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ON THE CONSISTENCY OF ESTIMATES

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The relations between four types of consistent estimates (see Definition) are studied. Especially, it is proved that the existence of a [wide sense] consistent estimate implies the existence of a [wide sense] superconsistent estimate. It is also shown that the existence of a wide sense [super]consistent estimate does not guarantee the existence of a [super]consistent estimate.

Throughout this paper the letter N will denote the set of all positive integers. If A is a set then the symbol A^∞ will denote the infinite dimensional Cartesian product

$$A^\infty = \mathop{\mathsf{X}}_{i=1}^{\infty} A_i \quad \text{where} \quad A_i = A \quad \text{for} \quad i = 1, 2, \ldots$$

Further, if $\mathcal{S}$ is a σ-algebra of subsets of the set A and $n \in N$ then the symbol $\mathcal{S}^n$ will denote the minimal σ-algebra over the class of all subsets of the set A^∞ which are of type $\mathop{\mathsf{X}}_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{S}$ for $i = 1, 2, \ldots, n$ and $E_i = A$ for $i = n + 1, n + 2, \ldots$, and the symbol $\mathcal{S}^\infty$ will denote the minimal σ-algebra over the class $\bigcup_{i=1}^{\infty} \mathcal{S}^i$. If P is a probability measure on the σ-algebra $\mathcal{S}$ then P^∞ will denote the probability measure on $\mathcal{S}^\infty$ defined by the implication

$$\mathop{\mathsf{X}}_{i=1}^{\infty} E_i \in \mathcal{S}^\infty \Rightarrow P^\infty\Big(\mathop{\mathsf{X}}_{i=1}^{\infty} E_i\Big) = \prod_{i=1}^{\infty} P(E_i)$$

and symbols $\bar{P}^\infty$ and $\underline{P}^\infty$ will denote the outer and the inner measure induced by P^∞, respectively. The characteristic function (indicator) of a set E will be denoted by χ_E.

Throughout this paper it will be assumed that we are given a measurable space $(X, \mathcal{X})$ such that $X \in \mathcal{X}$, a class $\mathcal{P}$ of probability measures on $\mathcal{X}$ and a mapping φ of $\mathcal{P}$ into some separable metric space (M, ϱ). To simplify our wording, the letter $\mathcal{M}$ will always denote the σ-algebra of all Borel subsets of M and the letter $\mathbf{X}$ the class of all subsets of the set X. Let us remark that the mapping f of X^∞ into M is $\mathbf{X}^n$-measurable if and only if the following implication holds: If $x = (x_1, x_1, \ldots)$, $y = (y_1, y_1, \ldots) \in \in X^\infty$ and $x_i = y_i$ for $i = 1, 2, \ldots, n$ then $f(x) = f(y)$.

With the above exceptions the usual terminology and notation as e.g. in [1] and [2] is used without further reference.

Definition. Let $\{f_n\}_{n\in N}$ be a sequence of mappings of X^∞ into M. Consider the following conditions:

(i) for every $n \in N$, f_n is $\mathbf{X}^n$-measurable

(is) for every $n \in N$, f_n is $\mathscr{X}^n$-measurable

(ii) for every $\varepsilon > 0$ and every $P \in \mathscr{P}$ there holds

$$\lim_{n\to\infty} \bar{P}^\infty\{x : \varrho(f_n(x), \varphi(P)) \geqq \varepsilon\} = 0$$

(iis) for every $\varepsilon > 0$ and every $P \in \mathscr{P}$ there holds

$$\lim_{m\to\infty} \bar{P}^\infty(\bigcup_{n=m}^{\infty} \{x : \varrho(f_n(x), \varphi(P)) \geqq \varepsilon\}) = 0$$

The sequence $\{f_n\}_{n\in N}$ is called a wide sense consistent estimate for φ on $\mathscr{P}$ if (i) and (ii) hold.

The sequence $\{f_n\}_{n\in N}$ is called a wide sense superconsistent estimate for φ on $\mathscr{P}$ if (i) and (iis) hold.

The sequence $\{f_n\}_{n\in N}$ is called a consistent estimate for φ on $\mathscr{P}$ if (is) and (ii) hold.

The sequence $\{f_n\}_{n\in N}$ is called a superconsistent estimate for φ on $\mathscr{P}$ if (is) and (iis) hold.

Evidently, $\{f_n\}_{n\in N}$ is a superconsistent estimate for φ on $\mathscr{P}$ if and only if (is) holds and if, for every $P \in \mathscr{P}$,

$$P^\infty\{x : \lim_{n\to\infty} f_n(x) = \varphi(P)\} = 1\,. \tag{1}$$

If $\{f_n\}_{n\in N}$ is a wide sense superconsistent estimate then, for every $P \in \mathscr{P}$,

$$\underline{P}^\infty\{x : \lim_{n\to\infty} f_n(x) = \varphi(P)\} = 1\,. \tag{2}$$

However, if conditions (i) and (2) hold, $\{f_n\}_{n\in N}$ need not be a wide sense superconsistent estimate (it even need not be a wide sense consistent estimate) since condition (2) does not imply condition (ii).

Nor the simultaneous fulfilling of conditions (2), (i) and (ii) guarantees that $\{f_n\}_{n\in N}$ is a wide sense superconsistent estimate for φ on $\mathscr{P}$. These facts are illustrated by the following example.

Example 1. Let $X = (0, 1\rangle \times (0, 1\rangle$, $\mathscr{X}$ be the class of all sets of the form $(0, 1\rangle \times \times E$ where E is a Lebesgue measurable subset of $(0, 1\rangle$, P be the restriction of the two-dimensional Lebesgue measure to $\mathscr{X}$, $\mathscr{P} = \{P\}$ and $\varphi(P) = 0$.

For $x \in X^\infty$, let us define

$$f_n(x) = \chi_{(0,1/i_n\rangle \times ((j_n - 1)/i_n, j_n/i_n\rangle}(x_1)$$
$$g_n(x) = \max \left\{ f_t(x) : \frac{(i_n - 1)\, i_n}{2} < t \leqq \frac{i_n(i_n + 1)}{2} \right\}$$

where x_1 denotes the first coordinate of the point x and i_n and j_n are positive integers defined by

$$i_n = \min \left\{ i : i \in N,\ n \leqq \frac{i(i+1)}{2} \right\}, \quad j_n = n - \frac{i_n(i_n - 1)}{2}.$$

Obviously, for every $x \in X$,

$$\lim_{n\to\infty} f_n(x) = 0\,, \quad \lim_{n\to\infty} g_n(x) = 0\,.$$

Therefore

$$\underline{P}^\infty\{x : \lim_{n\to\infty} f_n(x) = 0\} = 1\,, \quad \underline{P}^\infty\{x : \lim_{n\to\infty} g_n(x) = 0\} = 1\,.$$

Further, for every $0 < \varepsilon \leqq 1$

$$\{x : |f_n(x)| \geqq \varepsilon\} = \left(0, \frac{1}{i_n}\right\rangle \times \left(\frac{j_n - 1}{i_n}, \frac{j_n}{i_n}\right\rangle \times X \times X \times \ldots,$$
$$\{x : |g_n(x)| \geqq \varepsilon\} = \left(0, \frac{1}{i_n}\right\rangle \times (0, 1\rangle \times X \times X \times \ldots$$

Hence

$$\lim_{n\to\infty} \bar{P}^\infty\{x : |f_n(x)| \geqq \varepsilon\} = \lim_{n\to\infty} \frac{1}{i_n} = 0\,,$$
$$\lim_{m\to\infty} \bar{P}^\infty\Big(\bigcup_{n=m}^{\infty} \{x : |f_n(x)| \geqq \varepsilon\}\Big) = 1\,,$$
$$\lim_{n\to\infty} \bar{P}^\infty\{x : |g_n(x)| \geqq \varepsilon\} = 1\,.$$

Now we proceed to the study of relations between the four types of consistent estimates defined above. The following assertions are immediate consequences of our definition.

Proposition. *Every superconsistent estimate for φ on $\mathscr{P}$ is also a wide sense superconsistent estimate for φ on $\mathscr{P}$ and, at the same time, a consistent estimate for φ on $\mathscr{P}$. Every consistent estimate is also a wide sense consistent estimate.*

A nontrivial question is whether the existence of a wide sense consistent estimate for φ on $\mathscr{P}$ guarantees the existence of a consistent estimate for φ on $\mathscr{P}$. The answer is in the negative. Even the existence of a wide sense superconsistent estimate does not imply the existence of a consistent estimate as the following example illustrates.

Example 2. Let $X = (-\infty, +\infty)$, $\mathscr{X}$ be the class of all Borel subsets of X, (M, ϱ) be the one-dimensional Euclidean space, $\mathscr{P}$ be the class of all probability measures on $\mathscr{X}$ and B be an unmeasurable subset of X, i.e. $B \notin \mathscr{X}$. Let us define a mapping φ of $\mathscr{P}$ into M by putting

$$\begin{aligned} \varphi(P) &= 1 \quad \text{if there is an} \quad a \in B \quad \text{such that} \quad P\{a\} = 1\,, \\ \varphi(P) &= 0 \quad \text{in the other cases}\,. \end{aligned}$$

If, for $n \in N$ and $x = (x_1, x_2, \ldots) \in X^\infty$, $\tilde{f}_n(x)$ is defined by

$$\tilde{f}_n(x) = \chi_B(x_1) \prod_{i=2}^{n} \chi_{\{x_{i-1}\}}(x_i)$$

then $\{\tilde{f}_n\}_{n\in N}$ is obviously a wide sense superconsistent estimate for φ on $\mathscr{P}$.

Let us assume now that a consistent estimate for φ on $\mathscr{P}$ exists. Let $\{f_n\}_{n\in N}$ be such an estimate. Let us denote

$$C = \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{x : |f_n(x)| < \frac{1}{k}\right\}.$$

The $\mathscr{X}^n$-measurability of f_n implies that

$$C \in \mathscr{X}^n\,. \tag{3}$$

Let us mention that if $\{h_n\}_{n\in N}$ is a sequence of measurable functions that converge to a constant c_0 in probability measure μ then

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{x : |h_n(x) - c| < \frac{1}{k}\right\}\right) = \begin{cases} 1 & \text{if} \quad c = c_0 \\ 0 & \text{if} \quad c \neq c_0\,. \end{cases}$$

Therefore

$$\begin{aligned} P^\infty(C) &= 1 \quad \text{if} \quad \varphi(P) = 0\,, \\ P^\infty(C) &= 0 \quad \text{if} \quad \varphi(P) = 1\,. \end{aligned}$$

For $a \in X$ let P_a denote the probability measure on $\mathscr{X}$ such that $P_a\{a\} = 1$. Obviously $P_a^\infty\{(a, a, \ldots, a, \ldots)\} = 1$ and since $\varphi(P_a) = 0$ for $a \in X - B$ and $\varphi(P_a) = 1$ for $a \in B$, we have

$$\begin{aligned} (a, a, \ldots, a, \ldots) &\in C \quad \text{for} \quad a \in X - B\,, \\ (a, a, \ldots, a, \ldots) &\notin C \quad \text{for} \quad a \in B\,. \end{aligned} \tag{4}$$

Let $\mathscr{F}$ be the class of all subsets E of X^∞ such that

$$\{a : (a, a, \ldots, a, \ldots) \in E\} \in \mathscr{X} .$$

From (4) it follows that

$$C \notin \mathscr{F} . \tag{5}$$

It is easily seen that $\mathscr{F}$ is a σ-algebra containing all sets of the form $E = \bigtimes_{i=1}^{\infty} E_i$ where $E_i \in \mathscr{X}$, $i = 1, 2, \ldots$ Hence $\mathscr{X}^\infty \subset \mathscr{F}$. But this contradicts to (3) and (5). Thus we have proved that the consistent estimate for φ on $\mathscr{P}$ does not exist.

We proceed now to establish the fundamental result concerning the relation between [wide sense] consistent and [wide sense] superconsistent estimates. First of all, we state two lemmas.

Lemma 1. *Let ψ be a continuous mapping of a separable metric space (M, ϱ) into some separable metric space (M_1, ϱ_1). If $\{f_n\}_{n\in N}$ is a consistent [wide sense consistent, wide sense superconsistent or superconsistent] estimate for φ on $\mathscr{P}$ then $\{\psi(f_n)\}_{n\in N}$ is a consistent [wide sense consistent, wide sense superconsistent or superconsistent] estimate for $\psi(\varphi)$ on $\mathscr{P}$.*

Proof. Since ψ is continuous, there is for every $\varepsilon > 0$ and $P \in \mathscr{P}$ a positive number $\delta(\varepsilon, P)$ such that

$$\{x : \varrho_1(\psi(f_n(x)), \psi(\varphi(P))) \geqq \varepsilon\} \subset \{x : \varrho(f_n(x), \varphi(P)) \geqq \delta(\varepsilon, P)\} .$$

Obviously, $\psi(f_n)$ is an $\mathbf{X}^n$-measurable mapping into $(M_1 \mathscr{M}_1)$ where $\mathscr{M}_1$ denotes the σ-algebra of all Borel sets of the metric space (M_1, ϱ_1) and if f_n is $\mathscr{X}^n$-measurable then $\psi(f_n)$ is also $\mathscr{X}^n$-measurable.

Hence, assertions of the lemma follow from the monotony of outer measures.

Lemma 2. *Let f be a real measurable function defined on the measurable space $(X^\infty, \mathscr{X}^n)$ such that*

$$0 \leqq f(x) \leqq 1$$

and let for every nonnegative integer i T^i be a mapping of X^∞ into itself defined by

$$T^i(x_1, x_2, \ldots) = (x_{i+1}, x_{i+2}, \ldots) .$$

Then, for every $\varepsilon > 0$, every $k \in N$ and every probability measure P on $\mathscr{X}$,

$$P^\infty \left\{x : \left|\frac{1}{k} \sum_{i=0}^{k-1} f(T^{ni}x) - \int f \,\mathrm{d}P^\infty\right| \geqq \varepsilon\right\} \leqq \frac{1}{\varepsilon^2 k} .$$

Proof. It is clear that functions g_i defined on the probability space $(X^\infty, \mathscr{X}^\infty, P^\infty)$ by

$$g_i(x) = f(T^{n^i}x), \quad i = 1, 2, \ldots$$

are independent equally distributed random variables with

$$\mathsf{E}g_i = \int f \, \mathrm{d}P^\infty, \quad \mathsf{D}g_i \leqq 1.$$

Hence, our assertion follows from Tchebycheff's inequality.

Theorem. *If there exists a wide sense consistent estimate for φ on $\mathscr{P}$ then there exists also a wide sense superconsistent estimate for φ on $\mathscr{P}$. If there exists a consistent estimate for φ on $\mathscr{P}$ then there exists also a superconsistent estimate for φ on $\mathscr{P}$.*

Proof. Let $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ be two sequences of positive numbers such that

$$\lim_{n\to\infty} \alpha_n = 0$$

and

$$\sum_{n=1}^{\infty} \beta_n < \infty \tag{6}$$

and let k_n be an increasing sequence of positive integers such that

$$k_n \geqq \frac{1}{\beta_n \alpha_n^2} \quad n = 1, 2, \ldots \tag{7}$$

I. We shall first suppose that M is the interval $\langle 0, 1\rangle$ and the metric ϱ in M is defined by

$$\varrho(m, m') = |m - m'|.$$

Let $\{f_n\}_{n\in N}$ be a wide sense consistent estimate for φ on $\mathscr{P}$. We shall prove that for every $\varepsilon > 0$ and $P \in \mathscr{P}$,

$$\lim_{m\to\infty} \bar{P}^\infty \left(\bigcup_{n=m}^{\infty} \left\{ x : \left| \frac{1}{k_n} \sum_{i=0}^{k_n - 1} f_n(T^{n^i}x) - \varphi(P) \right| \geqq \varepsilon \right\} \right) = 0 \tag{8}$$

where T^k is the mapping defined in Lemma 2.

Since $\{f_n\}_{n\in N}$ fulfils condition (ii), there is for every $P \in \mathscr{P}$ a sequence $\{\varepsilon_n(P)\}_{n\in N}$ such that

$$\lim_{n\to\infty} \varepsilon_n(P) = 0$$

and

$$\lim_{n\to\infty} \bar{P}^\infty\{x : |f_n(x) - \varphi(P)| \geqq \varepsilon_n(P)\} = 0.$$

If $F \in \mathbf{X}^n$ then, obviously,

$$\bar{P}^\infty(F) = \inf \{P^\infty(E) : F \subset E \in \mathscr{X}^n\} .$$

Hence it follows that for every $P \in \mathscr{P}$ and $n \in N$ there exists a set $F_{n,P} \in \mathscr{X}^n$ such that

$$\{x : |f_n(x) - \varphi(P)| \geqq \varepsilon_n(P)\} \subset F_{n,P}$$

and

$$P^\infty(F_{n,P}) < \bar{P}^\infty\{x : |f_n(x) - \varphi(P)| \geqq \varepsilon_n(P)\} + \varepsilon_n(P) .$$

For every $n \in N$ and $P \in \mathscr{P}$ let us define mappings $h_{n,P}^+$ and $h_{n,P}^-$ of X^∞ into M by

$$h_{n,P}^+ = \min \{\varphi(P) + \varepsilon_n(P), 1\} \chi_{X^\infty - F_{n,P}} + \chi_{F_{n,P}} ,$$
$$h_{n,P}^- = \max \{\varphi(P) - \varepsilon_n(P), 0\} \chi_{X^\infty - F_{n,P}} .$$

Obviously, both $h_{n,P}^+$ and $h_{n,P}^-$ are $\mathscr{X}^n$-measurable and it holds

$$h_{n,P}^- \leqq f_n \leqq h_{n,P}^+ \tag{9}$$

$$\lim_{n \to \infty} \mathsf{E}_P h_{n,P}^+ = \lim_{n \to \infty} \mathsf{E}_P h_{n,P}^- = \varphi(P) \tag{10}$$

where the symbol E_P denotes the expectation with respect to the probability measure P.

It follows from (9) that

$$\left\{x : \frac{1}{k_n} \sum_{i=0}^{k_n - 1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^+ \geqq \alpha_n\right\} \subset \left\{x : \frac{1}{k_n} \sum_{i=0}^{k_n - 1} h_{n,P}^+(T^{ni}x) - \mathsf{E}_P h_{n,P}^+ \geqq \alpha_n\right\}$$

and according to Lemma 2 and (7) the P^∞-measure of the set on the right hand side of this inclusion is less than β_n and consequently

$$\bar{P}^\infty \left\{x : \frac{1}{k_n} \sum_{i=0}^{k_n - 1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^+ \geqq \alpha_n\right\} \leqq \beta_n .$$

Analogously we can prove the inequality

$$\bar{P}^\infty \left\{x : \frac{1}{k_n} \sum_{i=0}^{k_n - 1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^- \leqq -\alpha_n\right\} \leqq \beta_n .$$

According to (10) there is for every $P \in \mathscr{P}$ and $\varepsilon > 0$ an integer $n_0(P, \varepsilon)$ such that for all $n \geqq n_0(P, \varepsilon)$

$$\varphi(P) - \mathsf{E}_P h_{n,P}^+ + \varepsilon > \alpha_n , \quad \varphi(P) - \mathsf{E}_P h_{n,P}^- - \varepsilon < -\alpha_n .$$

Evidently, for $n \geqq n_0(P)$ we can write

$$\bar{P}^\infty\left\{x : \left|\frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) - \varphi(P)\right| \geqq \varepsilon\right\} \leqq$$

$$\leqq \bar{P}^\infty\left\{x : \frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^+ \geqq \varphi(P) - \mathsf{E}_P h_{n,P}^+ + \varepsilon\right\} +$$

$$+ \bar{P}^\infty\left\{x : \frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^- \leqq \varphi(P) - \mathsf{E}_P h_{n,P}^- - \varepsilon\right\} \leqq$$

$$\leqq \bar{P}^\infty\left\{x : \frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^+ \geqq \alpha_n\right\} +$$

$$+ \bar{P}^\infty\left\{x : \frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) - \mathsf{E}_P h_{n,P}^- \leqq -\alpha_n\right\} < 2\beta_n\,.$$

This inequality together with the countable subadditivity of outer measures and with assumption (6) immediately imply (8).

II. Now, let us assume that (M, ϱ) is a subspace of the Hilbert cube. Therefore every point $m \in M$ is a sequence of real numbers $m = \{m_t\}_{t\in N}$ such that $0 \leqq m_t \leqq 1/t$, $t = 1, 2, \ldots$, and the metric ϱ is defined by

$$\varrho(m, m') = \|m - m'\| = \sqrt{\left(\sum_{t=1}^{\infty}(m_t - m_t')^2\right)}.$$

Let $\{f_n\}_{n\in N}$ be a wide sense consistent estimate for φ on $\mathscr{P}$. Let us define the mappings g_k, $k = 1, 2, \ldots$ of X^∞ into M by

$$g_k(x) = f_k(x) \qquad \text{for} \quad k = 1, 2, \ldots, k_1 - 1\,,$$

$$g_k(x) = \frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) \quad \text{for} \quad nk_n \leqq k < (n+1)\,k_{n+1}\,.$$

Obviously, g_k is $\mathbf{X}^k$-measurable. If f_k is $\mathscr{X}^k$-measurable then g_k is also $\mathscr{X}^k$-measurable. Evidently, we can write

$$\lim_{m\to\infty} \bar{P}^\infty\left(\bigcup_{k=m}^{\infty}\{x : \|g_k(x) - \varphi(P)\| \geqq \varepsilon\}\right) = \tag{11}$$

$$= \lim_{m\to\infty} \bar{P}^\infty\left(\bigcup_{n=m}^{\infty}\left\{x : \left\|\frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n(T^{ni}x) - \varphi(P)\right\| \geqq \varepsilon\right\}\right) \leqq$$

$$\leqq \lim_{m\to\infty} \bar{P}^\infty\left(\bigcup_{n=m}^{\infty}\left\{x : \sum_{t=1}^{i_\varepsilon}\left(\frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n^{(t)}(T^{ni}x) - \varphi^{(t)}(P)\right)^2 \geqq \frac{\varepsilon^2}{4}\right\}\right) \leqq$$

$$\leqq \sum_{t=1}^{i_\varepsilon} \lim_{m\to\infty} \bar{P}^\infty\left(\bigcup_{n=m}^{\infty}\left\{x : \left|\frac{1}{k_n}\sum_{i=0}^{k_n-1} f_n^{(t)}(T^{ni}x) - \varphi^{(t)}(P)\right| \geqq \frac{\varepsilon}{2i_\varepsilon}\right\}\right)$$

where i_ε is such an integer that

$$\sum_{i=i_\varepsilon+1}^{\infty} \frac{1}{i^2} < \frac{\varepsilon^2}{4}$$

and $f_n^{(t)}(x)$ and $\varphi^{(t)}(P)$ denote the t-coordinate of $f_n(x)$ and $\varphi(P)$ respectively.

Since (8) implies that every summand on the right hand side of the second inequality in (11) is equal to zero, Theorem is proved provided that (M, ϱ) is a subspace of the Hilbert cube. However, according to Urysohn's Embedding Theorem, every separable metric space is homeomorphic to a subspace of the Hilbert cube. Consequently, in view of Lemma 1, it follows that Theorem holds in the general case.

Let us remark that the second assertion of Theorem is proved in a different way in [3] (see Theorem 2.9) where necessary and sufficient conditions for the existence of a superconsistent estimates are also given.

References

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