in A_i(C)$. Then $\varphi_i(c) = c$, $\varphi_j(c) = 0$.*

Proof. There exist elements $a \in A_i$, $d \in D$ such that $a = c + d$. From this we obtain $a_i = c_i + d_i$. Since $a \in A_i$, we have $a_i = a$, hence

$$c + d = c_i + d_i . \tag{3.3}$$

According to 3.1 $d_i \in D(A_i) = D \cap A_i$, thus $d_i(C) = 0$. From this and from (3.3) we get $c(C) = c_i(C)$. Since $c(C) = c$, we have $\varphi_i(c) = c$. Further we have $0 = a_j = c_j + d_j$, $0 = c_j(C) + d_j(C)$. But $d_j \in D$ implies $d_j(C) = 0$ and therefore $\varphi_j(c) = c_j(C) = 0$.

According to 1.2 it follows from 3.5 and 3.6:

3.7. Theorem. *If (3.1) and (3.2) are fulfilled, then*

$$C = \Omega_{i \in I} A_i(C) .$$

Now we shall consider another decomposition with two factors

$$G = A \circ B . \tag{3.4}$$

The following two statements were proved in [5] (under more general conditions):

3.8. ([5], 9 and 11.) *Let* (3.1) *and* (3.4) *be valid. Then either* $D \subset B$ *or* $B \subset D$. *If* $D \subset B$, *then* $B = B(C) \circ D$, $B(C) = B \cap C$.

3.9. ([5], 13.4) *Let* $G = A \circ B$, $G = C \circ B$ *hold. The mapping* $f : A \to C$ *defined by* $f(a) = a(C)$ *is an isomorphism of the partially ordered group* A *onto* C.

4. ISOMORPHIC REFINEMENTS

Let us consider the decompsitions α, β (cf. section 2). Throughout the whole paper we will assume that $A_i \neq \{0\}$, $B_j \neq \{0\}$ for each $i \in I$ and each $j \in J$. Let $j_0 \in J$ be fixed. By 2.14

$$B(j_0) = \Omega_{i \in I}(B(j_0) \cap A_i) . \tag{4.1}$$

Obviously

$$B(j_0) = B_{j_0} \circ B'(j_0) . \tag{4.2}$$

Then according to 3.7 we have

$$B_{j_0} = \Omega_{i \in I}((B(j_0) \cap A_i)(B_{j_0})) . \tag{4.3}$$

For any $i \in I$, $j \in J$ denote

$$(B(j) \cap A_i)(B_j) = C_{ji} . \tag{4.3'}$$

From the decomposition β and from (4.3) it follows

$$G = \Omega_{j \in J}\, \Omega_{i \in I} C_{ji} . \tag{4.3''}$$

The right hand side member of (4.3″) can be written in the form $\Omega C_{ji}((j, i) \in J \circ I)$. If we denote $(J \circ I)' = \{(j, i) \in J \circ I : C_{ji} \neq \{0\}\}$ (cf. 1.6), then we can write

$$G = \Omega C_{ji}((j, i) \in (J \circ I)') . \tag{4.4}$$

4.1. *Let* $(j_0, i_0) \in (J \circ I)'$. *The partially ordered group* $C_{j_0 i_0}$ *is directed.*

Proof. Let $x \in C_{j_0 i_0}$, $x \mid 0$. Then there exists $a \in B(j_0) \cap A_{i_0}$ such that $a_{j_0} = x$. By (4.2) $a \mid 0$. According to 2.12 there exists $a^1 \in B(j_0) \cap A_{i_0}$ such that $a^1 > 0$, $a^1 > a$. Using (4.2) once more we get $a = a_{j_0} + a'_{j_0}$, $a^1 = a^1_{j_0} + (a^{1'})_{j_0}$ where $a'_{j_0}, (a^{1'})_{j_0} \in B'(j_0)$. Now the relation $a \mid 0$ implies $a^1_{j_0} > x$, $a^1_{j_0} > 0$.

Let us consider the partially ordered set $(J \circ I)'$.

4.2. *Let* $(j_1, i_1) \in (J \circ I)'$, $j_2 < j_1$, $i_2 > i_1$. *Then* $(j_2, i_2) \notin (J \circ I)'$.

Proof. Let us suppose that $(j_2, i_2) \in (J \circ I)'$ holds. Then by 4.1 there exist elements

$x \in C_{j_1 i_1}$, $y \in C_{j_2 i_2}$, $x > 0$, $y > 0$. From this it follows that there exist elements $a \in B(j_1) \cap A_{i_1}$, $b \in B(j_2) \cap A_{i_2}$ such that

$$(4.5) \qquad a_{j_1} = x\,, \quad b_{j_2} = y\,.$$

Since $a \in B(j_1)$, $a_{j_1} > 0$, we have $\min J(a, 0) = \{j_1\}$, hence $a > 0$. Analogously, $\min J(b, 0) = \{j_2\}$, $b_{j_2} > 0$, hence $b > 0$. From $j_2 < j_1$ we get $\{j_2\} = \min J(a, b)$, and since $a_{j_2} = 0 < b_{j_2}$, $b > a$ holds. Moreover, since $a \in A_{i_1}$, $b \in A_{i_2}$ and since $i_1 < i_2$, $a > b$ is true; a contradiction.

4.3. *Let $(j_1, i_1) \in (J \circ I)'$, $j_2 < j_1$, $i_1 \mid i_2$. Then $(j_2, i_2) \notin (J \circ I)'$.*

Proof. Assume that $(j_2, i_2) \in (J \circ I)'$ and let x, y, a, b have the same meaning as in the proof of Lemma 4.2. From $j_2 < j_1$ we get $b > a$ nad from $i_1 \mid i_2$ it follows $a \mid b$, which is a contradiction.

4.4. *Let $(j_1, i_1) \in (J \circ I)'$, $i_1 < i_2$, $j_1 \mid j_2$. Then $(j_2, i_2) \notin (J \circ I)'$.*

Proof. Let us suppose that $(j_2, i_2) \in (J \circ I)'$. Under the same notation as in the proof of 4.2 we have $a > b$. Since $\min J(a, b) = \{j_1, j_2\}$ and $a_{j_1} > 0 = b_{j_1}$, $b_{j_2} > 0 = a_{j_2}$, $a \mid b$ holds; a contradiction.

From 4.2, 4.3 and 4.4 it follows:

4.5. *Let $(j_1, i_1), (j_2, i_2) \in (J \circ I)'$, $j_1 \neq j_2$, $i_1 \neq i_2$. Let $s \in \{<, >, |\}$. Then $j_1 \, s \, j_2 \Leftrightarrow i_1 \, s \, i_2$.*

Let us now denote $(I \circ J)^* = \{(i, j) \in I \circ J : (j, i) \in (J \circ I)'\}$ and consider the transformation $\chi : (j, i) \to (i, j)$ of the set $(J \circ I)'$ onto $(I \circ J)^*$.

4.6. *χ is an isomorphism with respect to the partial order.*

Proof. Let $(j_1, i_1), (j_2, i_2) \in (J \circ I)'$. If $j_1 \neq j_2$, $i_1 \neq i_2$, then by 4.5

$$(4.6) \qquad (j_1, i_1) < (j_2, i_2) \Leftrightarrow (i_1, j_1) < (i_2, j_2)\,.$$

If $j_1 = j_2$ or $i_1 = i_2$, then (4.6) obviously holds.

By changing the roles of A_i and B_j, we get analogously as in (4.4)

$$(4.7) \qquad G = \Omega E_{ij}((i, j) \in (I \circ J)')$$

where

$$(4.7') \qquad E_{ij} = (A(i) \cap B_j)\,(A_i)\,,$$
$$(I \circ J)' = \{(i, j) \in I \circ J : E_{ij} \neq \{0\}\}\,.$$

Now we intend to prove that C_{ji} and E_{ij} are isomorphic. Let $i_0 \in I$, $j_0 \in J$ be fixed elements and denote

$$X = A(i_0) \cap B(j_0),$$
$$D = \Omega_{i > i_0}(A_i \cap X),$$
$$D' = \Omega_{j > j_0}(B_j \cap X).$$

4.7. $X = \Omega_{i \in I}(A_i \cap X) = \Omega_{i \geqq i_0}(A_i \cap X) = (A_{i_0} \cap X) \circ D$.

Proof. It suffices to prove the first equality, since $A_i \cap X = \{0\}$ for $i \ngeqq i_0$. Let $x \in X$. Then $x \in B(j_0)$, hence by 2.4 (and replacing A_i and B_j) $x_i \in B(j_0)$. Since $x \in A(i_0)$, obviously $x_i \in A(i_0)$, thus $x_i \in A_i \cap X$. We get $x \in \Omega_{i \in I}(A_i \cap X)$. Conversely, let $x \in G$ and let $x_i \in A_i \cap X$ for each $i \in I$. This implies $x \in A(i_0)$ and according to 2.13 $x \in B(j_0)$, hence $x \in X$.

Analogously we have

$$X = \Omega_{j \in J}(B_j \cap X) = \Omega_{j \geqq j_0}(B_j \cap X) = (B_{j_0} \cap X) \circ D'.$$

4.7.1. $D = A'(i_0) \cap X$.

Proof. Let $x \in D$. Then, clearly, $x \in A'(i_0)$. By 4.7 $x \in X$, hence $x \in A'(i_0) \cap X$. Conversely, let $x \in A'(i_0) \cap X$. By 4.7 $x_i \in A_i \cap X$ for each $i \in I$; moreover, $x_i = 0$ for any $i \ngtr i_0$. From this it follows $x \in D$.

Analogously $D' = B'(j_0) \cap X$.

4.8. $X = E_{i_0 j_0} \circ (D \cup D')$.

Proof. Consider the decompositions

$$X = (X \cap A_{i_0}) \circ D, \quad X = (X \cap B_{j_0}) \circ D'.$$

By 3.7

$$(X \cap B_{j_0})(X \cap A_{i_0}) = (A(i_0) \cap B_{j_0})(A_{i_0}) = E_{i_0 j_0},$$
$$X = E_{i_0 j_0} \circ [D'(X \cap A_{i_0})] \circ D.$$

According to 4.7 and 3.8 either $D \subset D'$ or $D' \subset D$. In the first case we have by 3.8

$$D' = [D'(X \cap A_{i_0})] \circ D,$$

hence $X = E_{i_0 j_0} \circ D' = E_{i_0 j_0} \circ (D \cup D')$. In the latter case, by 4.7

$$D'(X \cap A_{i_0}) \subset D(A_{i_0} \cap X) = \{0\}.$$

From this it follows

$$X = E_{i_0 j_0} \circ \{0\} \circ D = E_{i_0 j_0} \circ (D \cup D').$$

Replacing A_i by B_j we get

4.8.1. $X = C_{j_0 i_0} \circ (D \cup D')$.

4.9. *The partially ordered groups $C_{j_0 i_0}$ and $E_{i_0 j_0}$ are isomorphic.*
This follows from 4.8, 4.8.1 and 3.9.

4.10. Theorem. *Let two decompositions of a partially ordered group*

$$(\alpha) \quad G = \Omega_{i \in I} A_i \,, \qquad (\beta) \quad G = \Omega_{j \in J} B_j$$

be given, where all factors A_i, B_j are directed and distinct from $\{0\}$. Then the decomposition

$$(\gamma) \quad G = \Omega E_{ij}((i, j) \in (I \circ J)')$$

is a refinement of α and the decomposition

$$(\delta) \quad G = \Omega C_{ji}((j, i) \in (J \circ I)')$$

is a refinement of β (E_{ij} and C_{ij} being defined by (4.7′) and (4.3′), respectively). The decompositions γ and δ are isomorphic.

Proof. From the construction of δ it follows that δ is a refinement of β; analogously, γ is a refinement of α. By 4.9 $(I \circ J)^* = (I \circ J)'$, hence by 4.6 $\chi : (j, i) \to (i, j)$ is an isomorphism of the partially ordered set $(J \circ I)'$ onto $(I \circ J)'$. Since, by 4.9, C_{ji} and E_{ij} are isomorphic, the proof is complete.

5. EQUIVALENT DECOMPOSITIONS

In this section we shall consider pairs of decompositions α, β which are reproduced by the construction of isomorphisms from the theorem 4.10, i.e., for which $\gamma = \alpha$, $\delta = \beta$ is fulfilled.

Let α and β have the same meaning as in Theorem 4.10 and let the suppositions of this theorem be satisfied. The decompositions α and β are said to be equivalent (this fact we denote by $\alpha \sim \beta$) if there exists an isomorphism ψ of the partially ordered set I onto J such that

$$A(i) = B(\psi(i))\,, \tag{5.1}$$

$$A'(i) = B'(\psi(i)) \tag{5.2}$$

holds for each $i \in I$.

5.1. *Equivalent decompositions are isomorphic.*

Proof. Let $\alpha \sim \beta$. From

$$A_i \circ A'(i) = A(i) = B(\psi(i)) = B_{\psi(i)} \circ B'(\psi(i)) = B_{\psi(i)} \circ A'(i)$$

and from 3.9 it follows that the partially ordered groups A_i and $B_{\psi(i)}$ are isomorphic.

5.1.1. Remark. Two isomorphic decompositions α, β need not be equivalent.

Example: Let K be the set of all real numbers. For any $k \in K$ let G_k be the additive group of all integers (with the natural ordering). Put $G = [\Omega_{k \in K} G_k]$. Let I and J be the set of all even integers or odd integers, respectively, and for any $i \in I, j \in J$ put

$$A_i = \{x \in G : x_k = 0 \text{ for } k \notin [i, i + 2)\},$$
$$B_j = \{x \in G : x_k = 0 \text{ for } k \notin [j, j + 2)\}.$$

Then the following decompsitions hold:

$$(\alpha) \quad G = \Omega_{i \in I} A_i, \qquad (\beta) \quad G = \Omega_{j \in J} B_j.$$

Consider the transformation $\psi(i) = i + 1$. ψ is an isomorphism of I onto J and the partially ordered groups A_i and $B_{\psi(i)}$ are isomorphic. Thus the decompositions α and β are isomorphic. α and β are not equivalent, since $A(i) \neq B(j)$ for any $i \in I$ and any $j \in J$.

Let us now consider the decompositions γ, δ from 4.10. For $(i, j) \in (I \circ J)'$ and $(j, i) \in (J \circ I)'$ let the symbols $E(i, j)$, $E'(i, j)$ or $C(j, i)$, $C'(j, i)$ have analogous meaning as $A(i)$, $A'(i)$ (for example, if $(i_0, j_0) \in (I \circ J)'$, then $E(i_0, j_0) = \Omega E_{ij}((i, j) \in (I \circ J)', (i, j) \geqq (i_0, j_0))$.

5.2. *Let $(i, j) \in (I \circ J)'$. Then $C(j, i) = E(i, j) = X$, $C'(j, i) = E'(i, j) = D \cup D'$ (where X, D, D' are the same as in Section 4 for $i = i_0$, $j = j_0$).*

Proof. Since $X = A(i) \cap B(j)$ and since $A(i)$, $B(j)$ are convex subsets of G, X is a convex subset of G as well. Moreover, by 4.8 $X = E_{ij} \circ (D \cup D')$. According to 4.1 there exist strictly positive elements in E_{ij}. Thus by 2.17 $E(i, j) = X$, $E'(i, j) = D \cup D'$. By the same argument we can prove $C(j, i) = X$, $C'(j, i) = D \cup D'$.

5.2.1. *If $(i, j) \in (I \circ J)'$, then $E'(i, j) = [A'(i) \cup B'(j)] \cap E(i, j)$.*

Proof. According to 5.2 and 4.7.1 we have

$$E'(i, j) = D \cup D' = (A'(i) \cap X) \cup (B'(j) \cap X) = [A'(i) \cup B'(j)] \cap E(i, j).$$

5.2.2. *If $(i, j) \in (I \circ J)'$, then $A'(i) \subset E'(i, j)$, $E(i, j) \subset A(i)$.*

This follows from 2.17.2 and from the fact that E_{ij} is a factor in A_i.

Let us denote $\gamma = f(\alpha, \beta)$. Under this notation, clearly, $\delta = f(\beta, \alpha)$.

5.3. $f(\alpha, \beta) \sim f(\beta, \alpha)$.

This follows from 5.2, since the mapping $(j, i) \to (i, j)$ is an isomorphism of $(J \circ I)'$ onto $(I \circ J)'$.

Now we can formulate a strengthened version of Theorem 4.10 (cf. 5.1 and 5.1.1):

5.4. *The decompositions α and β have equivalent refinements.*

A new characterization of equivalent decompositions is given by

5.5. Theorem. $\alpha \sim \beta \Leftrightarrow f(\alpha, \beta) = \alpha, f(\beta, \alpha) = \beta$.

Proof. Assume that $\alpha \sim \beta$. Then according to 5.1 it can be supposed that $J = I$ and $A(i) = B(i)$, $A'(i) = B'(i)$ for each $i \in I$. By (4.7') we have

$$E_{ii} = (A(i) \cap B_i)(A_i) = (B(i) \cap B_i)(A_i) = B_i(A_i)\,. \tag{5.3}$$

From $A_i \circ A'(i) = B_i \circ B'(i) = B_i \circ A'(i)$ and from 3.9 it follows $B_i(A_i) = A_i$. Thus by (5.3) $A_i = E_{ii}$. Since $A_i = \Omega E_{ij}\ (j \in I)$, $E_{ii} \cap E_{ij} = \{0\}$ for any $j \in I, j \neq i$. Hence $E_{ij} = E_{ij} \cap A_i = E_{ij} \cap E_{ii} = \{0\}$ for each $j \neq i$. This shows that $f(\alpha, \beta) = \alpha$. Analogously, $f(\beta, \alpha) = \beta$. Conversely, if $f(\alpha, \beta) = \alpha, f(\beta, \alpha) = \beta$, then by 5.3 $\alpha \sim \beta$.

From 5.3 and 5.5 it follows:

5.6. $f(f(\alpha, \beta), f(\beta, \alpha)) = f(\alpha, \beta)$.

Let α_1 and β_1 be decompositions of G such that all factors occuring in these decompositions are directed and non-trivial.

5.7. *If $\alpha_1 \sim \alpha$, $\beta_1 \sim \beta$, then $f(\alpha_1, \beta_1) \sim f(\alpha, \beta)$.*

Proof. Let $\alpha_1 \sim \alpha$. Then α_1 can be written in the form

$$(\alpha_1) \quad G = \Omega_{i \in I} A_i^1$$

where $A^1(i) = A(i)$, $A^{1\prime}(i) = A'(i)$ for each $i \in I$. Let us denote by E_{ij}^1 the factors of the decomposition $f(\alpha_1, \beta)$. Then by 5.2

$$E^1(i, j) = A^1(i) \cap B(j) = A(i) \cap B(j) = E(i, j)\,.$$

From this and from $A'(i) = A^{1\prime}(i)$ with regard to 5.2.1 we get $E^{1\prime}(i, j) = E'(i, j)$. This proves that $f(\alpha, \beta) \sim f(\alpha_1, \beta)$. Analogously, $f(\alpha_1, \beta) \sim f(\alpha_1, \beta_1)$. The relation $\sim$ being transitive, $f(\alpha, \beta) \sim f(\alpha_1, \beta_1)$.

5.8. *The following conditions are equivalent:*

(a) $f(\alpha, \beta) = \alpha$,

(b) *to each $i \in I$ there exists an element $\psi(i) \in J$ such that $B'(\psi(i)) \subset A'(i)$, $A(i) \subset B(\psi(i))$.*

Proof. Let (a) be fulfilled. Since $A_i = \Omega_{j\in J} E_{ij}$, there exists $j_1 \in J$ such that $A_i = E_{ij_1}$ and $E_{ij} = \{0\}$ for each $j \in J$, $j \neq j_1$. Put $j_1 = \psi(i)$. With the aid of 5.2.2,

$$B'(j_1) \subset C'(j_1, i)\,, \quad C(j_1, i) \subset B(j_1)\,.$$

Moreover, by 5.2 we have $E'(i, j_1) = C'(j_1, i)$, $E(i, j_1) = C(j_1, i)$. From $A_i = E_{ij_1}$ it follows $E'(i, j_1) = A'(i)$, $E(i, j_1) = A(i)$; hence (b) holds.

Conversely, let us suppose that (b) is true. Let $i \in I$ be fixed and denote $\psi(i) = j$. Obviously $E_{ij} = (A(i) \cap B_j)(A_i) \subset A_i$. Let $a \in A_i$. According to (b) $a \in B(j)$, hence $a = x + y$, $x \in B_j$, $y \in B'(j)$. Since $x = a_j$ and $a \in A(i)$, it follows from 2.4 that $x \in A(i)$, whence $x \in A(i) \cap B_j$. Moreover, by (b) $y \in A'(i)$, thus $y(A_i) = 0$. Therefore

$$a = a(A_i) = x(A_i) + y(A_i) = x(A_i) \in (A(i) \cap B_j)(A_i) = E_{ij}\,.$$

This implies $E_{ij} = A_i$. Then we have $E_{ij_1} = \{0\}$ for any $j_1 \in J$, $j_1 \neq j$; thus $f(\alpha, \beta) = \alpha$.

5.9. *Let α be a refinement of β. Then $f(\alpha, \beta) = \alpha$.*

Proof. Let $i \in I$. There exists $j_1 \in J$ such that A_i is a factor of B_{j_1}. Hence according to 2.17.2 $B'(j_1) \subset A'(i)$, $A(i) \subset B(j_1)$. Therefore by 5.8, $f(\alpha, \beta) = \alpha$.

Remark. From $f(\alpha, \beta) = \alpha$ it does not follow that α is a refinement of β.

Example: Let G be the set of all pairs (x, y) of real numbers with the group operation $+$ that is performed component-by-component and with the lexicographic order. Put $A = \{(x, y) \in G : y = 0\}$, $B = \{(x, y) \in G : x = 0\}$, $C = \{(x, y) \in G : x = y\}$. Then we have the decompositions (α) $G = A \circ B$, (β) $G = C \circ B$. The decompositions α and β are equivalent, hence $f(\alpha, \beta) = \alpha$, but neither α is a refinement of β nor β is a refinement of α. It is easy to see that α and β have no common refinement.

6. THE PARTIALLY ORDERED SET $\bar{\mathscr{G}}$

Let $G \neq \{0\}$ be a partially ordered group. Let $\mathscr{G}$ be the set of all mixed product decompositions α of G such that each factor occuring in α is directed and non-trivial. By $\bar{\mathscr{G}}$ we shall denote the system of all classes of the partition of the set $\mathscr{G}$ that is defined by the equivalence relation $\sim$. For $\alpha \in \mathscr{G}$ we put $\bar{\alpha} = \{\alpha_1 \in G : \alpha_1 \sim \alpha\}$ and for $\bar{\alpha}, \bar{\beta} \in \bar{\mathscr{G}}$ we put $\bar{\alpha} \leqq \bar{\beta}$ if and only if there exist elements $\alpha_1 \in \bar{\alpha}$, $\beta_1 \in \bar{\beta}$ such that α_1 is a refinement of β_1.

6.1. *Let $\bar{\alpha}, \bar{\beta} \in \bar{\mathscr{G}}$. Then $\bar{\alpha} \leqq \bar{\beta}$ if and only if $f(\alpha, \beta) = \alpha$.*

Proof. Let $\bar{\alpha} \leqq \bar{\beta}$. Then there exist elements $\alpha_1, \beta_1 \in \mathscr{G}$ such that $\alpha_1 \in \bar{\alpha}$, $\beta_1 \in \bar{\beta}$ and α_1 is a refinement of β_1. According to 5.9 $f(\alpha_1, \beta_1) = \alpha_1$, hence α_1 and β_1 satisfy the condition (b) of Lemma 5.8. Since $\alpha \sim \alpha_1$, $\beta \sim \beta_1$, the condition (b) holds for the decompositions α and β as well. Therefore by 5.8 $f(\alpha, \beta) = \alpha$. Conversely, let $f(\alpha, \beta) = \alpha$ be fulfilled. According to 5.3 $f(\alpha, \beta) \sim f(\beta, \alpha)$ and $f(\beta, \alpha)$ is a refinement of β, thus $\bar{\alpha} \leqq \bar{\beta}$.

6.2. *$(\mathscr{G}, \leqq)$ is a partially ordered set.*

Proof. The relation $\leqq$ is reflexive. Let $\bar{\alpha} \leqq \bar{\beta}$, $\bar{\beta} \leqq \bar{\gamma}$ where γ has the form

$$(\gamma) \quad G = \Omega_{k \in K} F_k .$$

Then by 6.1 and 5.8 the condition (b) of 5.8 holds and to each $j \in J$ there exists $\chi(j) \in K$ such that

$$F'(\chi(j)) \subset B'(j), \quad B(j) \subset F(\chi(j)) .$$

From this it follows

$$F'(\chi(\psi(i))) \subset A'(i), \quad A(i) \subset F(\chi(\psi(i))) ,$$

hence by 6.1 and 5.8 $\bar{\alpha} \leqq \bar{\gamma}$. If $\bar{\alpha} \leqq \bar{\beta}$, $\bar{\beta} \leqq \bar{\alpha}$, then by 6.1 $f(\alpha, \beta) = \alpha$, $f(\beta, \alpha) = \beta$, and thus, according to 5.3, $\bar{\alpha} = \bar{\beta}$.

For $\bar{\alpha}, \bar{\beta} \in \mathscr{G}$ put $f(\bar{\alpha}, \bar{\beta}) = \overline{f(\alpha, \beta)}$ (by 5.7, $\overline{f(\alpha, \beta)}$ does not depend on the choice of $\alpha \in \bar{\alpha}$, $\beta \in \bar{\beta}$).

6.3. *Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathscr{G}$, $\bar{\alpha} \leqq \bar{\beta}$. Then $f(\bar{\gamma}, \bar{\alpha}) \leqq f(\bar{\gamma}, \bar{\beta})$.*

Proof. We can suppose that α is a refinement of β. The factors on the decomposition $f(\gamma, \alpha)$ are

$$(F(k) \cap A_i)(F_k) = T_{ki}$$

and, analogously, the factors of $f(\gamma, \beta)$ are

$$(F(k) \cap B_j)(F_k) = S_{kj} .$$

For each $i \in I$ there exists $\psi(i) \in J$ such that A_i is a factor of B_j, hence $A_i \subset B_{\psi(i)}$. Therefore we have $T_{ki} \subset S_{k\psi(i)}$. Thus by 2.17.2

$$S'(k, \psi(i)) \subset T'(k, i), \quad T(k, i) \subset S(k, \psi(i)) .$$

According to 5.8 and 6.1 this implies $\overline{f(\gamma, \alpha)} \leqq \overline{f(\gamma, \beta)}$.

6.3.1. *Under the same assumptions as in 6.3 $f(\bar{\alpha}, \bar{\gamma}) \leqq f(\bar{\beta}, \bar{\gamma})$ holds.*

Proof. The assertion follows from 6.3 and from $f(\bar{\alpha}, \bar{\gamma}) = f(\bar{\gamma}, \bar{\alpha})$, $f(\bar{\beta}, \bar{\gamma}) = f(\bar{\gamma}, \bar{\beta})$ (cf. 5.3).

6.4. Theorem. $f(\bar{\alpha}, \bar{\beta}) = \bar{\alpha} \wedge \bar{\beta}$ *for any* $\bar{\alpha}, \bar{\beta} \in \bar{\mathscr{G}}$.

Proof. Since $f(\alpha, \beta)$ is a refinement of α, we have $f(\bar{\alpha}, \bar{\beta}) \leqq \bar{\alpha}$. Analogously, $f(\bar{\beta}, \bar{\alpha}) \leqq \bar{\beta}$ and thus according to 5.3 $f(\bar{\alpha}, \bar{\beta}) \leqq \bar{\beta}$. Let $\bar{\gamma} \leqq \bar{\alpha}$, $\bar{\gamma} \leqq \bar{\beta}$. Then by 6.3 and 6.3.1 $f(\bar{\alpha}, \bar{\beta}) \geqq f(\bar{\gamma}, \bar{\beta}) \geqq f(\bar{\gamma}, \bar{\gamma}) = \overline{f(\gamma, \gamma)} = \bar{\gamma}$.

Let $\alpha, \beta \in \mathscr{G}$ and let us denote $\gamma = f(\alpha, \beta)$. Now we intend to construct a new decomposition ε of the form

$$G = \Omega_{t \in T} F_t \tag{6.1}$$

such that $\bar{\varepsilon} = \bar{\alpha} \vee \bar{\beta}$ be valid.

We define a binary relation $\approx$ on the set $(I \circ J)'$ as follows: $(i, j) \approx (i', j')$ if there exists a finite sequence of elements of the set $(I \circ J)'$

$$(i, j) = (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n) = (i', j')$$

such that either $i_s = i_{s+1}$ or $j_s = j_{s+1}$ holds for $s = 1, \ldots, n - 1$. Obviously $\approx$ is an equivalence relation on the set $(I \circ J)'$; the class of the corresponding partition that contains the element (i, j) will be denoted by $t(i, j)$ and let T be the system of all such classes. For $t(i_1, j_1), t(i, j) \in T$ we put $t(i_1, j_1) < t(i, j)$, if $i_2 < i_3$ and $j_2 < j_3$ holds for each element $(i_2, j_2) \in t(i_1, j_1)$ and each $(i_3, j_3) \in t(i, j)$. The relation $<$ determines a partial order on the set T.

6.5. *Let* $(i_1, j_1), (i_2, j_2) \in (I \circ J)'$, $i_1 < i_2$, $j_1 < j_2$, $t(i_1, j_1) \neq t(i_2, j_2)$. *Then* $t(i_1, j_1) < t(i_2, j_2)$.

Proof. Let $(i_1, j_3) \in (I \circ J)'$. Consider the elements (i_1, j_3), (i_2, j_2). If $j_3 = j_2$, then $(i_1, j_1) \approx (i_2, j_2)$, hence $t(i_1, j_1) = t(i_2, j_2)$, a contradiction. Thus $j_3 \neq j_2$ holds. Since $i_1 < i_2$, it follows by 4.5 that $j_3 < j_2$. Analogously we can prove: if $(i_3, j_1) \in$ $\in (I \circ J)'$, then $i_3 < i_2$. From this we get by induction that $i_4 < i_2, j_4 < j_2$ is true for each element $(i_4, j_4) \in t(i_1, j_1)$. In a similar manner it can be proved that $i_4 < i_5$, $j_4 < j_5$ for any $(i_5, j_5) \in t(i_2, j_2)$.

For a fixed $t_0 = t(i_0, j_0) \in T$ we denote

$$F_{t_0} = \Omega E_{ij}((i, j) \in t(i_0, j_0))\,; \tag{6.2}$$

further we put

$$H = [\Omega_{t \in T} F_t]\,.$$

Let $t_0 = t(i_0, j_0) \in T$, $g \in G$. We shall denote by g_{t_0} the element of F_{t_0} satisfying $g(E_{ij}) = g_{t_0}(E_{ij})$ for each $(i, j) \in t_0$. Clearly there exists exactly one element of F_{t_0} fulfilling this condition. For each $t \in T$ consider the mapping $\varphi_t : G \to F_t$ defined by $\varphi_t(g) = g_t$ for any $g \in G$. If $g \in F_t$, $t' \in T$, $t' \neq t$, then $\varphi_t(g) = g$, $\varphi_{t'}(g) = 0$.

For completing the proof that (6.1) is valid it remains to show that the mapping

$$\varphi(g) = (\ldots, g_t, \ldots)\,(t \in T)$$

is an isomorphism of the partially ordered group G onto H (cf. 1.2).

Let $g \in G$ and consider the decomposition $\gamma = f(\alpha, \beta)$. Since $(I \circ J)'(g)$ satisfies the descending chain condition, according to 6.5 the set $\{t \in T : g_t \neq 0\}$ fulfils this condition, too. From this it follows $\varphi(g) \in H$ for each $g \in G$. Clearly φ is a homomorphism with respect to the group operation. Let $h = (\ldots, h^t, \ldots) \in H$. For any $(i_0, j_0) \in (I \circ J)'$ put $t_0 = t(i_0, j_0)$ and denote $h^{t_0}(E_{i_0 j_0}) = h^{i_0 j_0}$. Let $M = \{(i, j) \in (I \circ J)' : h^{ij} \neq 0\}$, $(i_n, j_n) \in M$ $(n = 1, 2, 3, \ldots)$, $(i_1, j_1) \geqq (i_2, j_2) \geqq \ldots$ According to 6.5 and 4.5 we have then $t_1 \geqq t_2 \geqq \ldots$ where $t(i_n, j_n) = t_n \in T(h)$. Since this set satisfies the descending chain condition, there exists a positive integer m such that $t_n = t_m$ for $n \geqq m$. Hence $h^{t_n} = h^{t_m}$, $h^{i_n j_n} = h^{t_m}(E_{i_n j_n})$ for $n \geqq m$. Since $h^{t_m} \in F_{t_m} \subset G$, the set $M_1 = (I \circ J)'(h^{t_m})$ satisfies the descending chain condition and $(i_n, j_n) \in M_1$ for $n \geqq m$. Thus there exists a positive integer $m_1 \geqq m$ such that $(i_n, j_n) = (i_{m_1}, j_{m_1})$ for $n \geqq m_1$. This proves that M fulfils the descending chain condition and there exists an element $g \in G$ satisfying $g_{ij} = h^{ij}$ for each $(i, j) \in (I \circ J)'$; then $\varphi(g) = h$ holds. If $g \in G$, $\varphi(g) = 0$, then $g_t = 0$ for each $t \in T$, hence for $(i, j) \in t$ we have $g(E_{ij}) = g_t(E_{ij}) = 0$; this implies $g = 0$.

Let $g \in G$, $g > 0$, $\varphi(g) = (\ldots, g_t, \ldots) = h$. Let $(i_0, j_0) \in \min (I \circ J)'(g_t, 0)$. Then $t_0 = t(i_0, j_0) \in T(h, 0)$ and $(i_0, j_0) \in \min (I \circ J)'(g, 0)$. This implies $g_{i_0 j_0} > 0$, hence $g_{t_0} > 0$, $h > 0$. Conversely, let $h > 0$ and let $(i_0, j_0) \in \min (I \circ J)'(g, 0)$. Then $t_0 = t(i_0, j_0) \in \min T(h, 0)$, $(i_0, j_0) \in \min (I \circ J)'(g_{t_0}, 0)$, thus $g_{t_0} > 0$ and $g_{i_0 j_0} > 0$. Therefore $g > 0$ holds.

We have proved that (6.1) is valid. Let us denote this decomposition by $\varepsilon = f_1(\alpha, \beta)$. Since E_{ij} are directed nontrivial factors, each F_t is directed and nontrivial, hence ε belongs to $\mathscr{G}$.

6.6. *The decomposition α is a refinement of $f_1(\alpha, \beta)$.*

Proof. For any $i_0 \in I$ and any $j_1, j_2 \in J$ such that $(i_0, j_1), (i_0, j_2) \in (I \circ J)'$ we have $(i_0, j_1) \approx (i_0, j_2)$, hence

$$A_{i_0} = \Omega_{j \in J} E_{i_0 j} \subset \Omega E_{ij}((i, j) \in t_0) = F_{t_0}$$

where $t(i_0, j_1) = t_0$.

6.7. $\bar{\beta} \leqq \bar{\varepsilon}$.

Proof. According to 6.1 and 5.8 it suffices to verify that for each $j_0 \in J$ there exists $t_0 = \psi(j_0) \in T$ such that $F'(\psi(j_0)) \subset B'(j_0)$, $B(j_0) \subset F(\psi(j_0))$. Since $B_{j_0} \neq \{0\}$, by (4.3) there exists $i_0 \in I$ such that $C_{j_0 i_0} \neq \{0\}$. By 4.9 we have $E_{i_0 j_0} \neq \{0\}$, hence $(i_0, j_0) \in (I \circ J)'$. Let one such i_0 be fixed and denote $\psi(j_0) = t_0 = t(i_0, j_0)$.

Let $x \in F'(\psi(j_0))$ and let $(i_1, j_0) \in (I \circ J)'$. Then $(i_1, j_0) \in t_0$, hence by (6.2) $E_{i_1 j_0} \subset \subset F_{t_0}$ and therefore according to 2.17.2, $F'(t_0) \subset E'(i_1, j_0)$. By 5.2 $E'(i_1, j_0) = = C'(j_0, i_1)$. Thus we have

$$x(C_{j_0 i}) = 0 \tag{6.3}$$

for each $i \in I$. (If $(i, j_0) \notin (I \circ J)'$, then $E_{ij_0} = \{0\}$, $C_{j_0 i} = \{0\}$ and $x(C_{j_0 i}) = 0$.) Clearly $C'(j_0, i) \subset C(j_0, i)$ and by 2.17.2 $C(j_0, i) \subset B(j_0)$, hence $x \in B(j_0)$. Thus $x_j = 0$ for any $j \ngeqq j_0$. Let us now consider the component x_{j_0}. By the construction of the decomposition (4.3″) for any $z \in G$ we compute $z(C_{j_0 i})$ as follows: we find at first the element $z(B_{j_0}) = z_{j_0}$ and then we construct the component of z_{j_0} in $C_{j_0 i}$ with respect to the decomposition (4.3); hence $z(C_{j_0 i}) = z_{j_0}(C_{j_0 i})$. By (6.3) $x(C_{j_0 i}) = 0$ for each $i \in I$, thus $x_{j_0}(C_{j_0 i}) = 0$ for each $i \in I$. From this we get $x_{j_0} = 0$ according to (4.3), hence $x_j = 0$ for any $j \in J$, $j \ngtr j_0$. This proves that $x \in B'(j_0)$.

Let $x \in B(j_0)$ and let $j \in J$, $i \in I$, $x(C_{ji}) \neq 0$. Since $x(C_{ji}) = x_j(C_{ji})$, we have $x_j \neq 0$, hence $j \geqq j_0$. Put $t = t(i, j)$. If $t \neq t_0$, then $j > j_0$ and by 4.5 $i > i_0$, thus $t > t_0$. Further we have $x(C_{ji}) \in C_{ji} \subset C(j, i) = E(i, j)$, and since $(i, j) \in t$, $E_{ij} \subset F_t$, by 2.17.2 $E(i, j) \subset F(t) \subset F(t_0)$. Therefore $x(C_{ji}) \in F(t_0)$ for each $i \in I$ and each $j \in J$. Then by 2.13, $x \in F(t_0)$ holds.

6.8. *Suppose that the decomposition*

$$(\varkappa)\; G = \Omega_{s \in S} H_s$$

belongs to $\mathscr{G}$ *and that* $\bar{\varkappa} \geqq \bar{\alpha}$, $\bar{\varkappa} \geqq \bar{\beta}$. *Then* $\bar{\varkappa} \geqq \bar{\varepsilon}$.

Proof. Let $t_0 \in T$, $(i, j) \in t_0$. By 6.1 and 5.8 there exist elements $s_1, s_2 \in S$ such that

$$H'(s_1) \subset A'(i)\,, \quad A(i) \subset H(s_1)\,,$$
$$H'(s_2) \subset B'(j)\,, \quad B(j) \subset H(s_2)\,.$$

At the same time we have

$$A'(i) \subset E'(i, j)\,, \quad E(i, j) \subset A(i)\,,$$
$$B'(j) \subset E'(i, j)\,, \quad E(i, j) \subset B(j)\,.$$

Any $x \in E_{ij}$, $x \neq 0$ belongs to $E(i, j) \setminus E'(i, j)$, hence

$$x \in [H(s_1) \setminus H'(s_1)] \cap [H(s_2) \setminus H'(s_2)]\,.$$

According to 2.18 $s_1 = s_2$. If $E_{ij_1} \neq \{0\}$ or $E_{i_1 j} \neq \{0\}$, then, as we have already proved,

$$H'(s_1) \subset E'(i, j_1)\,, \quad E(i, j_1) \subset H(s_1)\,,$$
$$H'(s_1) \subset E'(i_1, j)\,, \quad E(i_1, j) \subset H(s_1)\,.$$

By induction we get

$$(6.4) \qquad H'(s_1) \subset E'(i_2, j_2), \quad E(i_2, j_2) \subset H(s_1)$$

for any $(i_2, j_2) \approx (i, j)$. Let $x \in F_{t_0}$. For each $(i_2, j_2) \in t_0$ we have by (6.4) $x \in H(s_1)$. According to (6.2), for $(i_3, j_3) \notin t_0$ $x(E_{i_3 j_3}) = 0$ holds. By 2.13 $x(E_{ij}) \in H(s_1)$, thus $F_{t_0} \subset H(s_1)$. Since $H(s_1)$ is a convex subgroup of G, it follows from 2.17

$$(6.5) \qquad F(t_0) \subset H(s_1)\,.$$

Let $x \in H'(s_1)$, $t \in T$, $x_t \neq 0$. Then there exists $(i_3, j_3) \in t$ such that $x_{i_3 j_3} \neq 0$. We have $x_{i_3 j_3} \in H'(s_1)$ and by (6.4) $x_{i_3 j_3} \in E'(i_2, j_2)$. Therefore $(i_3, j_3) > (i_2, j_2)$ for each $(i_2, j_2) \in t_0$. This implies $t > t_0$. Hence $x \in F'(t_0)$ and thus

$$(6.6) \qquad H'(s_1) \subset F'(t_0)\,.$$

By 6.1 and 5.8 from (6.5) and (6.6) it follows $\bar{\varepsilon} \leqq \bar{\varkappa}$.

From 6.6, 6.7 and 6.8 we get:

6.9. *If $\alpha, \beta \in \mathscr{G}$, then $\overline{f_1(\alpha, \beta)} = \bar{\alpha} \vee \bar{\beta}$.*

6.9.1. Corollary. *If $\alpha, \beta, \alpha_1, \beta_1 \in \mathscr{G}$, $\alpha \sim \alpha_1$, $\beta \sim \beta_1$, then $f_1(\alpha_1, \beta_1) \sim f_1(\alpha, \beta)$.*

From 6.4 and 6.9 it follows:

6.10. Theorem. *The partially ordered set $\overline{\mathscr{G}}$ is a lattice.*

7. SOME GENERALIZATIONS AND PROBLEMS

7.1. Let σ be an ordinal with the property that the sum and product of any two ordinals less than σ are again less than σ. Let

$$G_2 = [\Omega_{i \in I} A_i]$$

be the mixed product of directed groups A_i. If $f \in G_2$, $R \subset I$ and if R is a chain, then (since $I(f)$ satisfies the descending chain condition) the set $I(f) \cap R$ is well-ordered. Let G_3 be the system of all $f \in G_2$ such that the order type of $I(f) \cap R$ is less than σ for any chain $R \subset I$. Then G_3 is the mixed σ-product of partially ordered groups A_i; we shall denote it by

$$G_3 = [(\sigma)\, \Omega_{i \in I} A_i]$$

(cf. [6] and [4] for the case of a linearly ordered set I). Analogously as in 1.2 we can define now a mixed σ-decomposition of a partially ordered group

$$G = (\sigma)\, \Omega_{i \in I} A_i\,;$$

the only difference consists in taking $[(\sigma)\,\Omega_{i\in I}A_i]$ instead of $[\Omega_{i\in I}A_i]$ in the condition (b) of Definition 1.2.

Let there be given two σ-decompositions

$$(\alpha) \quad G = (\sigma)\,\Omega_{i\in I}A_i\,, \qquad (\beta) \quad G = (\sigma)\,\Omega_{j\in J}B_j\,.$$

It can be easily verified that the constructions described in Sections 2–6 applied on these σ-decompositions lead to σ-decompositions $f(\alpha, \beta), f(\beta, \alpha), f_1(\alpha, \beta)$ and $f_1(\beta, \alpha)$. In this manner, each proposition from Sections 2–6 can be replaced by the corresponding "σ-proposition" concerning σ-decompositions. Then the σ-theorem 4.10 generalizes Theorem 2 of Mal'cev [6] and Theorem 9 of Fuchs [4, Chap. II].

7.2. Let $(G; +, \leqq)$ be a gruppoid with respect to the operation $+$ (neither the associativity nor the commutativity of $+$ are assumed) that is partially ordered and satisfies

$$xsy \Leftrightarrow (x + z)\,s(y + z)\,, \quad xsy \Leftrightarrow (z + x)\,s(z + y)$$

for any $x, y, z \in G$ and any $s \in \{<, >, \|\}$. If there exists $0 \in G$ such that $x + 0 = 0 + x = x$ for any G, then G is called a u_1-gruppoid [5]. For a u_1-gruppoid G we can define a mixed product decomposition $G = \Omega_{i\in I}A_i$ analogously as in 1.2. Consider the following condition for G:

(C) if A_i, B_j are factors of G, then $A_i^+ \subset B_j^+ \Rightarrow A_i \subset B_j$; $A_i^- \subset B_j^- \Rightarrow A_i \subset B_j$. (For any subset $X \subset G$ we put $X^+ = \{x \in X : x \geqq 0\}$, $X^- = \{x \in X : x \leqq 0\}$.) It can be proved that if a u_1-gruppoid G satisfies (C), then the propositions from Section 2 are true for mixed decompositions of G (some, but not all, proofs remain verbatim valid).

P r o b l e m 1. In what extent the results of Sections 3–6 remain true for u_1-gruppoids satisfying the condition (C)? (Cf. [5] for the case of decompositions $G = \Omega_{i\in I}A_i$ where I is linearly ordered.)

7.3. Let G be a partially ordered group. Let $\mathscr{F}$ be the system of all factors A_i in G for which there exists a decomposition $\alpha \in \mathscr{G}$ such that A_i is a factor of α. For $A_i, B_j \in \mathscr{F}$ put $A_i \sim B_j$, if $A(i) = B(j)$, $A'(i) = B'(j)$. Then $\sim$ is an equivalence relation on $\mathscr{F}$; the class of the corresponding partition containing the element $A_i \in \mathscr{F}$ will be denoted by $t(A_i)$ and the system of all such classes by $\overline{\mathscr{F}}$. We define a partial order on the set $\overline{\mathscr{F}}$ by

$$t(A_i) \leqq t(B_j) \Leftrightarrow B'(j) \subset A'(i)\,, \quad A(i) \subset B(j)\,.$$

P r o b l e m 2. Under which conditions is $\overline{\mathscr{F}}$ a lattice?

7.4. P r o b l e m 3. Characterize the class of lattices L for which there exists a partially ordered group G such that L is isomorphic to the corresponding $\overline{\mathscr{G}}$.

References

[1] *G. Birkhoff:* Lattice theory, New York 1948.
[2] *P. Conrad:* Representation of partially ordered abelian groups as groups of real valued functions. Acta math. *116* (1966), 199—221.
[3] *P. Conrad, J. Harvey, C. Holland:* The Hahn embedding theorem for lattice ordered groups. Trans. Amer. Math. Soc. *108* (1963), 143—169.
[4] *L. Fuchs:* Частично упорядоченные алгебраические системы. Москва 1965. (Partially ordered algebraic systems. Oxford 1963).
[5] *J. Jakubík:* Лексикографические произведения частично упорядоченных группоидов. Чехосл. мат. журн. *14* (*89*) (1964), 281—305.
[6] *А. И. Мальцев:* Об упорядоченных группах. Известия Акад. наук СССР, серия матем. *13* (1949), 473—482.

Author's address: Košice, Nám. Februárového víťazstva 9, ČSSR (Vysoká škola technická).