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THE CLASS OF FUNCTIONS FULFILLING THE INEQUALITY $\|f(x + z) - f(x) - f(y + z) + f(y)\| \leqq \|x - y\| \, \omega(\|z\|)$

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The condition (1,1) on a function $f(x)$ is considered in connection with the exponential stability [1]. Another condition that can be used for the same purpose is that $f(x)$ has continuous derivatives. A question arises, namely, what is the relation between the two mentioned classes of functions. The problem is formulated generally for transformations from linear spaces with semi-norms [2] into Banach spaces. It is shown that a transformation fulfilling (1,1) and some weak additional assumptions permits the unique extension such that the Gateaux differential T_x exists and T_x depends continuously on x in the sense of (11,1) (for the exact formulation see Theorems 1 and 2).

1. Definition and basic concepts. In what follows the symbol *meas* denotes always Lebesgue measure in Euclidean space of an arbitrary dimension. Instead of Lebesgue measurable function we shall speak about measurable function only. Let G be an open set of the n-dimensional Euclidean space E_n. We say that a measurable set D is almost everywhere (a.e.) in G, if meas $(G - D) = 0$. An open set G is called the carrier of D if it is the largest open set such that D is a.e. in G. In case of the one-dimensional Euclidean space the components of the carrier G will be called carrier-intervals.

Let A be a linear set. We denote by $A(x; h_1, \ldots, h_n)$, $x \in A$, $h_i \in A$ the linear set of all elements of A of the form $x + \sum_{i=1}^{n} \lambda_i h_i$ where λ_i are real numbers. There exists a one-to-one transformation $\mathcal{O}$ of $A(x; h_1, \ldots, h_n)$ onto the n-dimensional Euclidean space E_n: $\mathcal{O}(x + \sum_{i=1}^{n} \lambda_i h_i) = [\lambda_1, \ldots, \lambda_n]$ provided that h_i are linearly independent. If B is a subset of $A(x; h_1, \ldots, h_n)$, then $\mathcal{O}(B) = \{\mathcal{O}(z) : z \in B\}$.

Let A be a linear set with a semi-norm [2]. We denote by $U(x; h_1, \ldots, h_n; \delta)$ a δ-neighbourhood of x in $A(x; h_1, \ldots, h_n)$, i.e. $U(x; h_1, \ldots, h_n; \delta) = \{z : z \in A(x; h_1, \ldots, h_n), \|z - x\| < \delta\}$.

Definition 1. Let D be a subset of A. The set D is a.e. in $U(x; h_1, \ldots, h_n; \delta)$ if $\mathcal{O}(D \cap U(x; h_1, \ldots, h_n; \delta))$ is a.e. in $\mathcal{O}(U(x; h_1, \ldots, h_n; \delta))$ (h_i are linearly independent).

Definition 2. A subset S of A is called the basis of A if every element $x \in A$ is a linear combination of elements of S.

Definition 3. Let A be a linear set with a semi-norm, D a subset of A and S a basis of A. The set D has property (A), if there exist positive functions $\delta(x; h_1, \ldots, h_n)$ defined for all integers $n \geqq 2$, all $x \in D$ and all linearly independent $h_i \in S$, $i = 1, \ldots, n$ such that D is a.e. in every $U(x; h_1, \ldots, h_n; \delta(x; h_1, \ldots, h_n))$.

Definition 4. Let A be a linear set with a semi-norm, D, Q subsets of A. The set Q is called the star-neighbourhood of D if there exists a positive function $\eta(x, h)$, $x \in D$, $h \in A$ such that $x + \lambda h \in Q$ provided that $x \in D$, $h \in A$, $\|\lambda h\| < \eta(x, h)$.

Definition 5. Let A be a linear set with a semi-norm, D a subset of A with Property (A). We denote by $\hat{L}(D)$ the set of all triplets $[x, h, \delta]$ where $x \in D$, $h \in A$, δ is a positive number such that D is a.e. in $U(x; h; \delta)$. Denote by $L(D)$ the set

$$\{y : \exists\{[x, h, \delta],\ \lambda : [x, h, \delta] \in \hat{L}(D),\ y = x + \lambda h,\ \|\lambda h\| < \delta\}\},$$

i.e. the set of all points of A which belong at least to one interval $U(x; h; \delta)$ fulfilling $[x, h, \delta] \in \hat{L}(D)$.

Let B be a Banach space. In this paper we shall deal with transformations $f : D \to B$ fulfilling the inequality

$$\|f(x + h) - f(x) - f(y + h) + f(y)\| \leqq \|x - y\|\ \omega(\|h\|) \tag{1,1}$$

for $x \in D$, $x + h \in D$, $y \in D$, $y + h \in D$ where $\omega(\eta)$ is defined and continuous for $\eta \geqq 0$, $\omega(0) = 0$.

Definition 6. We say that $f^*(x)$ is an extension of $f(x)$ on $L(D)$ if $f^*(x)$ fulfils (1,1) on $L(D)$ and $f^*(x) = f(x)$ on D. The extension on $L(D)$ is unique if any two extensions are equal on $L(D)$.

Definition 7. Let f be a transformation $f : D \to B$ where B is a Banach space, D is a subset of a linear set A. Let $x \in D$. If D is a star-neighbourhood of the point x and if

$$T_x h = \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h) - f(x))$$

exists for all $h \in A$ and T_x is a linear operator $T_x : A \to B$, then T_x is called the (Gateaux) differential of f at the point x.

The semi-norm of A induces a certain topology on A. Although the topology need not be Hausdorff topology, we say that a transformation $f: D \to B$, $D \subset A$ is locally bounded (locally Lipschitz continuous) if to every $x \in D$ there exists a region G_x, $x \in G_x \subset A$ such that f is bounded (Lipschitz continuous) in G_x.

2. Let $[x, h, \delta] \in \hat{L}(D)$ then we can define $\varphi(x; h; \lambda) = f(x + \lambda h)$. The domain of definition of $\varphi(x; h; \lambda)$ (as a function of λ) is $\mathcal{O}(D \cap A(x; h))$. With respect to Definition 5, D is a.e. in the $U(x; h; \delta)$ and this means (Definition 1) that $\mathcal{O}(D \cap A(x; h))$ is a.e. in some open interval (a, b). The next Lemma enables us to consider only transformations $\varphi : (a, b) \to B$ where (a, b) is an open interval.

Lemma 1. *Let A be a linear set with a semi-norm, B a Banach space, $D \subset A$, D have Property (A), f be a transformation $f: D \to B$ fulfilling* (1,1). *Let $[x, h, \delta] \in \hat{L}(D)$, then the function $\varphi(x; h; \lambda)$ is defined on $H = \mathcal{O}(D \cap A(x; h))$ which is a.e. in some carrier-interval (a, b) containing* 0 *and*

$$(2,1) \qquad \|\varphi(x; h; \lambda + \gamma) - \varphi(x; h; \lambda) - \varphi(x; h; \mu + \gamma) + \varphi(x; h; \mu)\| \leqq \leqq |\lambda - \mu| \, \|h\| \, \omega(|\gamma| \, \|h\|) \leqq |\lambda - \mu| \, \omega^*(|\gamma|)$$

for $\lambda \in H$, $\lambda + \gamma \in H$, $\mu \in H$, $\mu + \gamma \in H$, $\omega^*(\eta) = \|h\| \max\limits_{0 \leqq \zeta \leqq \eta} \omega(\zeta \|h\|)$.

The proof follows immediately from Definition 5 and from the relation $\varphi(x; h; \lambda) = f(x + \lambda h)$.

3. For the following result we need some statement about measurable sets.

Lemma 2. *Let H be a.e. in an interval (a, b), $-\infty < a < b < \infty$. Let numbers u, v, ε fulfil $0 \leqq u < (b - a)/4$, $0 \leqq v < (b - a)/4$, $\varepsilon \geqq 0$ and let a measurable set Q fulfil $Q \subset (a, b)$,* $\operatorname{meas}(Q) \geqq b - a - \varepsilon$. *Then the set $F = \{x : a < x \leqq (a + 3b)/4,\ x + u \in Q,\ x + v \in H\}$ is measurable and* $\operatorname{meas}(F) \geqq 3(b - a)/4 - \varepsilon$.

Proof. Let $\pi_u(x) = x - u$ and $\pi_v(x) = x - v$. Denote $\pi_u(Q) = \{y : \exists\{x : x \in Q, y = x - u\}\}$ and similarly $\pi_v(H) = \{y : \exists\{x : x \in H,\ y = x - v\}\}$. Obviously

$$(3,1) \qquad F = \{x : a < x \leqq (a + 3b)/4,\ x + u \in Q,\ x + v \in H\} = = \{x : a < x \leqq (a + 3b)/4\} \cap \pi_u(Q) \cap \pi_v(H)\,.$$

Since H is a.e. in (a, b) and $0 \leqq v < (b - a)/4$, the set $\pi_v(H)$ is a.e. in the interval $\langle a, (a + 3b)/4\rangle$. Since $\operatorname{meas}(Q) \geqq b - a - \varepsilon$ and $0 \leqq u < (b - a)/4$ we have $\operatorname{meas}(Q \cap \{x : a + u < x \leqq u + (a + 3b)/4\}) \geqq 3(b - a)/4 - \varepsilon$ and $\operatorname{meas}(\pi_u(Q) \cap \cap \{x : a < x \leqq (a + 3b)/4\}) = \operatorname{meas} \pi_u(Q \cap \{x : a + u < x \leqq u + (a + 3b)/4\}) \geqq \geqq 3(b - a)/4 - \varepsilon$. From this inequality, (3,1) and from the fact that $\pi_v(H)$ is a.e. in $\langle a, (a + 3b)/4\rangle$ we obtain $\operatorname{meas}(F) \geqq 3(b - a)/4 - \varepsilon$. Lemma 2 is proved.

4. Lemma 3. *Let H be a.e. in an interval (a, b), $-\infty < a < b < \infty$, B a Banach space, $\varphi(\lambda)$ a transformation $\varphi : H \to B$ fulfilling (2,1) on H. If the function $\|\varphi(\lambda)\|$ is measurable then φ is bounded on $H \cap (a, b)$.*

Proof. We shall assume that φ is not bounded on $H \cap (a, b)$. We choose a number N such that meas $(Q) > 3(b - a)/4$ where $Q = \{\lambda : \lambda \in H \cap (a, b), \|\varphi(\lambda)\| \leqq N\}$. Since φ is assumed not to be bounded, hence there exists a number $\bar{\lambda} \in H \cap (a, b)$ such that $\|\varphi(\bar{\lambda})\| > 3N + \omega^*((b - a)/4)(b - a)/4 = M$. We shall assume $\bar{\lambda} \in H \cap \cap (a, (3a + b)/4)$ without loss of generality. If $\varphi(\lambda)$ is bounded on some subinterval of (a, b) then by Lemma 5 it is locally Lipschity continuous on this subinterval and by (2,1) is locally Lipschitz continuous on the whole (a, b). The proof of Lemma 5 is independent of Lemma 3. The set $Q^* = \{\lambda : \lambda \in H \cap (a, (a + 3b)/4), \|\varphi(\lambda)\| \leqq N\}$ has the measure meas $(Q^*) > (b - a)/2$ so that $Q^* \cap (a, (3a + b)/4)$ is nonempty. We choose $\mu \in Q^* \cap (a, (3a + b)/4)$. Using Lemma 2 with $u = \mu - a$, $v = \bar{\lambda} - a$ we obtain that the set $F = \{\lambda : \lambda \in (a, (a + 3b)/4), \|\varphi(\lambda + \mu - a)\| \leqq N, \lambda + \bar{\lambda} - a \in \in H\}$ has the measure meas $(F) \geqq (b - a)/2$. Using (2,1) we obtain

$$\|\varphi(\lambda + \bar{\lambda} - a) - \varphi(\bar{\lambda}) - \varphi(\lambda + \mu - a) + \varphi(\mu)\| \leqq$$
$$\leqq |\mu - \bar{\lambda}|\, \omega^*(|\lambda - a|) \leqq \frac{b - a}{4}\, \omega^*\left(\frac{b - a}{4}\right)$$

and

$$\|\varphi(\lambda + \bar{\lambda} - a)\| \geqq \|\varphi(\bar{\lambda})\| - \|\varphi(\mu)\| - \|\varphi(\lambda + \mu - a)\| - \frac{b - a}{4}\, \omega^*\left(\frac{b - a}{4}\right) >$$
$$> M - 2N - \frac{b - a}{4}\, \omega^*\left(\frac{b - a}{4}\right) \geqq N \quad \text{i.e.} \quad \|\varphi(\lambda + \bar{\lambda} - a)\| > N \quad \text{for} \quad \lambda \in F\,.$$

($\mu \in Q^*$ which implies $\|\varphi(\mu)\| \leqq N$).

The set $Q^{**} = \{\lambda + \bar{\lambda} - a : \lambda \in F\} = \pi_{a-\bar{\lambda}}(F)$ for which meas $(Q^{**}) =$ meas $(F) > > (b - a)/2$ is a subset of the complement of the set Q and therefore $(b - a)/2 < <$ meas $(Q^*) \leqq$ meas $(\langle a, b\rangle - Q^{**}) < (b - a)/2$. This contradiction proves Lemma 3.

5. We shall need one more statement about measurable sets.

Lemma 4. *Let H be a.e. in (a, b), then there exists a set $\tilde{H}$ which is a.e. in $(a, b) \times \times (a, b)$ and fulfils*

i) *if $(\lambda, \mu) \in \tilde{H}$ then $(\mu, \lambda) \in \tilde{H}$,*
ii) *if $(\lambda, \mu) \in \tilde{H}$ and $\lambda + k(\mu - \lambda) \in (a, b)$ for any integer k, then $\lambda + k(\mu - \lambda) \in H$,*
iii) *if $(\lambda, \mu) \in \tilde{H}$, then $\lambda \in H$, $\mu \in H$.*

Proof. Denote $J = (a, b)$, $J^2 = (a, b) \times (a, b)$. Let $\tilde{\tilde{H}}$ be the set

$$\tilde{\tilde{H}} = \{(\lambda, \mu) : (\lambda, \mu) \in J^2, \lambda \in H, \mu \in H\} + \{(\lambda, \mu) : \lambda \in HJ, \mu \notin J\} +$$
$$+ \{(\lambda, \mu) : \lambda \notin J, \mu \in HJ\} + \{(\lambda, \mu) : \lambda \notin J, \mu \notin J\}\,.$$

The set $\tilde{\tilde{H}}$ is a.e. in the plane $(-\infty, \infty) \times (-\infty, \infty)$. Denote

$$\tilde{\tilde{H}}_k = \left\{(\lambda, \mu) : \lambda = \frac{1}{2k-1}((k-1)x + ky), \right.$$

$$\left. \mu = \frac{1}{2k-1}(kx + (k-1)y), \ (x, y) \in \tilde{\tilde{H}} \right\}$$

for every integer k. The sets $\tilde{\tilde{H}}_k$ are also a.e. in the plane. It means that $\tilde{H} = \bigcap_k \tilde{\tilde{H}}_k \cap J^2$ is a.e. in J^2.

The item i) is fulfilled since all $\tilde{\tilde{H}}_k$ fulfil i). Let $(\lambda, \mu) \in \tilde{H}$ and let $\lambda + k(\mu - \lambda) \in J$. Obviously $(\lambda, \mu) \in \tilde{\tilde{H}}_k$. By the definition of $\tilde{\tilde{H}}_k$ there exists a point $(x, y) \in \tilde{\tilde{H}}$ such that

$$x = \lambda + k(\mu - \lambda), \quad y = \mu + k(\lambda - \mu), \quad (x, y) \in \tilde{\tilde{H}}, \quad x \in J$$

which means with respect to the definition of $\tilde{\tilde{H}}$ that $x \in H$. Condition iii) is a simple consequence of conditions i) ii) since if $(\lambda, \mu) \in \tilde{H}$ we can apply ii) with $k = 0$. It implies $\lambda \in H$. To prove that also $\mu \in H$ we can use condition i).

6. Lemma 5. *Let H be a.e. in (a, b), $-\infty < a < b < \infty$. Let B be a Banach space, $\varphi(\lambda)$ a transformation $\varphi : H \to B$ fulfilling* (2,1) *on H. If the function $\|\varphi(\lambda)\|$ is bounded then φ is Lipschitz continuous on $H \cap (a, b)$.*

Proof. First we prove a proposition.

Proposition 1. *Let the assumptions of Lemma* 5 *be fulfilled, then there exists a number $L > 0$ such that*

$$\|\varphi(\lambda) - \varphi(\mu)\| \leqq L|\lambda - \mu| \quad \textit{if} \quad (\lambda, \mu) \in \tilde{H}$$

where $\tilde{H}$ is the set constructed in Lemma 4.

Proof of the Proposition. We choose a number $\delta > 0$ such that $\omega^*(\delta) < \frac{1}{2}$, $\delta_1 = \frac{1}{3}\min(\delta, (b-a)/2)$ and put $L = \max(1, 2N/\delta_1)$ where N is the upper bound of φ : $\|\varphi(\lambda)\| \leqq N$. Assume that there exists $(\lambda_1, \lambda_2) \in \tilde{H}$ such that $\|\varphi(\lambda_1) - \varphi(\lambda_2)\| > L|\lambda_1 - \lambda_2|$. This inequality yields $|\lambda_1 - \lambda_2| \leqq 2N/L \leqq \delta_1$. Without loss of generality we assume $\lambda_1 < \lambda_2$ and $2\lambda_1 \leqq a + b$ (if $2\lambda_1 > a + b$ then we consider $\tilde{\varphi}(\lambda) = \varphi(-\lambda)$, $\tilde{a} = -b$, $\tilde{b} = -a$, $\tilde{\lambda}_1 = -\lambda_2$, $\tilde{\lambda}_2 = -\lambda_1$; since $a + b < 2\lambda_1 < 2\lambda_2$ we obtain $2\tilde{\lambda}_1 < 2\tilde{\lambda}_2 < \tilde{a} + \tilde{b}$ so that the additional assumption is fulfilled for $\tilde{\varphi}(\lambda), \tilde{\lambda}_1, \tilde{\lambda}_2)$. We find the largest integer k such that $\lambda_k = \lambda_1 + (k-1)(\lambda_2 - \lambda_1) \in (a, b)$, $\lambda_k < \lambda_1 + \delta$. Since $\delta_1 \leqq \delta/3$, $\delta_1 \leqq (b-a)/6$ and $\lambda_1 \leqq (a+b)/2$ we obtain

$$\delta > \lambda_k - \lambda_1 = (k-1)(\lambda_2 - \lambda_1) \geqq 2\delta_1 . \tag{6,1}$$

By Lemma 4 we know that $\lambda_i \in H$ for $i = 1, \ldots, k$ and using (2,1) we obtain

$$\|\varphi(\lambda_k) - \varphi(\lambda_1)\| =$$
$$= \|(k-1)(\varphi(\lambda_2) - \varphi(\lambda_1)) + \sum_{l=1}^{k-1}(\varphi(\lambda_{l+1}) - \varphi(\lambda_l) - \varphi(\lambda_2) + \varphi(\lambda_1))\| \geqq$$
$$\geqq (k-1)\|\varphi(\lambda_2) - \varphi(\lambda_1)\| - |\lambda_2 - \lambda_1| \sum_{l=1}^{k-1} \omega^*((l-1)|\lambda_2 - \lambda_1|) >$$
$$> (k-1) L|\lambda_2 - \lambda_1| - |\lambda_2 - \lambda_1| \sum_{l=1}^{k-1} \omega^*((l-1)|\lambda_2 - \lambda_1|) \geqq$$
$$\geqq L|\lambda_2 - \lambda_1| \sum_{l=1}^{k-1}(1 - \omega^*((l-1)|\lambda_2 - \lambda_1|)) \geqq$$
$$\geqq L|\lambda_2 - \lambda_1| \sum_{l=1}^{k-1}(1 - \omega^*(\delta)) \geqq L|\lambda_2 - \lambda_1|(k-1)/2 \geqq 2N .$$

The two last inequalities follow from (6,1) and from the definition of L. The former inequality gives $\|\varphi(\lambda_k)\| > 2N - \|\varphi(\lambda_1)\| \geqq N$ and finally $\|\varphi(\lambda_k)\| > N$ which is in contradicition with the boundedness of φ. Proposition 1 is proved.

Let H^* be the set of all λ such that $(\lambda, \mu) \in \tilde{H}$ for almost all $\mu \in (a, b)$. Since $\tilde{H}$ is a.e. in J^2 we obtain by Fubini's theorem that H^* is a.e. in $J = (a, b)$. By Lemma 4 iii) we have $H^* \subset H$. Let $\lambda_i \in H^*$, $i = 1, 2$, $\lambda_1 < \lambda_2$. Put $Z_i = \{\lambda : (\lambda_i, \lambda) \in \tilde{H}, \lambda_1 \leqq \lambda \leqq \lambda_2\}$, $i = 1, 2$. The sets Z_i are a.e. in (λ_1, λ_2) so that there exists $\mu \in \bigcap_{i=1}^{2} Z_i$. It means $(\lambda_i, \mu) \in \tilde{H}$, $i = 1, 2$, $\lambda_1 \leqq \mu \leqq \lambda_2$. Using Proposition 1 we obtain $\|\varphi(\lambda_2) - \varphi(\lambda_1)\| \leqq \|\varphi(\lambda_2) - \varphi(\mu)\| + \|\varphi(\mu) - \varphi(\lambda_1)\| \leqq L(\lambda_2 - \mu) + L(\mu - \lambda_1) = L(\lambda_2 - \lambda_1)$. We have already proved that $\varphi(\lambda)$ is Lipschitz continuous on H^*, $H^* \subset H$, H^* is a.e. in (a, b). It remains to prove that $\varphi(\lambda)$ is Lipschitz continuous on H. To prove this we shall need another proposition.

Proposition 2. *For every $\lambda \in H - H^*$ there exists a sequence λ_n, $\lambda_n \in H^*$, $\lambda_n \to \lambda$ such that $\varphi(\lambda_n) \to \varphi(\lambda)$.*

Proof of Proposition 2. Suppose that Proposition 2 is not true, then there exists $\bar{\lambda} \in H - H^*$ and a number $\delta > 0$ such that

$$\|\varphi(\bar{\lambda}) - \varphi(\lambda)\| > \delta \quad \text{for} \quad \lambda \in H^* , \quad |\bar{\lambda} - \lambda| \leqq \delta . \tag{6,2}$$

Choose a number δ_1, $0 < \delta_1 < \delta$, $\frac{1}{2}\delta_1(L + \omega^*(\frac{1}{2}\delta_1)) < \delta$ and μ, $\mu \in H^* \cap (\bar{\lambda}, \bar{\lambda} + \delta_1/4)$. Using Lemma 2 (where instead of (a, b) is $(\bar{\lambda}, \bar{\lambda} + \delta_1)$, instead of the sets H, Q is the set H^*, $u = 0$, $v = \mu - \bar{\lambda}$) we obtain: There exists γ fulfilling $\gamma \in H^* \cap (\bar{\lambda}, \bar{\lambda} + \frac{1}{2}\delta_1)$, $\gamma + \mu - \bar{\lambda} \in H^* \cap (\bar{\lambda}, \bar{\lambda} + \delta_1)$. With respect to (2,1) there is

$$\|\varphi(\gamma) - \varphi(\bar{\lambda})\| \leqq \|\varphi(\gamma + \mu - \bar{\lambda}) - \varphi(\mu)\| + \frac{\delta_1}{2} \omega^*\left(\frac{\delta_1}{2}\right) \leqq \tag{6,3}$$
$$\leqq L|\gamma - \bar{\lambda}| + \frac{\delta_1}{2} \omega^*\left(\frac{\delta_1}{2}\right) \leqq \frac{\delta_1}{2}\left(L + \omega^*\left(\frac{\delta_1}{2}\right)\right) < \delta .$$

But $\gamma \in H^* \cap (\bar{\lambda}, \bar{\lambda} + \delta_1/2)$ implies $|\gamma - \bar{\lambda}| \leqq \delta_1/2 < \delta$, hence (6,3) is in contradiction with (6,2). Proposition 2 is proved.

Let $\bar{\lambda} \in H$, $\bar{\bar{\lambda}} \in H$. By Proposition 2 there exist sequences $\bar{\lambda}_n$,

$$\bar{\lambda}_n \to \bar{\lambda}\,, \quad \bar{\lambda}_n \in H^*\,, \quad \|\varphi(\bar{\lambda}_n) - \varphi(\bar{\lambda})\| \to 0$$

and $\bar{\bar{\lambda}}_n$,

$$\bar{\bar{\lambda}}_n \to \bar{\bar{\lambda}}, \bar{\bar{\lambda}}_n \in H^*\,, \quad \|\varphi(\bar{\bar{\lambda}}_n) - \varphi(\bar{\bar{\lambda}})\| \to 0\,.$$

According to the fact that $\varphi(\lambda)$ is Lipschitz continuous on H^* we obtain

$$\|\varphi(\bar{\lambda}) - \varphi(\bar{\bar{\lambda}})\| \leqq \|\varphi(\bar{\lambda}) - \varphi(\bar{\lambda}_n)\| + L|\bar{\lambda}_n - \bar{\bar{\lambda}}_n| + \|\varphi(\bar{\bar{\lambda}}_n) - \varphi(\bar{\bar{\lambda}})\|\,.$$

Thus $\varphi(\lambda)$ is Lipschitz continuous on H.

Remark 1. The Lipschitz coefficient $L = \max(1, 2N/\delta_1)$ depends on the length of the interval (a, b), on ω^* and on N which bounds $\varphi(\lambda)$. The constant N can be determined by $N = \inf_{z \in B} \sup_{y \in H(a,b)} \|\varphi(\lambda) - z\|$.

7. **Remark 2.** Let the statement of Lemma 5 hold. Since the space B is complete we can extend $\varphi(\lambda)$ onto the whole carrier-interval of H (may be $(-\infty, \infty)$) so that $\varphi(\lambda)$ is continuous and locally Lipschitz continuous on the carrier-interval.

Let λ_i, $i = 1, \ldots, 4$ belong to the carrier-interval and $\lambda_2 - \lambda_1 = \lambda_4 - \lambda_3$. Consider numbers $\lambda, \lambda + \lambda_2 - \lambda_1, \lambda + \lambda_3 - \lambda_1, \lambda + \lambda_4 - \lambda_1$. Since H is a.e. in the carrier-interval there exists a sequence λ_n, $n > 4$ such that $\lambda_n \to \lambda_1$ for $n \to \infty$, $\lambda_n \in H$, $\lambda_n + \lambda_2 - \lambda_1 \in H$, $\lambda_n + \lambda_3 - \lambda_1 \in H$, $\lambda_n + \lambda_4 - \lambda_1 \in H$ (this can be proved similarly as Lemma 2). As (2,1) is fulfilled for

$$\lambda_n, \lambda_n + \lambda_2 - \lambda_1\,, \quad \lambda_n + \lambda_3 - \lambda_1\,, \quad \lambda_n + \lambda_4 - \lambda_1$$

and ω^* is continuous, inequality (2,1) holds for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, too.

This extension is obviously unique as a continuous extension, but we shall prove that the extension is unique in the sense of Definition 6. Let (a, b) be a part of the carrier-interval and $\tilde{\varphi}(\lambda)$ be another extension of φ. Put $\Delta(\lambda) = \varphi(\lambda) - \tilde{\varphi}(\lambda)$. The function $\Delta(\lambda)$ fulfils (2,1) on (a, b) (see Definition 6) and $\Delta(\lambda) = 0$ for $\lambda \in H$, H is a.e. in (a, b). Let $\bar{\lambda} \in (a, b) - H$. We choose μ, $\mu \in H \cap (a, \min(b, a + b - \bar{\lambda}))$. Since $a < \min(b, a + b - \bar{\lambda})$ and H is a.e. in this interval, such μ exists. We have $\mu + \bar{\lambda} - \lambda \in H \cap (a, b)$ for almost all λ, $a < \lambda < \bar{\lambda}$. Since H is a.e. in $(a, \bar{\lambda})$ there exists a sequence λ_n, $\lambda_n \to \bar{\lambda}$, $\lambda_n \in H \cap (a, \bar{\lambda})$, $\mu + \bar{\lambda} - \lambda_n \in H$. It means $\Delta(\lambda_n) = \Delta(\mu + \bar{\lambda} - \lambda_n) = \Delta(\mu) = 0$. If we use (2,1) for $\lambda_n, \bar{\lambda}, \mu, \mu + \bar{\lambda} - \lambda_n$ we obtain $\Delta(\bar{\lambda}) = 0$.

8. We have defined the differential of $f(x)$ as an operator. Since the domain of definition of φ is a one-dimensional interval the domain of the differential operator of φ is the real line. It is clear that there exists an element ζ of B such that $T_x\lambda = \zeta\lambda$. In case of the transformation φ we shall define the derivative $\varphi'(x)$ by $\varphi'(x) = \zeta$.

Lemma 6. *Let $-\infty < a < b < \infty$, B be a Banach space, $\varphi(\lambda)$ be a transformation $\varphi : (a, b) \to B$. If $\varphi(\lambda)$ is Lipschitz continuous and fulfils (2,1) on (a, b), then $\varphi(\lambda)$ has the derivative $\varphi'(\lambda)$ at every point of (a, b) and $\|\varphi'(\lambda) - \varphi'(\mu)\| \leqq \omega^*(|\lambda - \mu|)$.*

Proof. Let $\xi \in (a, b)$ be given. Let $\delta \leqq \min(\xi - a, (1 + 2(b - \xi) - \sqrt{[1 + 4(b - \xi)]})/2)$ and let $\lambda_i, \tilde{\lambda}_i, i = 1, 2$ fulfil $|\lambda_i - \xi| < \delta/2$, $\lambda_1 < \lambda_2$, $\lambda_1 \leqq \xi \leqq \lambda_2$ and $|\tilde{\lambda}_i - \xi| < \delta/2$, $\tilde{\lambda}_1 < \tilde{\lambda}_2$, $\tilde{\lambda}_1 \leqq \xi \leqq \tilde{\lambda}_2$. Put $\lambda_k = \lambda_1 + (k - 1)(\lambda_2 - \lambda_1)$, $\tilde{\lambda}_l = \tilde{\lambda}_1 + (l - 1)(\tilde{\lambda}_2 - \tilde{\lambda}_1)$. Consider the largest k such that $\lambda_k \in \langle \xi + \sqrt{\delta}, \xi + \delta + \sqrt{\delta} \rangle$, and the largest l such that $\tilde{\lambda}_l \in \langle \xi + \sqrt{\delta}, \xi + \delta + \sqrt{\delta} \rangle$. For $\lambda_i, \tilde{\lambda}_j$ we have

$$(8,1) \qquad |\lambda_1 - \tilde{\lambda}_1| < \delta/2\,, \quad |\tilde{\lambda}_l - \lambda_k| \leqq \delta\,, \quad \sqrt{\delta} \leqq |\lambda_k - \lambda_1| \leqq 2\delta + \sqrt{\delta}\,,$$
$$\sqrt{\delta} \leqq |\tilde{\lambda}_l - \tilde{\lambda}_1| \leqq 2\delta + \sqrt{\delta}\,.$$

By (2,1) we obtain

$$\left\| \frac{\varphi(\lambda_k) - \varphi(\lambda_1)}{\lambda_k - \lambda_1} - \frac{\varphi(\lambda_2) - \varphi(\lambda_1)}{\lambda_2 - \lambda_1} \right\| \leqq \frac{1}{k - 1} \sum_{s=1}^{k-1} \omega^*((s - 1)\,|\lambda_2 - \lambda_1|) \leqq \omega^*(2\delta + \sqrt{\delta})\,,$$

$$\left\| \frac{\varphi(\tilde{\lambda}_l) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_l - \tilde{\lambda}_1} - \frac{\varphi(\tilde{\lambda}_2) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \right\| \leqq \frac{1}{l - 1} \sum_{s=1}^{l-1} \omega^*((s - 1)\,|\lambda_2 - \lambda_1|) \leqq \omega^*(2\delta + \sqrt{\delta})\,.$$

These inequalities imply

$$(8,2) \qquad \left\| \frac{\varphi(\lambda_2) - \varphi(\lambda_1)}{\lambda_2 - \lambda_1} - \frac{\varphi(\tilde{\lambda}_2) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \right\| \leqq$$
$$\leqq \left\| \frac{\varphi(\lambda_k) - \varphi(\lambda_1)}{\lambda_k - \lambda_1} - \frac{\varphi(\tilde{\lambda}_l) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_l - \tilde{\lambda}_1} \right\| + 2\omega^*(2\delta + \sqrt{\delta})\,.$$

Using (8,1) we obtain

$$\left\| \frac{\varphi(\tilde{\lambda}_l) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_l - \tilde{\lambda}_1} - \frac{\varphi(\tilde{\lambda}_l) - \varphi(\tilde{\lambda}_1)}{\lambda_k - \lambda_1} \right\| =$$
$$= \|\varphi(\tilde{\lambda}_l) - \varphi(\tilde{\lambda}_1)\| \frac{|\lambda_k - \tilde{\lambda}_l - \lambda_1 + \tilde{\lambda}_1|}{(\lambda_k - \lambda_1)(\tilde{\lambda}_l - \tilde{\lambda}_1)} \leqq 2L(2\delta + \sqrt{\delta})$$

and by (8,2)

$$\left\| \frac{\varphi(\lambda_2) - \varphi(\lambda_1)}{\lambda_2 - \lambda_1} - \frac{\varphi(\tilde{\lambda}_2) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \right\| \leqq \frac{1}{\sqrt{\delta}} \|\varphi(\lambda_k) - \varphi(\lambda_1) - \varphi(\tilde{\lambda}_l) + \varphi(\tilde{\lambda}_1)\| +$$
$$+ 2L(2\delta + \sqrt{\delta}) + 2\omega^*(2\delta + \sqrt{\delta}) \leqq 4L(2\delta + \sqrt{\delta}) + 2\omega^*(2\delta + \sqrt{\delta})\,.$$

Thus we have proved the inequality

$$\left\| \frac{\varphi(\lambda_2) - \varphi(\lambda_1)}{\lambda_2 - \lambda_1} - \frac{\varphi(\tilde{\lambda}_2) - \varphi(\tilde{\lambda}_1)}{\tilde{\lambda}_2 - \tilde{\lambda}_1} \right\| \leqq 4L(2\delta + \sqrt{\delta}) + 2\omega^*(2\delta + \sqrt{\delta})\,.$$

Since B is a complete space there follows from the last inequality that $\varphi(\lambda)$ has the derivative at all points of (a, b). With respect to (2,1) we easily obtain $\|\varphi'(\lambda) - \varphi'(\mu)\| \leqq \omega^*(|\lambda - \mu|)$.

9. Now we return to the transformation $f(x)$. We have a set of extended functions $\varphi(x; h; \lambda)$. We may use this set of functions for the extension of $f(x)$ on $L(D)$ only if the following condition is fulfilled. Let $z \in L(D)$ and let $[x_i, h_i, \delta_i]$, λ_i, $i = 1, 2$ be given such that $[x_i, h_i, \delta_i] \in \hat{L}(D)$, $z = x_i + \lambda_i h_i$, $\|\lambda_i h_i\| < \delta_i$, $i = 1, 2$, then $\varphi(x_1; h_1; \lambda_1) = \varphi(x_2; h_2; \lambda_2)$ (h_i linearly independent).

If this condition is fulfilled, we can define the extension of $f(x)$ by $f(z) = \varphi(x_1; h_1; \lambda_1)$. To this problem the next Lemma is devoted.

Lemma 7. *The set of functions $\varphi(x; h; \lambda)$ determines the extension $f(x)$ on $L(D)$.*

Proof. Let $[x_i, h_i, \delta_i]$, λ_i, $i = 1, 2$ have the above formulated properties. Since $x_1 \in D$ there exists a constant $\delta = \delta(x_1; h_1, h_2)$ (cf. Definition 3) such that $\mathcal{O}(D \cap A(x_1; h_1, h_2))$ is a.e. in $\mathcal{O}(U(x_1; h_1, h_2; \delta))$. By Fubini theorem we obtain that there exists $\bar{\lambda}$, $\bar{\lambda}$ between 0 and λ_1 such that $x_1 + \bar{\lambda} h_1 \in D$, $x_1 + \bar{\lambda} h_1 + \mu h_2 \in D$ for almost all $|\mu| < \delta^*$ where δ^* is a positive number. (Since $\bar{\lambda}$ is between 0 and λ_1 it belongs to the same carrier-interval as 0 and λ_1.) Hence there exists a sequence of μ_n such that $x_1 + \bar{\lambda} h_1 + \mu_n h_2 \in D$ and $x_1 + \lambda_1 h_1 + \mu_n h_2 \in D$, $\mu_n \to 0$. Obviously we can choose λ, λ between 0, λ_1 such that $x_1 + \lambda h_1 \in D$ and $x_1 + (\lambda + \lambda_1 - \bar{\lambda})\, h_1 \in D$. As these four points belong to D we use (1,1) to obtain

$$(9,1) \qquad \|f(x_1 + (\lambda + \lambda_1 - \bar{\lambda})\, h_1) - f(x_1 + \lambda_1 h_1 + \mu_n h_2)\| \leqq$$
$$\leqq \|f(x_1 + (\lambda + \lambda_1 - \bar{\lambda})\, h_1) - f(x_1 + \lambda_1 h_1 + \mu_n h_2) - f(x_1 + \lambda h_1) +$$
$$+ f(x_1 + \bar{\lambda} h_1 + \mu_n h_2)\| + \|f(x_1 + \lambda h_1) - f(x_1 + \bar{\lambda} h_1 + \mu_n h_2)\| \leqq$$
$$\leqq (|\lambda - \bar{\lambda}| \times \|h_1\| + |\mu_n| \times \|h_2\|)\, \omega(|\lambda_1 - \bar{\lambda}| \times \|h_1\|) +$$
$$+ \|f(x_1 + \lambda h_1) - f(x_1 + \bar{\lambda} h_1 + \mu_n h_2)\|\,.$$

Denote $\bar{f}(z) = \lim_{n \to \infty} f(z + \mu_n h_2)$. Since $x_1 + \lambda_1 h = z = x_2 + \lambda_2 h_2$, there is $\bar{f}(z) = \lim_{n \to \infty} f(x_1 + \lambda_1 h_1 + \mu_n h_2)$. By (9,1) and by $x_1 + \bar{\lambda} h_1 \in D$, $x_1 + \bar{\lambda} h_1 + \mu h_2 \in L(D)$ for $|\mu| < \delta^*$, which means $[x_1 + \bar{\lambda} h_1, h_2, \delta^*] \in \hat{L}(D)$, we obtain

$$(9,2) \quad \|f(x_1 + (\lambda + \lambda_1 - \bar{\lambda})\, h_1) - \bar{f}(z)\| \leqq |\lambda - \bar{\lambda}| \times \|h_1\|\, \omega(|\lambda_1 - \bar{\lambda}| \times \|h_1\|) +$$
$$+ \|f(x_1 + \lambda h_1) - f(x_1 + \bar{\lambda} h_1)\| \leqq |\lambda - \bar{\lambda}| \times \|h_1\|\, (\omega(|\lambda_1 - \bar{\lambda}| \times \|h_1\|) + L)\,.$$

Since $\mathcal{O}(D \cap A(x_1; h_1))$ is a.e. on $\mathcal{O}(U(x_1; h_1; \delta_1))$ we can choose a sequence of λ_n such that $\lambda_n \to \bar{\lambda}$ and $x_1 + \lambda_n h_1 \in D$, $x_1 + (\lambda_n + \lambda_1 - \bar{\lambda})\, h_1 \in D$. With respect to (9,2) we obtain $\lim_{n \to \infty} f(x_1 + (\lambda_n + \lambda_1 - \bar{\lambda})\, h_1) = \bar{f}(z) = \lim_{n \to \infty} f(z + \mu_n h_2)$. Since $x_1 + (\lambda_n + \lambda_1 - \bar{\lambda})\, h_1$ converges to z the uniqueness is proved.

10. Lemma 8. *The set $L(D)$ is a star-neighbourhood of D.*

Proof. Let $x \in D$, $h \in A$. Since S is a basis of A (Definition 2) there are linearly independent h_i, $i = 1, \ldots, n$, $h_i \in S$ such that $h = \sum_{i=1}^{n} \lambda_i h_i$. Since the set D has Property (A) there exists a positive number $\delta(x; h_1, \ldots, h_n)$ (cf. Definition 3). We shall prove that the set $U(x; h_1, \ldots, h_n; \delta(x; h_1, \ldots, h_n))$ is a subset of $L(D)$. Let $y \in U(x; h_1, \ldots, h_n; \delta(x; h_1, \ldots, h_n))$. Since $\mathcal{O}(D \cap A(x; h_1, \ldots, h_n))$ is a.e. in $\mathcal{O}(U(x; h_1, \ldots, h_n; \delta(x; h_1, \ldots \ldots, h_n)))$ there exists $\tilde{h}$ which is a linear combination of $h_1, \ldots, h_n$ and there exists a number $\tilde{\delta} > 0$ such that $\mathcal{O}(D \cap A(y; \tilde{h}))$ is a.e. in $\mathcal{O}(U(y; \tilde{h}; \tilde{\delta}))$. It means that y belongs to $L(D)$. The statement is proved and from this statement Lemma immediately follows.

11. We have all prepared to present one of the main results.

Theorem 1. *Let A be a linear set with a semi-norm, B a Banach space, D a subset of A having property (A). Let f be a transformation $f : D \to B$ fulfilling* (1,1) *on D. If the condition:*

> (α) *$\|f(x + \lambda h)\|$ is a measurable function of λ for every $x \in D$, $h \in S$ for which the carrier-interval of $H = \mathcal{O}(D \cap A(x; h))$ containing* 0 *is nonempty (where S is basis in A)*

is fulfilled, then f can be extended onto the star-neighbourhood $L(D)$ of D and the extension has the differentials T_x at all points of D. We have

$$\|T_x h - T_y h\| \leqq \|h\| \, \omega(\|x - y\|) \quad for \quad x \in D\,, \quad y \in D\,. \tag{11,1}$$

The extension of f on $L(D)$ is unique in the sense of Definition 6.

Proof. Let a point z be from $L(D)$. By Definition 5 there exists $[x, h, \delta] \in \hat{L}(D)$, λ such that $z = x + \lambda h$, $\|\lambda h\| < \delta$ and $\mathcal{O}(D \cap A(x; h))$ is a.e. in $\mathcal{O}(U(x; h; \delta))$. It means that $\mathcal{O}(U(x; h; \delta))$ is a part of the carrier interval containing $H = \mathcal{O}(D \cap \cap A(x; h))$. The function $\varphi(x; h; \lambda)$ is defined on H. Hence recalling assumption (α) we can successively apply Lemmas 1, 3, 5. Since the function φ is Lipschitz continuous on H we can extend φ on the carrier-interval of H as in Remark 2. Then using Lemma 6 we obtain that φ has the derivatives at all points of the carrier-interval of H. By Lemma 7 we can define $f(z) = \varphi(x + \lambda h)$ where $z = x + \lambda h$. We have just extended the domain of definition of $f(x)$ onto the whole $L(D)$ (we shall denote this extension by the same letter f).

We pass to the proof that the extension f has the differentials T_x at all points of D. Let points $x \in D$ and $h \in A$ be given. Since $L(D)$ is a star-neighbourhood of D (cf. Lemma 8) there exists a positive δ such that $[x, h, \delta] \in \hat{L}(D)$. We shall consider the function $\varphi(x; h; \lambda) = f(x + \lambda h)$. By Lemma 6 there exists the derivative $\varphi'(x; h; 0)$. We put $T_x h = \varphi'(x; h; 0)$. We must still prove that T_x is a linear operator.

1. T_x is homogeneous. Let Θ be a real number, then

$$T_x\Theta h = \varphi'(x; \Theta h; 0) = \lim_{\lambda \to 0} \frac{1}{\lambda} (\varphi(x; \Theta h; \lambda) - \varphi(x; \Theta h; 0)) =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \Theta\lambda h) - f(x)) = \Theta \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h) - f(x)) =$$

$$= \Theta \lim_{\lambda \to 0} \frac{1}{\lambda} (\varphi(x; h; \lambda) - \varphi(x; h; 0)) = \Theta T_x h .$$

2. T_x is additive.

$$T_x(h_1 + h_2) = \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h_1 + \lambda h_2) - f(x)) =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h_1 + \lambda h_2) - f(x + \lambda h_1)) + \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h_1) - f(x)) =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h_2) - f(x)) + \lim_{\lambda \to 0} \frac{1}{\lambda} (f(x + \lambda h_1) - f(x)) + \lim_{\lambda \to 0} \psi(x, \lambda, h_1, h_2)) =$$

$$= \varphi'(x; h_2; 0) + \varphi'(x; h_1; 0) + \lim_{\lambda \to 0} \psi(x, \lambda, h_1, h_2) = T_x h_1 + T_x h_2 .$$

The last equality holds as

$$\psi(x, \lambda, h_1, h_2) = \frac{1}{\lambda} (f(x + \lambda h_1 + \lambda h_2) - f(x + \lambda h_1) - f(x + \lambda h_2) + f(x))$$

and by (1,1) $\|\psi(x, \lambda, h_1, h_2)\| \leqq \|h_1\| \, \omega(\|\lambda h_2\|)$. This proves that the operator T_x is linear. Theorem 1 is proved.

12. We have simultaneously proved

Theorem 2. *Let the assumptions of Theorem* 1 *be fulfilled with exception of assumption* (α) *which is replaced by*

(β) $\|f(x + \lambda h)\|$ *is locally bounded in* λ *for every* $x \in D$, $h \in S$ *for which the carrier-interval of* $H = \mathcal{O}(D \cap A(x; h))$ *containing* 0 *is nonempty,*

then the statement of Theorem 1 *is valid.*

The proof of this Theorem follows the same lines as the proof of Theorem 1. We only start with Lemma 5 since assumption (β) and Lemma 1 makes possible its direct application.

13. In this paragraph we introduce some consequences of these Theorems.

Corollary 1. *Let A be a linear set with a semi-norm, B a Banach space, D a region in A, f a transformation fulfilling* (1,1) *on D. If assumption* (α) *or* (β) *is fulfilled, then f has the differential T_x at every point of D and* (11,1) *holds.*

This Corollary follows immediately from the fact that the region D has property (A).

Corollary 2. *Let all assumptions of Theorem* 1 *be fulfilled except* (α). *If f is locally bounded in D, then f may be extended on a star-neighbourhood $L^*(D)$, $D \subset L^*(D) \subset \subset L(D)$ such that f is locally Lipschitz continuous on $L^*(D)$, f has the differentials T_x at all points of D, the differentials T_x are continuous operators and* (11,1) *is fulfilled.*

Obviously, if f is locally bounded on D, then for every $x \in D$ there exists a neighbourhood G_x, $x \in G_x \subset A$ such that $\|f(y)\| \leqq N$ in G_x. Hence $\|f(x + \lambda h)\| \leqq N$ for $x + \lambda h \in G_x$. By Theorem 2 we know that there exists the differential T_x. With respect to Remark 1 the Lipschitz coefficient depends explicitly on N, i.e., all $\varphi(x; h; \lambda) = f(x + \lambda h)$ have the same Lipschitz coefficient. It implies $\|\varphi'(x; h; 0)\| \leqq \leqq L\|h\|$. By Definition $T_x h = \varphi'(x; h; 0)$ so that T_x is a continuous operator. Put $G = \bigcup_{x \in D} G_x$ and $L^*(D) = L(D) \cap G$. $L^*(D)$ is obviously a star-neighbourhood of D. If $x \in L^*(D)$ then a point y exists, $y \in D$, $x \in G_y$. Since the extension of f is bounded in G_y and fulfils (1,1) it is Lipschitz continuous in G_y by Lemma 5.

Corollary 3. *Let the assumptions of Theorem* 1 *be fulfilled except* (α) *which is replaced by the assumption that f is locally bounded on D. If A is a Banach space, then f may be extended (uniquely as a continuous function) onto an open set G, $D \subset G \subset A$ such that f has the differentials T_x at every point of G, T_x are continuous operators fulfilling* (11,1) *on G and f is locally Lipschitz continuous on G.*

By Corollary 2 f may be extended onto the star-neighbourhood $L^*(D)$ so that f is locally Lipschitz continuous there. Let x be a given point, $x \in D$. Since $L^*(D)$ is the star-neighbourhood of D there exists a function $\eta(x, h) > 0$ such that $Q = \{y : y = = x + \lambda h, \ \|\lambda h\| < \eta(x, h)\} \subset L^*(D)$. We denote $Q_k = \{z : z = x + k(y - x), y \in Q\}$ for any nonnegative integer k. Obviously $\bigcup_k Q_k = A$. Recalling Bair Theorem Q cannot be a thin set. Hence an open set G_x^1 exists, $x \in G_x^1 \subset A$, such that Q is dense in G_x^1. Since the extension of f is locally Lipschitz continuous in $L^*(D)$ (cf. Corollary 2) an open set G_x^2, $x \in G_x^2$ exists such that f is Lipschitz continuous in G_x^2. Put $G_x^3 = G_x^1 \cap G_x^2$. We shall extend f onto G_x^3. Let $y \in G_x^3$. Since $L^*(D)$ is dense in G_x^3 a sequence x_n, $x_n \to \to y$, $x_n \in L^*(D)$ exists. As f is Lipschitz continuous in G_x^3 and B is complete there exists $\lim_{n \to \infty} f(x_n)$ and we put $f(y) = \lim_{n \to \infty} f(x_n)$. Put $G = \bigcup_{x \in D} G_x^3$. Obviously inequality (1,1) holds on G. Since the extension is locally bounded on G (the extension is

Lipschitz continuous on every G_x^3), Corollaries 1 and 2 imply Corollary 3 if these Corollaries are applied on the extension of f on G.

Corollary 4. *Let A be a Euclidean space, B a Banach space, G a region in A, $D \subset \subset G$, D be a.e. in G and let f be a transformation $f: D \to B$ fulfilling* (1,1) *on D. If $\|f(x)\|$ is measurable on D, then f can be uniquely extended on G so that the statement of Corollary* 3 *is valid.*

Since A is a Euclidean space, we consider an orthonormal basis $h_1, \ldots, h_n$. Let $x_0 \in G$, $\delta > 0$ be given such that $[x_0 + \lambda h_i] \in D$ for almost all λ, $|\lambda| < \delta$ and let $\|f(x_0 + \lambda h_i)\|$ be measurable for $|\lambda| < \delta$. Using Theorem 1 we obtain that $f(x_0 + \lambda h_i)$ is locally Lipschitz continuous and with respect to (1,1) $f(x + \lambda h_i)$ is locally Lipschitz continuous in λ for all $x \in G$. Since A is finite dimensional $f(x)$ is locally bounded and $G \subset L(D)$. If we used Corollary 2 on the extension of $f(x)$ onto $L(D)$ (i.e. onto G) we obtain the statement of Corollary 4.

Corollary 5. *Let the assumptions of Corollary* 4 *be fulfilled and let B be a Euclidean space, then the extension f on G has continuous partial derivatives in G. The function ω (from* (1,1)) *is the modulus of continuity of all partial derivatives.*

This is an obvious consequence of Corollary 4 and of Definition 7.

14. We shall show that under some additional assumption the converse result is valid.

Theorem 3. *Let A, B be linear sets with semi-norms. Let $D \subset A$ be convex and let f be a transformation $f: D \to B$ which has the differential T_x at every point x of D fulfilling*

$$(14,1) \qquad \|T_x h - T_y h\| \leqq \|h\| \, \omega(\|x - y\|) \quad \textit{for} \quad x, y \in D, \quad h \in A$$

where $\omega(\eta)$ is defined and continuous for $\eta \geqq 0$, $\omega(0) = 0$. Then the transformation f fulfils (1,1) *on D.*

Proof. Assume x, y, z to be given such that $x \in D$, $x + y \in D$, $x + z \in D$, $x + y + z \in D$. Let ε be a positive number. Recalling the definition of the differential (Definition 7) there exists a positive function $\delta(\mu)$ for every μ, $0 \leqq \mu \leqq 1$ such that

$$\|f(x + \mu z + \nu z) - f(x + \mu z) - \nu T_{x+\mu z} z\| < |\nu| \varepsilon ,$$

$$\|f(x + y + \mu z + \nu z) - f(x + y + \mu z) - \nu T_{x+y+\mu z} z\| < |\nu| \varepsilon \quad \text{for} \quad |\nu| < \delta(\mu) .$$

(Since the set D is convex the points $x + \mu z$ and $x + y + \mu z$ belong to D for $0 \leqq \leqq \mu \leqq 1$.) We can construct a finite sequence: $0 = \mu_0 < \zeta_0 < \mu_1 < \zeta_1 < \ldots$

$\ldots < \mu_{n-1} < \zeta_{n-1} < \mu_n = 1$ such that $\mu_{i+1} - \mu_i < \delta(\zeta_i)$. By means of the above inequalities we obtain

$$\|f(x + z) - f(x) - f(x + y + z) + f(x + y)\| \leqq$$

$$\leqq \sum_{i=0}^{n-1} \|f(x + \mu_{i+1}z) - f(x + \mu_i z) - f(x + y + \mu_{i+1}z) + f(x + y + \mu_i z)\| \leqq$$

$$\leqq \sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i) \|T_{x+\zeta_i z} z - T_{x+y+\zeta_i z} z\| + 2\varepsilon$$

and using (14,1) we obtain

$$\|f(x + z) - f(x) - f(x + y + z) + f(x + y)\| \leqq$$

$$\leqq \sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i) \|z\| \, \omega(\|y\|) + 2\varepsilon \leqq \|z\| \, \omega(\|y\|) + 2\varepsilon .$$

Since ε is an arbitrary number inequality (1,1) is proved.

15. It can be shown that the convexity of D cannot be ommited. This is shown by the following

Example. We construct the region in the plane which consists from five parts:

$$G_1 = \{[x, y] : |x| < 2, |y - \tfrac{5}{2}| < \tfrac{1}{2}\} ,$$

$$G_2 = \{[x, y] : |x - \tfrac{3}{2}| < \tfrac{1}{2}, |y - \tfrac{3}{2}| \leqq \tfrac{1}{2}\} ,$$

$$G_3 = \{[x, y] : |x + \tfrac{3}{2}| < \tfrac{1}{2}, |y - \tfrac{3}{2}| \leqq \tfrac{1}{2}\} ,$$

$$G_4 = \{[x, y] : 0 < x < 2, 0 < y < 1\} ,$$

$$G_5 = \{[x, y] : -2 < x < 0, 0 < y < 1\} ,$$

$G = \bigcup_i G_i$. We define a function $f(x, y)$ on G:

$$f(x, y) = 0 \quad \text{on} \quad G_1, f(x, y) = 1 \quad \text{on} \quad G_4, f(x, y) = -1 \quad \text{on} \quad G_5 ,$$

$$f(x, y) = (y - 2)^2 (2y - 1) \quad \text{on} \quad G_2, f(x, y) = -(y - 2)^2 (2y - 1) \quad \text{on} \quad G_3 .$$

We have $T_{[x,y]}(1, 0) = \partial f/\partial x \equiv 0$, $T_{[x,y]}(0, 1) = (\partial f/\partial y)(x, y)$ and $T_{[x,y]}h = (\partial f/\partial y)(x, y) h_2$ where h_2 is the second component of he vector h. Since $\|T_{[x,y]}h - T_{[u,v]}h\| = |h_2| \, |(\partial f/\partial y)(x, y) - (\partial f/\partial y)(u, v)| \leqq \|h\| \, |(\partial f/\partial y)(x, y) - (\partial f/\partial y)(u, v)| \leqq \|h\| (6|y - v| + \tfrac{3}{2}|u - x|) \leqq 12\|h\| \times \|[x - u, y - v]\|$, inequality (14,1) is fulfilled with $\omega(\eta) = 12\eta$.

On the other side we choose points $[-3\varepsilon, \frac{1}{2}]$, $[-\varepsilon, \frac{1}{2}]$, $[\varepsilon, \frac{1}{2}]$ for $\varepsilon < \frac{2}{3}$. Easily we obtain

$$|f(-\varepsilon, \tfrac{1}{2}) - f(-3\varepsilon, \tfrac{1}{2}) - f(\varepsilon, \tfrac{1}{2}) + f(-\varepsilon, \tfrac{1}{2})| = 2\,.$$

It means (1,1) is not valid.

Conditions (α) or (β) in Theorems 1 and 2, respectively are also necessary. It is a well known fact that an additive function $f(x)$ exists, $f(x + y) = f(x) + f(y)$ which is defined for all real x but is not continuous. Since additive functions obviously satisfy (1,1) this function cannot be neither bounded nor measurable according to Theorems 1 and 2. The other assumptions (i.e., except (α) and (β), respectively) of Theorems 1 and 2 are fulfilled.

References

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