Czechoslovak Mathematical Journal

Ladislav Bican

On splitting mixed abelian groups

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 1, 74–80

Persistent URL: http://dml.cz/dmlcz/100944

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

ON SPLITTING MIXED ABELIAN GROUPS

LADISLAV BICAN, Praha

(Received December 13, 1968)

The purpose of this note is to prove two theorems generalizing theorems A1, A2 from [6]. After that, theorems 13–15 from [2] are generalized by using these theorems and some theorems from [7].

By the word "group" we shall always mean an additively written abelian group. A group G is said to be split if its maximal torsion part is a direct summand of G. If H is a subgroup of a torsion free group G then $\{H\}_*^G$ means the pure closure of H in G, i.e. the intersection of all pure subgroups of G containing H. $\hat{\tau}$ denotes the type containing the characteristic τ, $T(G)$ denotes the set of the types of all direct summands J_ι of a completely decomposable group $G = \sum_{\iota \in I}{}_d J_\iota$. In the other cases we adopt the notation used in [1].

Let us note that a torsion free group A is called a K-group if, for every torsion group P, any group G splits whenever G is an extension of the group $H = A \dotplus P$ by a bounded group (see Procházka's paper [3]). In [4] there was proved that any torsion free group of finite rank is a K-group. Finally, let A be a K-group and P an arbitrary torsion group. It is easy to see that if H is a subgroup of $G = A \dotplus P$ such that G/H is bounded, then H splits.

Definition 1. Let H be a subgroup of a group G (mixed in general). We say that H is fully regular in G if the factor-group

$$S/\{S \cap H; T\} \tag{1}$$

is finite for every subgroups $T \subseteq S$ pure in G such that S/T is a torsion free group of finite rank.

Lemma 1. *Let H be a subgroup of a mixed group G such that G/H is a torsion group and P is the maximal torsion part of both groups G and H. Let $P \subseteq H_1 \subseteq \subseteq H_2$ be pure subgroups of H such that H_2/H_1 is of finite rank. Let G_1 and G_2 denote the subgroup of G such that $G_1/P = \{H_1/P\}_*^{G/P}$, $G_2/P = \{H_2/P\}_*^{G/P}$ respectively. Then $G_1 \subseteq G_2$ and G_2/G_1 is of finite rank.*

Proof. Let $g \in G_1$ and $\bar{g} = g + P \in G_1/P$. Then there exists an integer s such that $s\bar{g} \in H_1/P \subseteq H_2/P$. Hence it follows $\bar{g} \in G_2/P$ and $g \in G_2$ so that $G_1 \subseteq G_2$ is proved.

Assume that $r(H_2/H_1) = n - 1$ and let $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ be arbitrary elements of G_2/G_1. If $g_1, g_2, \ldots, g_n$ are representants of the cosets $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ then $g_i \in G_2$, $i = 1, 2, \ldots, n$ and from the periodicity of G/H the existence of an integer $m \neq 0$ such that $mg_i \in H_2$, $i = 1, 2, \ldots, n$ follows easily. From $r(H_2/H_1) = n - 1$ it is easy to derive the existence of integers λ_i, $i = 1, 2, \ldots, n$, not all equal to zero, such that $\sum_{i=1}^{n} \lambda_i m g_i \in H_1$. From $H_1 \subseteq G_1$ it follows now $\sum_{i=1}^{n} \alpha\lambda_i m\bar{g}_i = \bar{0}$ (in G_2/G_1) and the elements $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ are dependent in G_2/G_1 so that $r(G_2/G_1) \leqq n - 1$ and the proof of the lemma is finished.

Theorem 1. *Let G be a mixed group containing a splitting subgroup $H = P \dotplus A$, where P is a torsion group and A a direct sum of torsion free groups of finite rank. If H is fully regular in G then G splits.*

Proof runs on similar principles as the proof of Theorem Al from [6]. Suppose that $A = \sum_{\alpha<\sigma}^{d} A_\alpha$ where $r(A_\alpha) < \infty$ and σ is an arbitrary ordinal. Let T denote the maximal torsion subgroup of G and put $H' = T \dotplus A$ and $H'_\beta = T \dotplus \sum_{\alpha<\beta}^{d} A_\alpha$. Let us define the subgroups G_β of G by the formula $G_\beta/T = \{H'_\beta/T\}_*^{G/T}$. Then G_β is surely pure in G for every $\beta \leqq \sigma$. Finally, it is easy to see that H' is fully regular in G, too.

Using the method of transfinite induction we shall prove that G_β splits for every $\beta \leqq \sigma$, or more precisely that for every $\beta \leqq \sigma$ it is

(2) $$G_\beta = T \dotplus B_\beta \quad \text{and for every} \quad \gamma < \beta \quad \text{it is} \quad B_\gamma \subseteq B_\beta\,.$$

For $\beta = 0$ it is all evident. Firstly, we shall assume that $\beta - 1$ exists. Then by induction hypothesis it holds

(3) $$G_{\beta-1} = T \dotplus B_{\beta-1}\,.$$

Because $A_{\beta-1} \subseteq H'_\beta$ and $G_{\beta-1} \cap H'_\beta = H'_{\beta-1}$, it is true that $G_{\beta-1} \cap A_{\beta-1} = G_{\beta-1} \cap H'_\beta \cap A_{\beta-1} = H'_{\beta-1} \cap A_{\beta-1} = 0$ which implies that the factor-group $G_\beta/B_{\beta-1}$ is an extension of $(G_{\beta-1} \dotplus A_{\beta-1})/B_{\beta-1} = (T \dotplus B_{\beta-1} \dotplus A_{\beta-1})/B_{\beta-1} \cong T \dotplus A_{\beta-1}$ by

$$(G_\beta/B_{\beta-1})/((G_{\beta-1} \dotplus A_{\beta-1})/B_{\beta-1}) \cong G_\beta/(G_{\beta-1} \dotplus A_{\beta-1}) = G_\beta/\{G_{\beta-1},\, G_\beta \cap H'\}\,.$$

By Lemma 1, the factor-group $G_\beta/G_{\beta-1}$ is of finite rank, so that by Definition 1 and by hypothesis the factor-group $G_\beta/(G_{\beta-1} \dotplus A_{\beta-1})$ is finite.

The group $A_{\beta-1}$ as a rank finite group is a K-group (see e.g. Procházka's papers [3], [4]) so that $G_\beta/B_{\beta-1}$ splits,

(4) $$G_\beta/B_{\beta-1} = B_\beta/B_{\beta-1} \dotplus G_{\beta-1}/B_{\beta-1}$$

where $G_{\beta-1}/B_{\beta-1}$ is the maximal torsion subgroup of $G_\beta/B_{\beta-1}$. In fact, $G_{\beta-1}/B_{\beta-1}$ is a torsion group by (3) and it is maximal because $G_\beta/B_{\beta-1}/G_{\beta-1}/B_{\beta-1} \cong G_\beta/G_{\beta-1}$ is torsion free by Lemma 1. Clearly, $T \cap B_\beta = 0$, $B_{\beta-1} \subseteq B_\beta$. From (4) and (3) it may be easily derived that (2) is true.

Secondly, let β be a limit ordinal. Then clearly $G_\beta = \bigcup_{\gamma<\beta} G_\gamma$ and by induction hypothesis $G_\gamma = T \dotplus B_\gamma$ for all $\gamma < \beta$ and $B_\delta \subseteq B_\gamma$ for all $\delta < \gamma < \beta$ so that we can put $B_\beta = \bigcup_{\gamma<\beta} B_\gamma$. For an arbitrary $g \in G_\beta$ there exists $\gamma < \beta$ such that $g = t + b$, $t \in T$, $b \in B_\gamma \subseteq B_\beta$, i.e. $g \in T + B_\beta$. From this fact the splittingness of G_β easily follows.

In particular, for $\beta = \sigma$ it is $G = G_\sigma = T \dotplus B_\sigma$ so that the proof of Theorem 1 is finished.

Theorem 2. *Let $G = T \dotplus B$ be a splitting mixed group where T is a torsion group and B torsion free and H is a subgroup of G with the maximal torsion subgroup P. If either*

1) T/P is bounded and B is of finite rank,

or

2) $B = \sum_{\lambda\in\Lambda}{}_d B_\lambda$ is a direct sum of K-groups and for every $\lambda \in \Lambda$ the factor-group $B_\lambda/B_\lambda \cap H$ is bounded,

then H splits, too.

Proof. Firstly, let T/P be bounded and B be of finite rank. Put $K = \{T, H\}$ so that $K = T \dotplus K_1$, where $K_1 = K \cap B$. Further, $K/H = \{T, H\}\, H \cong T/T \cap H = T/P$ is bounded. K_1 as a subgroup of B is of finite rank, i.e. it is a K-group and thus H splits.

Secondly, we can assume that Λ is the set of ordinals $\alpha < \sigma$. Put

$$G_\beta = T \dotplus \sum_{\alpha<\beta}{}_d B_\alpha \tag{5}$$

and

$$H_\beta = G_\beta \cap H \tag{6}$$

for every ordinal $\beta \leqq \sigma$. Clearly, G_β is a pure subgroup of G for every $\beta \leqq \sigma$. Using the method of transfinite induction we shall prove that for every $\beta \leqq \sigma$ it is

$$H_\beta = P \dotplus A_\beta \quad \text{and for} \quad \gamma < \beta \quad \text{it is} \quad \alpha_\gamma \subseteq A_\beta\,. \tag{7}$$

For $\beta = 0$ it is all evident. Firstly, we shall assume that $\beta - 1$ exists. Then by induction hypothesis it holds

$$H_{\beta-1} = P \dotplus A_{\beta-1}\,. \tag{8}$$

By hypothesis and by (6) the factor-group $G_{\beta-1}/H_{\beta-1}$ is periodical so that to an arbitrary $g \in G_{\beta-1}$ there exists an integer $n \neq 0$ (depending on g) such that $ng \in H_{\beta-1}$. By (8) it is $ng = p + a$ where $p \in P$, $a \in A_{\beta-1}$. From the periodicity of P the existence of a non-zero integer m follows such that $mp = 0$. Altogether we have $mng = ma \in \in A_{\beta-1}$ so that the factor-group

$$(9) \qquad G_{\beta-1}/A_{\beta-1}$$

is a torsion group. Further, by (5) it is $G_\beta = G_{\beta-1} \dotplus B_{\beta-1}$. From $A_{\beta-1} \cap B_{\beta-1} \subseteq \subseteq H_{\beta-1} \cap B_{\beta-1} \subseteq G_{\beta-1} \cap B_{\beta-1} = 0$ it easily follows

$$(10) \qquad G_\beta/A_{\beta-1} = G_{\beta-1}/A_{\beta-1} \dotplus (B_{\beta-1} \dotplus A_{\beta-1})/A_{\beta-1}\,.$$

Due to the isomorphism

$$(11) \qquad (B_{\beta-1} \dotplus A_{\beta-1})/A_{\beta-1} \cong B_{\beta-1}$$

the factor-group $G_\beta/A_{\beta-1}$ splits by hypothesis and by (9) Put $K = \{H_\beta, B_{\beta-1}\}$. Then $B_{\beta-1} \dotplus A_{\beta-1} \subseteq K$ and

$$(12) \qquad K/A_{\beta-1} = (G_{\beta-1}/A_{\beta-1} \cap K/A_{\beta-1}) \dotplus (B_{\beta-1} \dotplus A_{\beta-1})/A_{\beta-1}\,.$$

Hence the factor-group $K/A_{\beta-1}$ splits by (11), (9) and its torsion free direct summand is a K-group by hypothesis. Further, $H_\beta/A_{\beta-1} \subseteq K/A_{\beta-1}$ and the factor-group $(K/A_{\beta-1})/(H_\beta/A_{\beta-1}) \cong K/H_\beta = \{H_\beta, B_{\beta-1}\}/H_\beta \cong B_{\beta-1}/B_{\beta-1} \cap H_\beta = B_{\beta-1}/B_{\beta-1} \cap H$ is bounded by hypothesis so that $H_\beta/A_{\beta-1}$ splits by the definition of a K-group. The maximal torsion subgroup of $H_\beta/A_{\beta-1}$ is $H_{\beta-1}/A_{\beta-1}$. In fact, $H_{\beta-1}/A_{\beta-1}$ is a torsion group by (8) and $(H_\beta/A_{\beta-1})/(H_{\beta-1}/A_{\beta-1})$ is torsion free because $(H_\beta/A_{\beta-1})/(H_{\beta-1}/A_{\beta-1}) \cong H_\beta/H_{\beta-1} = H_\beta/G_{\beta-1} \cap H_\beta \cong \{G_{\beta-1}, H_\beta\}/G_{\beta-1} \subseteq \subseteq G_\beta/G_{\beta-1} \cong B_{\beta-1}$. Then we can write

$$(13) \qquad H_\beta/A_{\beta-1} = H_{\beta-1}/A_{\beta-1} \dotplus A_\beta/A_{\beta-1}$$

where $A_\beta/A_{\beta-1}$ is a suitable torsion free subgroup of $H_\beta/A_{\beta-1}$.

Clearly, $A_\beta \cap P = 0$. If $h \in H_\beta$ is an arbitrary element, then $h + A_{\beta-1} = (a + + A_{\beta-1}) + (h' + A_{\beta-1})$, $a \in A_\beta$, $h' \in H_{\beta-1}$, so that (7) now easily follows in view of (8).

Secondly, let β be a limit ordinal. It is easy to see that $H_\beta = \bigcup_{\gamma<\beta} H_\gamma$ and by induction hypothesis $H_\gamma = P \dotplus A_\gamma$ for all $\gamma < \beta$ and $A_\delta \subseteq A_\gamma$ for all $\delta < \gamma < \beta$. Put $A_\beta = = \bigcup_{\gamma<\beta} A_\gamma$. For an arbitrary $h \in H_\beta$ there exists $\gamma < \beta$ such that $g = p + a$, $p \in P$, $a \in A_\gamma \subseteq A_\beta$, i.e. $h \in P \dotplus A_\beta$. From this fact the splittingness of H_β easily follows.

In particular, for $\beta = \sigma$ it is $H = H_\sigma = P \dotplus A_\sigma$ and the proof is now finished.

Definition 2. Let H be a subgroup of the group G. We say that H is strongly regular in G if the factor-group $S/S \cap H$ is finite for every torsion free subgroup S of finite rank pure in G.

Theorem 3. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = P \dotplus A$ where P is a torsion and A a direct sum of countably many rank finite groups. If $\{H, T\}/T$ is strongly regular in G/T then G splits.*

Proof. If A is of finite rank then $G(T \dotplus A) \cong (G/T)/((T \dotplus A)/T) = (G/T)/(\{H, T\}/T)$ is finite by hypothesis, and G splits by Theorem 3 from [5].

Let us suppose that $A = \sum_{n=1}^{\infty}{}_d A_n$, $r(A_n) < \infty$, $n = 1, 2, \ldots$ Put $H' = T \dotplus A$, $H = T + \sum_{i<n}{}_d A_i$ and let G_n be a pure subgroup of G defined by the formula $G_n/T = \{H_n'/T\}_*^{G/T}$. Now we shall proceed by induction by n. Firstly, $G_1 = T$ splits. If $G_{n-1} = T \dotplus B_{n-1}$ splits then for $K = G_{n-1} \dotplus A_{n-1}$, $(G_n/B_{n-1})/(K/B_{n-1}) \cong G_n/K \cong (G_n/T)/(K/T)$ is a finite group as a homomorphic image of $(G_n/T)/(H_n'/T)$. Then G_n/B_{n-1} splits by Theorem 3 from [5]. It is easy to see that

$$G_n/B_{n-1} = G_{n-1}/B_{n-1} \dotplus B_n/B_{n-1} \tag{14}$$

for a suitable subgroup $B_n \subseteq G_n$. Now the proof proceeds along the same lines as in Theorem 1 (among the limit ordinals only ω must be discussed).

Definition 3. We say that the subgroup H of the group G is regular in G, if the factor-group $S/S \cap H$ is finite for every torsion free rank one subgroup S pure in G.

Note that Baer introduced the following classes of torsion free groups (see e.g. [1], d. 174). Define Γ_1 as the set of all countable torsion free groups. If α is an ordinal, $\alpha > 1$, then we let the torsion free group G belong to Γ_α if $G \notin \Gamma_\beta$ for $\beta < \alpha$ and there exists a pure subgroup S of finite rank of G such that G/S is a direct sum of groups belonging to classes of indices less than α.

Now we shall formulate three theorems (without proofs) which were stated in [7].

Theorem A (see Theorem 4 from [7]): *Let G be a torsion free group containing a completely decomposable homogeneous subgroup H such that G/H is a torsion group. Then $G \cong H$ if and only if*

1) *$G \in \Gamma_\alpha$ for some ordinal α,*

2) *H is strongly regular in G.*

Theorem B (see Theorem 1 from [7]). *Let G be a torsion free group containing a completely decomposable subgroup H such that*

1) *$T(H)$ satisfies the maximum condition,*

2) *for any two incomparable types $\hat{\tau}_1, \hat{\tau}_2$ from $T(H)$ it is $\hat{\tau}_1 \vee \hat{\tau}_2 = \hat{R}$.*[1]*) If H is fully regular in G then $G \cong H$.*

[1]) $\hat{R}$ denotes the greatest element of the lattice of all types.

Theorem C (see Theorem 2 from [7]). *Let G be a completely decomposable torsion free group such that $T(G)$ satisfies conditions* 1) *and* 2) *stated in Theorem B. If H is regular in G then $G \cong H$.*

Now we are ready to prove several theorems, some of which are generalizations of the theorems 13–15 from [2]. This fact we shall not prove here, because it can be easily derived from some theorems and corollaries proved in [7].

Theorem 4. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = P \dotplus A$, where P is a torsion group and A a torsion free completely decomposable group such that $T(A)$ satisfies conditions* 1) *and* 2) *from Theorem B. If H is fully regular in G then G splits, $G = T \dotplus A_0$ and $A_0 \cong A$.*

Proof. G splits by Theorem 1, $G = T \dotplus A_0$. Further, $H \subseteq H_0 = T \dotplus A \subseteq G$ and hence $H_0 = T \dotplus A_0 \cap H_0$. Let $U \subseteq S$ be pure subgroups of A_0 such that S/U is a torsion free group of finite rank. From the purity of A_0 in G it follows by Definition 1 that the factor-group $S/\{S \cap H, U\} = S/\{S \cap (A_0 \cap H), U\}$ is finite. The inclusion $H \subseteq H_0$ shows that $A_0 \cap H_0$ is fully regular in A_0. As $A_0 \cap H_0 \cong H_0/T \cong$ $\cong A$ fulfils all the conditions of Theorem B, the isomorphism $A_0 \cong A_0 \cap H_0$ completes the proof.

Theorem 5. *Let G be a splitting group, $G = T \dotplus A_0$ where T is a torsion group and A_0 a completely decomposable torsion free group such that $T(A_0)$ satisfies conditions* 1) *and* 2) *from Theorem B. If H is a regular subgroup of G then H splits, $H = P \dotplus A$ and $A \cong A_0$.*

Proof. By Theorem 2 H splits, $H = P \dotplus A$. As in the preceding proof it is $H \subseteq$ $\subseteq H_0 = T \dotplus A = T \dotplus (A_0 \cap H_0)$ so that $A \cong A_0 \cap H_0$. It is not too difficult to show that $A_0 \cap H_0$ is regular in A_0, hence Theorem C completes the proof.

Theorem 6. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = A \dotplus P$ where P is a torsion group and A a homogeneous completely decomposable torsion free group. If G/T is countable and $\{H, T\}/T$ strongly regular in G/T then G splits, $G = A_0 \dotplus T$ and $A_0 \cong A$.*

Proof. Let us denote $H_0 = \{H, T\} = T \dotplus A \subseteq G$. Then $A \cong H_0/T \subseteq G/T$ is a direct sum of countably many rank one groups and H_0/T is clearly strongly regular in G/T. By Theorem 3 G splits, $G = T \dotplus A_0$. Now G/T is a torsion free countable group containing $H_0/T \cong A$ as a subgroup, so that by Theorem A (for $\alpha = 1$) it is $G/T \cong H_0/T$ and the theorem easily follows.

Theorem 7. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = A \dotplus P$ where P is a torsion group and A a homogeneous completely decomposable torsion free group. If G contains* *a subgroup G_1 such that $H \subseteq G_1 \subseteq G$, H is fully regular in G_1, $\{G_1, T\}/T$ is strongly regular in G/T and G/G_1 is countable, then G splits, $G = A_0 \dotplus T$ and $A_0 \cong A$.*

Proof. By Theorem 4 G_1 splits, $G_1 = Q \dotplus A_1$ and $A_1 \cong A$. If $g \in G \dot{-} G_1$ is an arbitrary element then by hypothesis it is $r\{g + T\}_*^{G/T} \subseteq \{G_1, T\}/T$ for a suitable non-zero integer r, i.e. $rg = a + t$, $a \in A_1$, $t \in T$. If s is the order of t then for $m = rs$ it is $mg \in A_1$, i.e. mg has a non-zero component in finitely many direct summands of a given complete decomposition of $A_1 = \sum_{\iota \in I}^d J_\iota$. Let us choose one element in each coset of G/G_1 and let us denote by M the set of all these elements. If we denote by I_1 the set of all indices $\iota \in I$ such that J_ι contains a non-zero component of at least one element mg, $g \in M$[2]) (m depending on g), then I_1 is clearly countable (because M is countable). Put $I_2 = I \dot{-} I_1$, $G' = \{T \dotplus \sum_{\iota \in I_1}^d J_\iota; M\}$, $G'' = \sum_{\iota \in I_2}^d J_\iota$. It is $G' \cap G'' = 0$, because for $g \in G' \cap G''$ it is $mg \in (\sum_{\iota \in I_1}^d J_\iota) \cap G'' = 0$ for a suitable integer m and hence the torsion free character of G'' implies $g = 0$. On the other hand $G = \{G_1, M\} = \{G', G''\}$ so that $G = G' \dotplus G''$.

Further, G'/T is countable because the elements from $\{\sum_{\iota \in I_1}^d J_\iota, M\}$ form the set of representatives of the cosets of G'/T. If we denote $G'_1 = T \dotplus \sum_{\iota \in I_1}^d J_\iota$, then clearly $G'_1 = G' \cap G_1$ and from Definition 2 now easily follows that G'_1/T is strongly regular in G'/T. By using Theorem 6 our assertion now follows without complications.

References

[1] *L. Fuchs:* Abelian groups, Budapest, 1958.

[2] *Л. Прохазка:* Об однородных абелевых группах без кручения, Чех. Мат. Ж. *14* (*89*), (1964), 171—202.

[3] *L. Procházka:* A note on quasi-splitting of abelian groups, Commenth. Math. Univ. Carol. *7*, (*2*) (1966), 227—235.

[4] *Л. Прохазка:* К проблеме расщепления некоторых абелевских расширений расщепляемых абелевых групп, Чех. Мат. Ж. *11* (*86*), (1961), 365—380.

[5] *Л. Прохазка:* Заметка о расщепляемости смешанных абелевых групп, Чех. Мат. Ж. *10* (*85*), (1960), 479—492.

[6] *L. G. Kovács:* On a paper of Ladislav Procházka, Czech. Math. J. *13* (*88*), (1963), 612—618.

[7] *L. Bican:* Some properties of completely decomposable torsion free abelian groups Czech. Math. J. *19* (*94*), (1969), 518—533.

[8] *John S. P. Wang:* On completely decomposable groups, Proc. Am. Math. Soc. *15* (*64*), 184—186.

Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU).

[2]) The set of those J_ι, $\iota \in I$ in which mg has a non-zero component does not depend on the choice of the integer m for which $mg \in A_1$. Surely, if t is the least positive integer for which $tg \in A_1$, then $m = tq + r$, $0 \leqq r < t$. For $r \neq 0$ it is $rg = mg - qtg \in A_1$ a contradiction and the assertion follows.