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SIMULTANEOUS EXTENSIONS OF TWO FUNCTIONALS

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With the present paper we resume the publication of a series of notes devoted to the open mapping theorem in inductive limits of Fréchet spaces. The first two notes [1] and [2] appeared in 1965 and 1966. Although the results of the present note existed in manuscript form at that time already, the final editing had to be postponed for different – partly non mathematical – reasons. The present note is direct continuation of [2] which represents the abstract background for the discussion of the problem to be treated here. This problem necessitates the study of a weaker notion of openness for linear mappings; this was done in the note [2]. Let us recall in a few words some of the properties of this notion. If E and F are two normed spaces, T a continuous linear mapping of E into F and if we denote by Q the canonical quotient mapping modulo $T^{-1}(0)$, openness of T may be characterized by the following two equivalent conditions

1° if $Tx_n \to 0$ then $Qx_n \to 0$,

2° $T'F'$ is $\sigma(E', E)$ closed in E'.

Let us state now another set of equivalent conditions which characterizes a somewhat weaker type of openness which we have called type $(\infty, 0)$ in [2].

1° If x_n is bounded and $Tx_n \to 0$ then $Qx_n \to 0$ weakly,

2° if x_n is bounded and $Tx_n \to 0$ weakly then $Qx_n \to 0$ weakly,

3° the norm closure of $T'F'$ is $\sigma(E', E)$ closed in E'.

This notion arises naturally in the study of simultaneous extensions of two functionals, the subject matter of the present note. Given a normed space F and two subspaces Y and R the following question may be asked: if $y' \in Y'$ and $r' \in R'$ coincide on $Y \cap R$, does there exist a functional $z' \in F'$ whose restrictions to Y and R coincide with y' and r'? It is easy to see that this is not always true; on the other hand, if the mutual position of Y and R is favourable, a simultaneous extension exists. There is, however, an intermediate situation: the simultaneous extension does not exist but the problem may be solved approximately. By this we mean the following: given $\varepsilon > 0$, there exists a $z' \in F'$ which is an extension of r' and such that its restriction to Y differs from y' less than ε in the norm of Y.

To illustrate the three possibilities mentioned above, let us consider the following example.

We shall denote by F the Banach space of all sequences $x = \{x_n;\ n \in N\}$ of real numbers such that $\lim x_n = 0$. The norm is defined by the formula

$$|x| = \max |x_n| .$$

We denote by f_j the coordinate functionals so that $f_j \in F'$ and $\langle x, f_j \rangle = x_j$. We denote by e_j the unit vectors in F so that

$$\langle e_i, f_j \rangle = \delta_{ij} .$$

We shall denote by R the closed subspace of F generated by the elements $e_1, e_3, e_5, \ldots$ We shall keep the space R fixed and consider three different spaces Y to produce examples of the situations possible.

I. Let Y be the closed subspace generated by the elements $e_2, e_4, e_6, \ldots$ Clearly $R \cap Y = \{0\}$ abd $R + Y = F$. Consider a $y' \in Y'$ and an $r' \in R'$. To simplify the notation, we decompose the set N of all natural numbers into N_0, the set of all even numbers and N_1, the set of all odd numbers: $N = N_1 \cup N_0$. It is easy to see that

$$\sum_{i \in N_1} \langle e_i, r' \rangle f_i + \sum_{i \in N_0} \langle e_i, y' \rangle f_i$$

is an element of F' which extends both y' and r'.

II. Let Y be the closed subspace generated by the elements $g_1 = e_1 + \frac{1}{2}e_2$, $g_3 = e_3 + \frac{1}{4}e_4, \ldots\ g_{2n-1} = e_{2n-1} + (1/2n)\, e_{2n}$. Let us show first that $R \cap Y = \{0\}$. We intend to prove that every $y \in Y$ may be written in the form $y = \sum_{i \in N_1} \langle y, f_i \rangle\, g_i$. Denote by G the set consisting of the g_i, $i \in N_1$ so that $Y = G^{00}$. We observe that G^0 contains all elements of the form $f_i - (i + 1) f_{i+1}$ for i odd. The difference $d = y - \sum_{i \in N_1} \langle y, f_i \rangle\, g_i$ is clearly annihilated by every f_i, $i \in N_1$ and also, d being an element of G^{00}, by every $f_i - (i + 1) f_{i+1}$, $i \in N_1$. It follows that $d = 0$. Suppose now that $y \in R \cap Y$. Since $y \in R$, we have $\langle y, f_k \rangle = 0$ for every even k. If j is odd, consider $\langle y, f_{j+1} \rangle = \langle (\sum_{i \in N_1} \langle y, f_i \rangle\, g_i), f_{j+1} \rangle = \langle y, f_j \rangle \langle g_j, f_{j+1} \rangle = \langle y, f_j \rangle (j + 1)^{-1}$ whence, $j + 1$ being even, $(j + 1)^{-1} \langle y, f_j \rangle = \langle y, f_{j+1} \rangle = 0$. It follows that $y = 0$. Further, $R + Y$ is dense in F since it contains all e_t, $t \in N$. We shall show, however, that $R + Y$ is different from F. Suppose that $R + Y = F$. It follows from the closed graph theorem that there exists an $\omega > 0$ such that $|r| + |y| \leqq \omega |r + y|$ for all $r \in R$ and $y \in Y$. In particular, $(1/2n)\, e_{2n} = g_{2n-1} - e_{2n-1}$ whence $2 = |e_{2n-1}| + |g_{2n-1}| \leqq \omega |-e_{2n-1} + g_{2n-1}| = \omega (1/2n)\, |e_{2n}| = \omega/2n$ which is a contradiction.

Let us show now that there exists a linear functional on Y such that no extension of it annihilates R. Let β_n be a sequence such that $\sum|\beta_n| < \infty$ and $\sum n|\beta_n| = \infty$. It is easy to see that there exists a $y' \in Y'$ such that $\langle g_{2n-1}, y'\rangle = \beta_n$. Suppose that $x' \in F'$, $x' = \sum\omega_n f_n$ and x' extends y'. It follows that $\beta_n = \langle g_{2n-1}, x'\rangle = \langle e_{2n-1} + (1/2n)\, e_{2n}, x'\rangle = \omega_{2n-1} + (1/2n)\,\omega_{2n}$. Suppose now that $x' \in R^0$; we have then $\omega_{2n-1} = \langle e_{2n-1}, x'\rangle = 0$ whence $\omega_{2n} = 2n\beta_n$. This is impossible since

$$\infty = \sum 2n|\beta_n| = \sum|\omega_{2n}| \leqq |x'| < \infty\,.$$

Similarly, there exists an $r' \in R'$ such that no extension of it belongs to Y^0. Indeed, it is easy to see that there exists an $r' \in R'$ for which $\langle e_{2n-1}, r'\rangle = \beta_n$. Suppose that $x' \in F'$, $x' = \sum\omega_n f_n$ and $x' \in Y^0$. It follows that $\omega_{2n-1} + (1/2n)\,\omega_{2n} = \langle g_{2n-1}, x'\rangle = 0$. If x' is an extension of r', we have $\omega_{2n-1} = \langle e_{2n-1}, x'\rangle = \langle e_{2n-1}, r'\rangle = \beta_n$ whence $\omega_{2n} = -2n\omega_{2n-1} = -2n\beta_n$, again a contradiction.

Consider now a fixed $r' \in R'$ and denote by P the set of all possible extensions of r' to the whole of F. We intend to show now that there exist extensions with arbitrarily small norms on Y, in other words, that

$$\inf\{|f|_Y; f \in P\} = 0\,.$$

If $f = \sum\omega_j f_j$ and $y \in Y$, we have $y = \sum\xi_{2i-1}g_{2i-1}$ $\lim \xi_{2i-1} = 0$ and $|y| = \max|\xi_{2i-1}|$. It follows that $|f|_Y = \sum|\langle g_{2i-1}, f\rangle| = \sum|\omega_{2i-1} + (1/2i)\,\omega_{2i}|$. If f is to be an extension of r', we must have $\omega_{2i-1} = \langle e_{2i-1}, r'\rangle$. To produce an extension of r' with small norm on Y, take a fixed natural p and define, for $i = 1, 2, \ldots, p$, the coefficient ω_{2i} so as to have $\omega_{2i-1} + (1/2i)\,\omega_{2i} = 0$; for $i > p$ set $\omega_{2i} = 0$. It follows that $|f|_Y = \sum_{i>p}|\omega_{2i-1}|$ which can be made arbitrarily small if p is chosen large enough.

III. Let Y be the closed subspace generated by the elements

$$h_n = -\frac{1}{2n-2}\,e_{2n-2} + e_{2n-1} + \frac{1}{2n}\,e_{2n} \quad \text{for} \quad n = 1, 2, \ldots$$

(we define $e_0 = 0$). We shall write H for the set consisting of the h_n, $n \in N$, so that $Y = H^{00}$. Let $y \in H^{00}$; we intend to show that $y = \sum_{n\in N}\langle y, f_{2n-1}\rangle\, h_n$. Write $d = y - \sum\langle y, f_{2n-1}\rangle\, h_n$; clearly $\langle d, f_{2n-1}\rangle = 0$ for each n. Also, $\langle d, H^0\rangle = 0$ since $d \in H^{00}$. If we show that the set consisting of all the f_{2n-1} together with H^0 is total in F' then $d = 0$ and our assertion is established. To see that, it suffices to show that H^0 contains all elements of the form $f_{2n-1} - 2n\, f_{2n} - f_{2n+1}$. This, however, is immediate since this functional annihilates h_n and h_{n+1}.

Let us show now that there exists an $r' \in R'$ such that every extension f of r' satisfies $|f|_Y \geqq 1$. Define r' by the equations $\langle e_{2n-1}, r'\rangle = 1/2^n$. It follows that $|r'| = 1$. Let f be any extension of r' to the whole of F. We intend to show that $|f|_Y \geqq 1$.

Indeed, if $f = \sum \omega_j f_j$, we have $\omega_{2n-1} = 1/2^n$. Given $\varepsilon > 0$, there exists an n such that $1/2 + \ldots + 1/2^n > 1 - \varepsilon$ and at the same time $|\omega_{2n}| < \varepsilon$. We have then, for $y = h_1 + \ldots + h_n$, the following

$$y \in Y, \quad |y| = 1, \quad \langle y, f \rangle = \frac{1}{2} + \ldots + \frac{1}{2^n} + \frac{1}{2n} \omega_{2n} \geqq 1 - 2\varepsilon .$$

It follows that $|f|_Y \geqq 1 - 2\varepsilon$ for each ε which completes the proof.

To sum up: in the first case, a simultaneous extension is always possible; the second example shows that simultaneous extensions need not always exist but that they may exist up to a small error; in the third case simultaneous extensions do not exist even approximately.

A systematic study of these phenomena forms the contents of the present note. The results are intended to be applied to the study of open mapping theorems in *LF*-spaces. The results are rather technical; some of the ideas used in the proofs, however, might be not entirely uninteresting.

1. NOTATION, TERMINOLOGY AND PRELIMINARIES

The term "convex space" is used for "locally convex Hausdorff topological linear space over the real or complex field". If E is a convex space, we denote by E' its dual. If $x' \in E'$, the value of the functional x' at the point x will be denoted by $\langle x, x' \rangle$. If Y is a subspace of E, we denote by $P(Y)$ the operator which assigns to every $x' \in E'$ its restriction to Y. In particular, if E is a normed space, we denote by $|x'|_Y$ the norm of $P(Y)\, x'$ as an element of Y', so that

$$|x'|_Y = |P(Y)\, x'| = \sup \{ |\langle y, x' \rangle| ; \; y \in Y, \; |y| \leqq 1 \} .$$

If V is a subspace of a normed space E and x a given element of E, we denote by $d(x, V)$ the distance of x from V, defined as

$$d(x, V) = \inf \{ |x - v| ; v \in V \} .$$

We shall also use some simple results about weakly open mappings and closed mappings. For these, we refer the reader to the first two sections of [1].

We conclude this section with two lemmas which will be used in the sequel.

(1,1) *Let E be a normed space, Y a subspace of E. For an $x' \in E'$ the following conditions are equivalent*

1° $|x'|_Y \leqq 1$,

2° *x' may be written in the form $x' = y^0 + m$ with $y^0 \in Y^0$ and $|m| \leqq 1$,*

3° $d(x', Y^0) \leqq 1$.

Proof. Suppose that 1° is satisfied. According to the Hahn-Banach theorem, there exists an extension z' of $P(Y)\,x'$ with norm $|z'| = |P(Y)\,x'| = |x'|_Y$. It follows that $x' - z' = y^0 \in Y^0$ so that $x' = y^0 + z'$ is a decomposition of the required form. The implication $2° \to 3°$ being immediate, it suffices to prove that 3° implies 1°. Assume 3° and take any $y \in Y$, $|y| \leqq 1$. Given $\varepsilon > 0$, there exists a $y^0 \in Y^0$ such that $|x' - y^0| < 1 + \varepsilon$. Hence $\langle y, x' \rangle = \langle y, x' - y^0 \rangle \leqq |y|\,|x' - y^0| < 1 + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $|x'|_Y \leqq 1$. This completes the proof.

(1,2) *Let E be a normed space, U its unit ball, A, B two subspaces of E' and β a real number, $0 < \beta < 1$. If $A \subset B$ and $B \cap U^0 \subset A + \beta U^0$ then $A \subset B \subset$ norm closure of A.*

Proof. Clearly it suffices to show that the inclusions $A \subset B$ and $B \cap U^0 \subset A + \beta U$ imply $B \cap U^0 \subset A + \beta^2 U^0$ as well. Indeed, if $b \in B$ and $|b| \leqq 1$, the vector b may be written in the form $b = a + \beta y$, $y \in U^0$. It follows that $y = (1/\beta)(b - a) \in B$ whence $y \in B \cap U^0$ so that $y = a_2 + \beta y_2$ for suitable $a_2 \in A$ and $y_2 \in U^0$. Hence

$$b = a + \beta(a_2 + \beta y_2) = (a + \beta a_2) + \beta^2 y_2 \in A + \beta^2 U^0 .$$

2. ORTHOGONAL PAIRS OF SUBSPACES

This section is devoted to the study of the simplest case where every pair of functionals has a common extension.

The results of this section are fairly obvious, the proofs straightforward; they are included for the sake of completeness. Also, we intend to compare them with analogous results in a more general situation in section four.

(2,1) Theorem. *Let F be a convex space, R and Y subspaces of F. Then the following conditions are equivalent.*

1° *the natural mapping of $R \oplus Y$ onto $R + Y$ is weakly open,*

2° *given any two $r^* \in R'$ and $y^* \in Y'$ which coincide on $R \cap Y$, there exists a simultaneous extension $x' \in F'$,*

21° *given any $r^* \in R' \cap (R \cap Y)^0$, there exists an extension $r' \in F'$ of r^* which is zero on Y,*

22° *given any $y^* \in Y' \cap (R \cap Y)^0$, there exists an extension $y' \in F'$ of y^* which is zero on R,*

3° $(R \cap Y)^0 = R^0 + Y^0$,

4° $(R \cap Y)^0 \subset R^0 + Y^0$,

Any of these conditions implies $\bar{R} \cap \bar{Y} \subset \overline{R \cap Y}$ and $(R^0 + Y^0)^{00} = R^0 + Y^0$.

If R and Y are closed then the preceding conditions are equivalent to the following

5° *$R^0 + Y^0$ is closed,*

6° *the natural mapping of F onto $F/R \oplus F/Y$ is weakly open.*

Proof. Consider first the mapping S which assigns to the pair $[r, y] \in R \oplus Y$ the sum $r + y$. First of all, the mapping S is weakly open if and only if each $x' \in S^{-1}(0)^0$ may be expressed in the form $x' = S'z'$ for some $z' \in F'$. Let us clear up the meaning of S'. If $S'z' = [r^*, y^*]$ we have, for any $r \in R$ and $y \in Y$, taking $x = [r, y]$,

$$\langle r, z' \rangle + \langle y, z' \rangle = \langle Sx, z' \rangle = \langle x, S'z' \rangle = \langle [r, y], [r^*, y^*] \rangle = \\ = \langle r, r^* \rangle + \langle y, y^* \rangle .$$

It follows that S' is the mapping which assigns to each $z' \in F'$ the couple consisting of its restrictions to R and Y. Further, it is easy to see that $S^{-1}(0)$ consists of all elements of the form $[t, -t]$ with $t \in R \cap Y$; hence $S^{-1}(0)^0$ is the set of those $[r^*, y^*]$ which coincide on $R \cap Y$. This proves the equivalence of 1° and 2°.

The inclusion $R^0 + Y^0 \subset (R \cap Y)^0$ being satisfied for any pair of subspaces, the equivalence of 3° and 4° is obvious.

To complete the proof, it suffices to prove the following two chains of implications. 4° → 2° → 22° → 4° and 4° → 2° → 21° → 4°. Since 21° is obtained from 22° by interchanging R and Y and both 4° and 2° are invariant with respect to this change, it will suffice to prove 4° → 2° and 22° → 4°, the implication 2° → 22° being immediate.

4° → 2°. Given y^* and r^* which coincide on $R \cap Y$, take y'_0 and r'_0 in F' which extend y^* and r^*. It follows that $y'_0 - r'_0 \in (R \cap Y)^0$ so that, by 3°,

$$y'_0 - r'_0 = y^0 + r^0$$

with $y^0 \in Y^0$ and $r^0 \in R^0$. It follows that $y'_0 - y^0 = r'_0 + r^0$ is the required extension.

22° → 4°. If $y' \in (R \cap Y)^0$ is given, there exists, by 22°, a $z' \in F'$ such that z' coincides with y' on Y and $z' = 0$ on R. We have thus $z' \in R^0$ an $z' - y' \in Y^0$ so that $y' = z' + (y' - z') \in R^0 + Y^0$.

It is easy to see that $(R^0 + Y^0)^0 = \bar{R} \cap \bar{Y}$. Hence if $(R \cap Y)^0 \subset R^0 + Y^0$, we have $(R^0 + Y^0)^0 \subset (R \cap Y)^{00} = \overline{R \cap Y}$ so that $\bar{R} \cap \bar{Y} \subset \overline{R \cap Y}$ whence equality. Condition 3° clearly implies that $R^0 + Y^0$ is closed.

Let us examine now condition 5°. If R and Y are any two subspaces of F, the following two inclusions are immediate

$$R^0 + Y^0 \subset (R^{00} \cap Y^{00})^0 , \quad (R^0 + Y^0)^0 \subset R^{00} \cap Y^{00} .$$

It follows that $R^0 + Y^0 \subset (R^{00} \cap Y^{00})^0 \subset (R^0 + Y^0)^{00}$; this proves, for closed R and Y, the equivalence of 3°, 4° and 5°.

Consider now condition 6°. The natural mapping Q of F onto $H = F/R^{00} \oplus F/Y^{00}$ is weakly open if and only if $Q^{-1}(0)^0 = Q'H'$. Clearly $Q^{-1}(0) = R^{00} \cap Y^{00}$ and $H' = R^0 \oplus Y^0$, Q' being the addition. Hence Q is weakly open if and only if $(R^{00} \cap \cap Y^{00})^0 = R^0 + Y^0$ which proves, for closed R and Y, the equivalence of 6° and 3°.

The following two lemmas contain additional information of a quantitative character in the case of normed spaces.

(2,2) *Let F be a Banach space, R and Y two closed subspaces of F. If we define the norm on $R \oplus Y$ as* $\max(|r|, |y|)$ *and the norm on $F/R \oplus F/Y$ as $d(x, R) + d(x, Y)$, then the following conditions are equivalent.*

1° *the natural mapping of $R \oplus Y$ onto $R + Y$ is of type $(\omega, 0)$;*
2° *if $r^* \in R'$ and $y^* \in Y'$ coincide on $R \cap Y$ and $|r^*| + |y^*| \leqq 1$ then there exists an $x' \in F'$ with $|x'| \leqq \omega$ which is a simultaneous extension of r^* and y^*.*

Further, the following two conditions are equivalent.

3° *the natural mapping of F onto $F/R \oplus F/Y$ is of type $(\omega, 0)$;*
4° *each $x' \in (R \cap Y)^0$, $|x'| \leqq 1$ may be written in the form $x' = r^0 + y^0$ with $|r^0| \leqq \omega$, $|y^0| \leqq \omega$, $r^0 \in R^0$, $y^0 \in Y^0$.*

Proof. According to a general proposition (see e.g. proposition (2,3) of [2]) a continuous mapping T of F into a normed space H is of type $(\omega, 0)$ if and only if $T^{-1}(0)^0 \cap U_F^0 \subset T'(\omega U_H^0)$.

(2,3) *Let F be a normed space, R and Y two subspaces of F. Then the following two conditions are equivalent:*

1° *$d(r, R \cap Y) \leqq \omega d(r, Y)$ for each $r \in R$;*
2° *for each $r^* \in R' \cap (R \cap Y)^0$ there exists an extension r' of r^* such that $|r'| \leqq \omega |r^*|$ and $r' \in Y^0$.*

Proof. Let $r^* \in R'$ be zero on $R \cap Y$. Define a linear form g on $R + Y$ by the formula $g(r + y) = \langle r, r^* \rangle$. This is possible since $r^* \in (R \cap Y)^0$. If $r + y = r_1 + y_1$, we have $r - r_1 = y_1 - y \in R \cap Y$ so that $\langle r_1, r^* \rangle = \langle r, r^* \rangle$. We have, r^* being zero on $R \cap Y$,

$$|g(r + y)| = |\langle r, r^* \rangle| \leqq d(r, R \cap Y) |r^*| \leqq \omega |r^*| \, d(r, Y) \leqq$$
$$\leqq \omega |r^*| \, |r + y|$$

so that g is continuous with norm $\leqq \omega |r^*|$.

On the other hand, if $r \in R$, we have $d(r, R \cap Y) = \sup \langle r, r^* \rangle$ where $r^* \in R'$ are zero on $R \cap Y$ and $|r^*| \leqq 1$. Take such an r^*. By 2° there exists an extension r' of r^* with norm $\leqq \omega$ such that $r' \in Y^0$. If $y \in Y$ is arbitrary, we have

$$|\langle r, r^* \rangle| = |\langle r, r' \rangle| = |\langle r - y, r' \rangle| \leqq \omega |r - y|$$

so that $|\langle r, r^* \rangle| \leqq \omega \, d(r, Y)$ whence $d(r, R \cap Y) \leqq \omega \, d(r, Y)$.

(2,4) Theorem. *Suppose that F is a Banach space and that both R and Y are closed in F; then the conditions of Theorem (2,1) are also equivalent to the following*

7° *$R + Y$ is closed in F;*
8° *the natural mapping of $R \oplus Y$ onto $R + Y$ is open;*
9° *there exists an $\omega > 0$ such that $d(r, R \cap Y) \leqq \omega d(r, Y)$ for all $r \in R$;*

10° *there exists an* $\omega > 0$ *such that* $d(y, R \cap Y) \leqq \omega d(y, R)$ *for all* $y \in Y$;

11° *there exists an* $\omega > 0$ *such that each* $r^* \in R' \cap (R \cap Y)^0$ *has an extension* r' *with* $|r'| \leqq \omega |r^*|$ *and* $r' \in Y^0$;

12° *there exists an* $\omega > 0$ *such that each* $y^* \in Y' \cap (R \cap Y)^0$ *has an extension* y' *with* $|y'| \leqq \omega |y^*|$ *and* $y' \in R^0$;

13° *there exists an* $\omega > 0$ *with the following property: given any two* $y^* \in Y'$ *and* $r^* \in R'$ *which coincide on* $R \cap Y$ *and* $|r^*| + |y^*| \leqq 1$, *there exists a simultaneous extension* $x' \in F'$ *with* $|x'| \leqq \omega$;

14° *the natural mapping of* F *into* $F/R \oplus F/Y$ *is open*;

15° *there exists an* $\omega > 0$ *such that each* $x' \in (R \cap Y)^0$ *may be written in the form* $x' = r^0 + y^0$ *with* $|r^0| \leqq \omega |x'|$, $|y^0| \leqq \omega |x'|$, $r^0 \in R^0$, $y^0 \in Y^0$.

Proof. First of all, 1° and 8° are equivalent according to a general theorem. The open mapping theorem gives the equivalence of 7° and 8°. The equivalence of 8° and 13° is a consequence of (2,2). By (2,2) the conditions 14° and 15° are equivalent. Further, 14° is equivalent to 6° according to a general theorem. To complete the proof, it will suffice to prove the following two chains of implications:

$$21° \to 11° \to 9° \to 11° \to 21° ; \quad 22° \to 12° \to 10° \to 12° \to 22° .$$

We observe first that it is sufficient to prove the first four implications only, the other chain being obtained from the first one by interchanging R and Y.

The equivalence of 11° and 9° is a consequence of (2,3) and 11° $\to$ 21° is immediate. Hence the proof will be complete if we show that 21° $\to$ 11°.

Denote by $P(R)$ the mapping which assigns to each $x' \in Y^0$ its restriction to R. If 21° is satisfied, $P(R)$ is a mapping onto $R' \cap (R \cap Y)^0$. By the open mapping theorem, there exists an $\omega > 0$ such that $d(x', P(R)^{-1}(0)) \leqq \omega |P(R)\, x'|$ for all $x' \in Y^0$. Clearly $P(R)^{-1}(0) = Y^0 \cap R^0$. If $r^* \in R' \cap (R \cap Y)^0$ is given, there exists, by 21°, an extension $z' \in F'$ of r^* which is zero on Y so that $P(R)\, z' = r^*$ and $z' \in Y^0$. We have thus $d(z', Y^0 \cap R^0) \leqq \omega |P(R)\, z'| = \omega |r^*|$. According to (1,1) there exists a $v' \in Y^0 \cap R^0$ such that $|z' - v'| = d(z', Y^0 \cap R^0)$. It follows that the functional $x' = z' - v'$ satisfies $|x'| \leqq \omega |r^*|$, $P(R)\, x' = P(R)\, z' - P(R)\, v' = P(R)\, z' = r^*$ and $x' = z' - v' \in Y^0$. The proof is complete.

(2,5) Definition. *Let R and Y be two closed subspaces of a convex space F. The pair R, Y will be called orthogonal if it satisfies one of the conditions of (2,1).*

In applications we shall encounter a situation where the existence of simultaneous extensions is deduced from stronger assumptions. The following lemma shows that this situation may be reduced to the case of orthogonality if we make appropriate changes in the spaces considered.

(2,6) *Let F be a normed space, R, S and Y three closed subspaces of F such that $R \cap Y \subset S \subset R$. Then the following conditions are equivalent.*

1° *every* $r^* \in R'$ *which annihilates* S *has an extension* $x' \in Y^0$;

2° $S^0 \subset R^0 + Y^0$;

3° *$Y + S$ is closed in $R + Y$ and $R, Y + S$ are orthogonal.*

Proof. Assume 1° and take an $s^0 \in S^0$. By condition 1°, there exists a $y^0 \in Y^0$ which extends $P(R)\, s^0$. It follows that $s^0 - y^0 \in R^0$ whence $s^0 \in Y^0 + R^0$. Suppose now that $S^0 \subset R^0 + Y^0$ and consider an $r_0 \in R$ outside S. Since S is closed in R, there exists an $r^* \in R'$ such that r^* annihilates S and $\langle r_0, r^* \rangle = 1$. According to 2°, r^* has an extension $r' \in Y^0$. Given a $y_0 \in Y$, we have, for each $y \in Y$

$$\langle y + s, r' \rangle = 0 \quad \text{while} \quad \langle y_0 + r_0, r' \rangle = \langle r_0, r' \rangle = 1 .$$

This proves that $Z = Y + S$ is closed in $Y + R$. To prove that the pair R, Z is orthogonal it suffices, by (2,1), to show that $(R \cap Z)^0 \subset R^0 + Z^0$. Since clearly $R \cap Z = S$ and $Z^0 = (Y + S)^0 = Y^0 \cap S^0$, this reduces to $S^0 \subset R^0 + (Y^0 \cap S^0)$. To prove that, take an arbitrary $s^0 \in S^0$ and write it in the form $s^0 = r^0 + y^0$, $r^0 \in R^0$ and $y^0 \in Y^0$. Since clearly $R^0 \subset S^0$, we have $y^0 = s^0 - r^0 \in S^0$ so that $y^0 \in Y^0 \cap S^0 = Z^0$. This proves 3°. The implication 3° $\to$ 1° being immediate, the proof is complete.

3. AN INEQUALITY

We devote a separate section to the proof of the equivalence of an interesting inequality and a quantitative statement about extensions (the first two conditions of the proposition below). This equivalence generalizes considerably lemma (2,3) to which it reduces in the case $\alpha = 0$.

(3,1) *Let F be a normed space, R and Y two subspaces of F. Then the following conditions are equivalent*

1° *for each $r \in R$ and each $y \in Y$*

$$d(r, R \cap Y) \leqq \omega |r - y| + \alpha |y| ,$$

2° *if $r^* \in R'$, $|r^*| \leqq 1$, $r^* \in (R \cap Y)^0$ then there exists an extension $x' \in E'$ such that $|x'| \leqq \omega$ and $|x'|_Y \leqq \alpha$;*

3° *if $r^* \in R'$ and $y^* \in Y'$ coincide on $R \cap Y$ and $|r^*| + |y^*| \leqq 1$ then there exists an $x' \in E'$ which extends r^* and $|x'| \leqq |y^*| + \omega$, $|x' - y^*|_Y \leqq \alpha$;*

4° $(R \cap Y)^0 \cap U^0 \subset R^0 + (Y^0 + \alpha U^0) \cap \omega U^0$.

If $\alpha < 1$ then these conditions are equivalent to those of theorems (2,1) *and* (2,4).

Proof. Assume 1° and put, for each $x \in F$

$$p(x) = \inf \{\omega |x - y| + \alpha |y|;\ y \in Y\} .$$

Clearly $p(\lambda x) = \lambda\, p(x)$ for $\lambda \geqq 0$. If x_1 and x_2 are given, take an $\varepsilon > 0$ and find

$y_1, y_2 \in Y$ such that

$$p(x_i) \leqq \omega|x_i - y_i| + \alpha|y_i| < p(x_i) + \varepsilon, \quad i = 1, 2.$$

It follows that $p(x_1 + x_2) \leqq \omega|x_1 + x_2 - (y_1 + y_2)| + \alpha|y_1 + y_2| \leqq \omega|x_1 - y_1| + \alpha|y_1| + \omega|x_2 - y_2| + \alpha|y_2| < p(x_1) + p(x_2) + 2\varepsilon$. Since ε was arbirary, the function p is subadditive. Consider now an $r^* \in R'$, $|r^*| \leqq 1$ and $r^* \in (R \cap Y)^0$; we have, for each $r \in R$ and each $z \in R \cap Y$

$$\langle r, r^* \rangle = \langle r - z, r^* \rangle \leqq \inf|r - z| = d(r, R \cap Y) \leqq$$
$$\leqq \inf\{\omega|r - y| + \alpha|y|;\ y \in Y\} = p(r).$$

It follows that the linear form r^* on R may be extended to a linear form f on the whole of F with $f(x) \leqq p(x)$ for all $x \in F$. Since $p(x) \leqq \omega|x|$, the linear form f is continuous. The rest follows from the fact that $p(y) \leqq \alpha|y|$ for $y \in Y$. This proves 2°.

Now assume 2° and suppose we are given an $r^* \in R'$ and a $y^* \in Y'$ such that they coincide on $R \cap Y$ and $|r^*| + |y^*| \leqq 1$. Consider an extension y' of y^* with norm $|y'| = |y^*|$. Then $r^* - P(R)\, y'$ has norm $\leqq 1$ and is zero on $R \cap Y$. According to 2°, there exists an extension z' of $r^* - P(R)\, y'$ with $|z'| \leqq \omega$ and $|P(Y)\, z'| \leqq \alpha$. Put $x' = y' + z'$. We have $P(R)\, x' = P(R)\, y' + (r^* - P(R)\, y') = r^*$ and

$$|P(Y)\, x' - y^*| = |P(Y)\, y' + P(Y)\, z' - y^*| = |P(Y)\, z'| \leqq \alpha.$$

Further, $|x'| \leqq |y'| + |z'| \leqq |y^*| + \omega$ which proves 3°. Since 3° implies 2° immediately, the equivalence of 2° and 3° is established.

Now assume 2° and let $z' \in (R \cap Y)^0 \cap U^0$. There exists, by condition 2°, an x' which extends $P(R)\, z'$ and $|x'| \leqq \omega$, $|x'|_Y \leqq \alpha$. Since $|x'|_Y \leqq \alpha$, the functional x' may be written in the form $x' = y^0 + m$, where $y^0 \in Y^0$ and $|m| \leqq \alpha$. Clearly $z' - x' \in R^0$ so that $z' = (z' - x') + (y^0 + m)$ is a decomposition of x' of the form required in 4°. To complete the proof, let us show that 4° implies 1°. If $r \in R$ is given, we have

$$d(r, R \cap Y) = \max\{\langle r, x' \rangle;\ x' \in (R \cap Y)^0 \cap U^0\}.$$

Now each such x' may be written in the form $x' = r^0 + y^0 + m$ with $|y^0 + m| \leqq \omega$ and $|m| \leqq \alpha$. Hence

$$\langle r, x' \rangle = \langle r, y^0 + m \rangle = \langle r - y, y^0 + m \rangle + \langle y, y^0 + m \rangle =$$
$$= \langle r - y, y^0 + m \rangle + \langle y, m \rangle \leqq \omega|r - y| + \alpha|y|.$$

The equivalence of the four conditions is thus established.

Condition 1° implies

$$d(r, R \cap Y) \leqq \omega|r - y| + \alpha|y| \leqq (\omega + \alpha)\,|r - y| + \alpha|r|.$$

If $\alpha < 1$, this implies $d(r, R \cap Y) \leqq ((\omega + \alpha)/(1 - d)\, d(r, Y))$ so that theorem (2,4) applies.

In this section we intend to investigate another set of equivalent conditions which characterize a slightly weaker notion than that of orthogonality. Roughly speaking, the openness of the mapping $R \oplus Y \to R + Y$ is replaced by the weaker type $(\infty, 0)$ and the existence of extensions by the existence of approximate extensions. Although a part of the results could be proved – similarly as in (2,1) – without the assumption that R and Y are closed, we restrict ourselves to closed R and Y: the following lemma indicates that this restriction is a natural one.

(4,1) *Let F be a normed space, R and Y two subspaces of F; suppose that $(R \cap Y)^0$ is contained in the norm closure of $R^0 + Y^0$. Then*

$$\overline{Y} \cap \overline{R} \subset \overline{Y \cap R}\,.$$

Proof. If x_0 lies outside $\overline{Y \cap R}$, there exists an $x' \in (Y \cap R)^0$ with $\langle x_0, x' \rangle = 1$. According to our assumption, x' may be written in the form $x' = y^0 + r^0 + m$ with $y^0 \in Y^0$, $r^0 \in R^0$ and $|x_0|\,|m| < 1$. Then x_0 cannot belong to $\overline{Y} \cap \overline{R}$ since otherwise

$$1 = \langle x_0, x' \rangle = \langle x_0, y^0 \rangle + \langle x_0, r^0 \rangle + \langle x_0, m \rangle = \langle x_0, m \rangle < 1\,,$$

a contradiction.

(4,2) Theorem. *Let F be a normed space, Y and R two closed subspaces of F. Then the following conditions are equivalent.*

1° *there exists an $\alpha < 1$ such that*

$$(R \cap Y)^0 \cap U^0 \subset R^0 + Y^0 + \alpha U^0\,;$$

2° *$(R \cap Y)^0$ is contained in (or equal to) the norm closure of $R^0 + Y^0$;*

3° *the norm closure of $R^0 + Y^0$ is weakly closed;*

41° *if $r^* \in R'$ and $y^* \in Y'$ coincide on $R \cap Y$ and $|r^*| + |y^*| \leqq 1$, there exists, for each $\varepsilon > 0$, an x' with $P(R)\,x' = r^*$ and $|P(Y)\,x' - y^*| \leqq \varepsilon$;*

42° *if $r^* \in R'$ and $y^* \in Y'$ coincide on $R \cap Y$ and $|r^*| + |y^*| \leqq 1$ there exists, for each $\varepsilon > 0$, an x' with $P(Y)\,x' = y^*$ and $|P(R)\,x' - r^*| \leqq \varepsilon$;*

51° *if $r^* \in R' \cap (R \cap Y)^0$, $|r^*| \leqq 1$ and $\varepsilon > 0$ there exists an extension x' of r^* such that $|P(Y)\,x'| \leqq \varepsilon$;*

52° *if $y^* \in Y' \cap (R \cap Y)^0$, $|y^*| \leqq 1$ and $\varepsilon > 0$, there exists an extension x' of y^* such that $|P(R)\,x'| \leqq \varepsilon$;*

6° *if $r^* \in R'$ and $y^* \in Y'$ coincide on $R \cap Y$ and $|r^*| + |y^*| \leqq 1$ there exists, for each $\varepsilon > 0$, an x' such that*

$$|P(R)\,x' - r^*| + |P(Y)\,x' - y^*| < \varepsilon\,;$$

7° *suppose that* r_n, $y_n \in U$ *and that* $r_n + y_n \to 0$; *then both* $Q(r_n)$ *and* $Q(y_n)$ *tend weakly to zero,* Q *being the quotient map modulo* $R \cap Y$;

8° $\overline{R} \cap \overline{Y} \subset \overline{R \cap Y}$, *the bar denoting* $\sigma(F'', F')$ *closure;*

9° *the natural mapping of* F *onto* $F/R \oplus F/Y$ *is of type* $(\infty, 0)$.

Proof. The implication 1° → 2° is a consequence of (1,2). The equivalence of 2° and 3° is a consequence of the following two inclusions (valid even without the assumption that R and Y are closed)

$$R^0 + Y^0 \subset (R \cap Y)^0, \quad (R^{00} \cap Y^{00})^0 \subset (R^0 + Y^0)^{00}.$$

2° → 41°. Take an extension r' of r^* with $|r'| = |r^*|$ and an extension y' of y^* with $|y'| = |y^*|$. We have $r' - y' \in (R \cap Y)^0$ and $|r' - y'| \leqq |r'| + |y'| \leqq 1$ so that, by 2°, $r' - y'$ may be written in the form $r' - y' = r^0 + y^0 + m$ where $|m| \leqq \varepsilon$. Put $x' = r' - r^0 = y' + y^0 + m$. Since $x' = r' - r^0$, we have $P(R)\, x' = r^*$. Since $x' - y' = y^0 + m$, we have $|P(Y)\, x' - y^*| \leqq \varepsilon$.

41° → 51°. Follows immediately if we take $y^* = 0$.

51° → 1°. Take an $r' \in (R \cap Y)^0 \cap U^0$ and an $\varepsilon > 0$. Take $P(R)\, r'$ and extend it to an x' with $|P(Y)\, x'| \leqq \varepsilon$. We have $r' - x' \in R^0$ and $x' \in Y^0 + \varepsilon U^0$ whence $r' \in x' + R^0 \subset R^0 + Y^0 + \varepsilon U^0$.

This proves the equivalence of 1°, 2°, 41°, 51°. Since 42° and 52° may be obtained from 41° and 51° by interchanging R and Y, they are also equivalent to 2°, condition 2° being invariant with respect to this change.

41° → 6°. Immediate.

6° → 51°. Let $r^* \in (R \cap Y)^0 \cap U^0$ and let $\varepsilon > 0$. Put $y^* = 0$. By 6°, there exists a v' such that

$$|P(R)\, v' - r^*| + |P(Y)\, v'| \leqq \varepsilon. \quad \text{Put} \quad \sigma = |P(R)\, v' - r^*|$$

and take an extension z' of $P(R)\, v' - r^*$ with $|z'| = \sigma$. Put $x' = v' - z'$. We have $P(R)\, x' = P(R)\, v' - (P(R)\, v' - r^*) = r^*$ and $|P(Y)\, x'| \leqq |P(Y)\, v'| + |z'| \leqq \varepsilon$.

7° → 2°. Let $x' \in (R \cap Y)^0 \cap U^0$ and $\varepsilon > 0$ be given. Suppose that x' non $\in R^0 + Y^0 + \varepsilon U^0$. Let n be a natural number. Since x' does not belong to the $\sigma(E', E)$ closed set $R^0 + (Y^0 \cap nU^0) + \varepsilon U^0$ there is an $x_n \in F$, $x_n \neq 0$ such that

$$\langle x_n, x' \rangle \geqq \langle x_n, R^0 + (Y^0 \cap nU^0) + \varepsilon U^0 \rangle.$$

We may clearly assume that $|x_n| = 1$. Since

$$1 \geqq \langle x_n, x' \rangle \geqq \langle x_n, R^0 \rangle$$

we have $x_n \in R$; we write, accordingly, $r_n = x_n$. Further, $1 \geqq \langle r_n, x' \rangle \geqq \langle r_n, \varepsilon U^0 \rangle$ so that it follows from $|r_n| = 1$ that $\langle r_n, x' \rangle \geqq \varepsilon$. We have

$$1 \geqq \langle r_n, x' \rangle \geqq \sup \langle r_n, Y^0 \cap nU^0 \rangle = nd(r_n, Y)$$

so that $d(r_n, Y) \leqq 1/n$.

We have $\langle r_n, x'\rangle \geqq \varepsilon > 0$ and $d(r_n, Y) \leqq 1/n$ so that there exist $y_n \in Y$ such that $|r_n + y_n| \leqq 2/n$. Hence $|y_n| \leqq 3$. Set $\hat{r}_n = \frac{1}{3}r_n$ and $\hat{y}_n = \frac{1}{3}y_n$ so that $\hat{r}_n \in U$, $\hat{y}_n \in U$, $|\hat{r}_n + \hat{y}_n| < 1/n$. Now $x' \in (R \cap Y)^0$ and $\langle \hat{r}_n, x'\rangle \geqq \frac{1}{3}\varepsilon > 0$. This contradicts 7°.

2° → 7°. Suppose that $r_n \in U$, $y_n \in U$ and $r_n + y_n \to 0$. Take an $x' \in (R \cap Y)^0$ and an $\varepsilon > 0$. According to condition 2°, there exist $r^0 \in R^0$, $y^0 \in Y^0$ and $m \in F'$ such that $x' = r^0 + y^0 + m$ and $|m| \leqq \varepsilon$. It follows that $\langle Q(r_n), x'\rangle = \langle r_n, x'\rangle = = \langle r_n, y^0 + m\rangle = \langle r_n + y_n, y^0 + m\rangle - \langle y_n, y^0 + m\rangle = \langle r_n + y_n, y^0 + m\rangle - - \langle y_n, m\rangle$. Given $\varepsilon > 0$, there exists an $n(\varepsilon)$ such that, for $n \geqq n(\varepsilon)$, we have $|r_n + y_n| \leqq \varepsilon/|y^0 + m|$. It follows that, for $n \geqq n(\varepsilon)$

$$|\langle Q(r_n), x'\rangle| \leqq |\langle r_n + y_n, y^0 + m\rangle| + |\langle y_n, m\rangle| \leqq 2\varepsilon .$$

This proves 7°.

To prove the equivalence of 2° and 8° let us recall first a general equality in convex spaces. If E is a convex space, P and Q two subspaces of E, then $(P + Q)^0 = P^0 \cap Q^0$. Now condition 2° says that the $\sigma(E', E'')$ closure of $R^0 + Y^0$ contains $(R \cap Y)^0$, in other words

$$(R^0 + Y^0)^{E''E'} \supset (R \cap Y)^{E'} .$$

Since $(R^0 + Y^0)^{E''} = (R^{E'} + Y^{E'})^{E''} = R^{E'E''} \cap Y^{E'E''} = \bar{R} \cap \bar{Y}$ the inclusion above may be rewritten as

$$(\bar{R} \cap \bar{Y})^{E'} \supset (R \cap Y)^{E'} .$$

We shall use condition 2° in this form.

Taking polars in E'', we obtain

$$(\bar{R} \cap \bar{Y}) = (\bar{R} \cap \bar{Y})^{E'E''} \subset (R \cap Y)^{E'E''} = \overline{R \cap Y}$$

so that 8° is satisfied.

On the other hand, condition 8° may be rewritten as $\bar{R} \cap \bar{Y} \subset (R \cap Y)^{E'E''}$; taking polars in E', we obtain

$$(\bar{R} \cap \bar{Y})^{E'} \supset (R \cap Y)^{E'E''E'} = (R \cap Y)^{E'}$$

so that 2° holds.

The equivalence of 9° and 2° is a consequence of a general theorem.

5. THE CHARACTERISTIC $\varepsilon(R, Y)$

Let F be a normed space; given two subspaces R and Y, we may consider the following problem: take a linear functional r^* on R of norm one and such that r^* is zero on $R \cap Y$; consider all possible extensions of r^* to the whole space and compute for each of them its norm on Y. What is the infimum of these norms? The answer is given in the following theorem.

(5,1) Definition. *Let R and Y be two subspaces of a normed space F. We define*

$$\varepsilon(R, Y) = \sup_{r^* \in R', r^* \in (R \cap Y)^\circ, |r^*| \leqq 1} \inf \{|x'|_Y;\ P(R)\, x' = r^*\}\,.$$

(5,2) Theorem. *The characteristic ε is symmetric, $\varepsilon(R, Y) = \varepsilon(Y, R)$. There are only two possible values for ε, zero or one.*

Proof. The symmetry is a consequence of the following equations:

$$\varepsilon(R, Y) = \sup_{r^* \in R', r^* \in (R \cap Y)^\circ, |r^*| \leqq 1} \ \inf_{P(R)x' = r^*} |x'|_Y = \sup_{r' \in (R \cap Y)^0 \cap U^0} \ \inf_{x' - r' \in R^0} |x'|_Y =$$

$$= \sup_{r \in (R \cap Y)^0 \cap U^0} \ \inf_{x' - r' \in R^0} \ \inf_{y^0 \in Y^0} |x' - y^0| = \sup_{r' \in (R \cap Y)^0 \cap U^0} \ \inf_{r^0 \in R^0, y^0 \in Y^0} |x' - r^0 - y^0| =$$

$$= \sup_{r' \in (R \cap Y)^0 \cap U^0} d(r', R^0 + Y^0)\,.$$

This expression is independent of the order of the subspaces considered. Also, it shows immediately that $0 \leqq \varepsilon(R, Y) \leqq 1$. Suppose now that $\varepsilon(R, Y) < 1$. It follows that $(R \cap Y)^0 \cap U^0 \subset R^0 + Y^0 + \beta U^0$ for some $\beta < 1$; the conclusion $\varepsilon(R, Y) = 0$ is then a consequence of the equivalence of conditions 1° and 2° of theorem (4,2).

(5,3) Definition. *If R and Y are two closed subspaces of a normed space F, then the pair R, Y will be called semiorthogonal if $\varepsilon(R, Y) = 0$ or, equivalently, if it satisfies one of the cvnditions of (4,2).*

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