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COUNTABLY COMPACT AND PSEUDOCOMPACT PRODUCTS

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Following FROLIK we use $\mathfrak{C}$ [resp. $\mathfrak{P}$] to denote the class of spaces X such that $X \times Y$ countably compact [resp. pseudocompact] whenever Y is. In the first section we show that countably compact k-spaces (indeed, countably compact spaces in which each non-P-point is a k-point) are in $\mathfrak{C}$. Corresponding results for pseudocompact spaces are given in the second section and in the third section we prove that the class $\mathfrak{P}$ is closed under arbitrary products.

1. COUNTABLY COMPACT SPACES

In this section, all hypothesized spaces are assumed to be T_1. Recall that a space X is called a k-space provided each subset of X which meets every compact set in a relatively closed set is itself closed, and that associated with each space X there is a unique k-space kX having the same underlying set and the same compact sets as X. (The space kX is formed by adjoining to the topology on X all those subsets whose complements meet each compact set in a relatively closed set.) When X is a T_1-space, kX is also a T_1-space; in fact, the identity map from kX to X is always continuous.

Let $\mathfrak{C}^*$ denote the class of spaces X with the property: Each infinite subset of X meets some compact subset of X in an infinite set. FROLÍK shows in [4] that the class of completely regular spaces in $\mathfrak{C}^*$ is precisely the class, $\mathfrak{P}_F$, of spaces each closed subspace of which is in $\mathfrak{P}$. Thus our first result implies that $\mathfrak{P}_F \subseteq \mathfrak{C} \cap \mathfrak{P}$. The containment is proper: this is shown, using the continuum hypothesis, in [6]; for an example which does not use the continuum hypothesis, let X be a space in $\mathfrak{C} \backslash \mathfrak{P}$ ([6] also contains a construction of such a space and apply the construction we give in 2.3. The resulting space, Y, is clearly in $\mathfrak{C} \cap \mathfrak{P}$ but not $\mathfrak{P}_F$.

1.1. Theorem. *If X is in $\mathfrak{C}^*$ and Y is countably compact then $X \times Y$ is countably compact.*

[1]) This paper is an outgrowth of a pair of results in the author's Ph. D. disseration, which was written at the University of Rochester under the direction of Professor W. W. COMFORT.

Proof. Suppose $X \times Y$ contains a countably infinite closed discrete subset $\{(x_n y_n)\}$, let K be a compact subset of X such that $K \cap \{x_n\} = \{x_{n_i}\}$ is infinite ($\{x_n\}$ must be infinite since Y is countably compact) and note that since $K \times Y$ is countably compact, $\{x_{n_i}, y_{n_i})\}$ must have a cluster point.

The value of the theorem above derives from the fact that many common countably compact spaces are in $\mathfrak{C}^*$, as the next theorem shows. Since sequential spaces (as well as Frechet spaces and locally compact spaces) are k-spaces, Theorems 1.1 and 1.2 combine to generalize Corollary 1.10.1 of [2].

1.2. Theorem. *A space X is in $\mathfrak{C}^*$ if and only if kX is countably compact. Thus each countably compact k-space is in $\mathfrak{C}^*$.*

Proof. If kX is countably compact, then any countable subset which meets each compact set in only finitely many points must be closed and hence, since each countable closed set is compact, finite. Conversely if kX contains an infinite closed discrete subspace D, then $D \cap K$ must be finite for each compact subset K of X.

We call a point x in X a k-point if each open subset of kX which contains x is a neighborhood of x. Clearly X is a k-space if and only if each point in X is a k-point. Recall that a point x in X is called a P-point if each G_δ containing x is a neighborhood of x. The following result should be compared with Tamano's Theorem 2 of [8] or our Theorem 2.2.

1.3. Corollary. *If X is countably compact and each point of X is either a P-point or a k-point then X is in $\mathfrak{C}^*$.*

Proof. If N is a countable subset of X whose intersection with each compact set is finite, then N is closed ($X \backslash N$ is a beighbourhood of each k-point and no P-point in $X \backslash N$ can be in the closure of N) and therefore finite.

2. PSEUDOCOMPACT SPACES

Henceforth hypothesized spaces are assumed to be completely regular (Hausdorff). A k-space X is characterized by the propety: A function with domain X is continuous if its restriction to each compact subset of X is continuous. A space is called a k_R-space if it has this property for real-valued functions, or equivalently, for functions with completely regular range. Associated with each (completely regular) space X there is a unique (completely regular) k_R-space, $k_R X$ having the same underlying set and the same compact subsets as X. (The space $k_R X$ has the smallest topology making continuous each real-valued function on X whose restriction to compact subsets is continuous.) In general, the set-identity maps: $kX \rightarrow k_R X \rightarrow X$ are continuous but not open; that is, X need not be a k_R-space and kX need not be completely regular.

A reasonable pseudocompact analog of the spaces considered in section 1 are the spaces with the property: Each infinite collection of disjoint open sets has an infinite subcollection each of which meets some fixed compact set. We denote the class of spaces with this peoperty as $\mathfrak{P}^*$. Frolik studies this class briefly in [4], where he shows that $\mathfrak{P}^* \subset \mathfrak{P}$. While we have not been able to decide if $\mathfrak{P}^*$ is the class of spaces X such that $k_R X$ is pseudocompact, it is true, as the next theorem shows, that X is in $\mathfrak{P}^*$ when $k_R X$ is pseudocompact. Thus pseudocompact spaces which are locally compact, or Frechet, or sequential, etc., are in $\mathfrak{P}^*$.

2.1. Theorem. *A k_R-space is pseudocompact if and only if it is in $\mathfrak{P}^*$. Thus if $k_R X$ is pseudocompact, X is in $\mathfrak{P}^*$.*

Proof. Clearly any space in $\mathfrak{P}^*$ is pseudocompact. Suppose X is a k_R-space which is not in $\mathfrak{P}^*$, and let $\{U_n\}$ be a countable collection of disjoint open subsets of X such that for each compact $K \subseteq X$, $K \cap U_n = \emptyset$ with only finitely many exceptions. For each n let f_n be a continuous real-valued function which maps $X \backslash U_n$ to zero and some point in U_n to n, and set $f = \sum_n f_n$. Then f is continuous on compact sets, hence continuous, and f is unbounded so X is not pseudocompact. The final statement follows from the fact that continuous (completely regular) images of members of $\mathfrak{P}^*$ are in $\mathfrak{P}^*$.

Combined with Frolík's result (that $\mathfrak{P}^* \subseteq \mathfrak{P}$) the theorem above generalizes a number of theorems including Tamano's result [8, Proposition 2] that each pseudocompact k-space is in $\mathfrak{P}$. In the same paper Tamano shows, as Theorem 2, that X is in $\mathfrak{P}^*$ if each point of X is either a "k-point" or a P-point. Tamano's definition of a k-point is less general than ours (his assertion that each point of a k-space is a "k-point" is erroneous) but the theorem remains true using our definition of k-point. In fact, calling a point x in X a k_R-point if each real-valued function on X which is continuous on compact sets is continuous at x, we have the following result.

2.2. Theorem. *If X is pseudocompact and each point of X is either a P-point or a k_R-point, then X is in $\mathfrak{P}^*$.*

Proof. Suppose X is not in $\mathfrak{P}^*$, let $\{U_n\}$ be a countable collection of disjoint open sets only finitely many of which meet any single compact set and construct an unbounded function f as in the proof of Theorem 2.1. Since f is continuous on compact sets it is continuous at each k_R-point of X, and f is continuous at P-points in $X \backslash \bigcup_n U_n$ since it is zero on a neighborhood of such a point. Finally, since $f|U_n = f_n$, f is continuous on $\bigcup_n U_n$ and therefore f is continuous. Since X is pseudocompact, this is a contradiction so X is in $\mathfrak{P}^*$.

Recall that a closed subspace of a k-space is a k-space and that each closed subspace of a space in $\mathfrak{C}$ is in $\mathfrak{C}$. The following construction shows that a pseudocompact k_R-space need not be a k-space (and need not be countably compact). It also shows that a countably compact k_R-space need not be in $\mathfrak{C}$.

2.3. Construction. *Each completely regular space can be embedded as a closed subspace of a pseudocompact k_R-space, and each countably compact completely* *regular space can re embedded as a closed subspace of a countably compact k_R-space.*

Proof. Let X be a completely regular space, let $\aleph_\alpha$ be a cardinal greater than the cardinality of βX such that $\alpha > 0$ is a non-limit ordinal, let ω_α be the smallest ordinal of cardinality $\aleph_\alpha$ and let $Y = (\beta X \times \omega_\alpha) \cup (X \times (\omega_\alpha + 1))$ topologized as a subspace of $\beta X \times (\omega_\alpha + 1)$.

To see that Y is a pseudocompact k_R-space, let f be a real-valued function on Y whose restriction to each compact subset of Y is continuous; then f is continuous on the locally compact subspace $\beta X \times \omega_\alpha$ and also, for x in X, on the compact subspaces $\{x\} \times (\omega_\alpha + 1)$. But since f must be eventually constant on $\{x\} \times \omega_\alpha$ for each x in βX, and since $\aleph_\alpha > \text{card}\,(\beta X)$, there exists an α_0 in ω_α such that, for each x in βX, f is constant on $(\{x\} \times [\alpha_0, \omega_\alpha]) \cap Y$. From this it is clear that f is continuous and bounded. Thus Y is a pseudocompact k_R-space.

It remains to show that Y is countably compact if X is. But since the closure of each countable subset of ω_α is compact, ω_α and hence $\beta X \times \omega_\alpha$ is countably compact, so this is clear.

3. INFINITE PSEUDOCOMPACT PRODUCTS

There are several theorems having the form: A product of pseudocompact spaces is pseudocompact if all but one of the factors has Q. For instance, [5, Theorem 4] with Q = "locally compact" or Q = "each point is either a P-point or has a countable neighbourhood base" and [7, Theorem 2] with Q = "sequentially compact". Restating these theorems as: Each product of spaces having Q is in $\mathfrak{P}$, we see that the best possible result of this type is:

3.1. Theorem. *The class $\mathfrak{P}$ is closed under arbitrary products.*

The proof of this theorem will use Frolík's characterization [4, Theorem 3.6] of members of $\mathfrak{P}$: A space X is in $\mathfrak{P}$ if and only if each infinite disjoint family of non-empty open subsets of X contains an infinite subfamily $\{U_n\}$ which satisfies (α), where (α) is the condition: For each filter Φ consisting of infinite subsets of N,

$$\bigcap_{N' \in \Phi} \text{cl}\,\Big(\bigcup_{n \in N'} U_n\Big) \neq \emptyset\,.$$

We will also use the following observations.

3.2. Lemma. *Let $\{U_n : n \in N\}$ be a family of subsets of X, with multiple indexing allowed.*

(i) *If $\{U_n\}$ satisfies (α), then so does each subfamily $\{U_n : n \in N'\}$ for $N' \subseteq N$ infinite.*

(ii) *If* $\{U_n\}$ *satisfies* (α), *if* S *is any finite collection of subsets of* X *and if* $\{S_n\}$ *is any indexing of* $\{U_n \cup S\}$ *by* N, *then* $\{S_n\}$ *satisfies* (α).

(iii) *If* Φ *is a filter of infinite subsets of* N *and* x *is in the closure of* $\bigcup_{n\in N'} U_n$ *for each* N' *in* Φ, *and if* $\{V_n\}$ *is a family of subsets of a space* Y *which satisfies* (α), *then there exists a point* y *in* Y *such that* (x, y) *is in the closure of* $\bigcup_{n\in N'} U_n \times V_n$ *for each* N' *in* Φ.

(iv) *If* X *is in* $\mathfrak{P}$ *and each* U_n *is nonempty and open, then for some infinite* $N' \subseteq N$, $\{U_n : n \in N'\}$ *satisfies* (α).

Proof. Parts (i) and (ii) are trivial. For (iii), let $\mathscr{B}$ be a neighborhood base for x and let $\Phi' = \{S(B, N') : B \in \mathscr{B},\ N' \in \Phi\}$ where $S(B, N') = \{n \in N' : B \cap U_n \neq \emptyset\}$. Evidently Φ' is a filter consisting of infinite subsets of N, so there exists a point y in Y which is in the closure of $\bigcup_{n\in S} V_n$ for each S in Φ'. But then for N' in Φ, (x, y) is in the closure of $\bigcup_{n\in N'} U_n \times V_n$ since for $B \times B'$ a neighborhood of (x, y) with B in $\mathscr{B}$, $B' \cap V_n \neq \emptyset$ for some n in $S(B, N')$ and hence $(B \times B') \cap (U_n \times V_n) \neq \emptyset$.

To prove (iv), choose x_1 in $U_1 = U_{n_1}$; if each neighborhood of x_1 meets all but finitely many of the U_n, we are done; otherwise there exists an open neighborhood $V_1 \subseteq U_1$ of x_1 and an infinite subset N_1 of N such that $V_1 \cap U_n = \emptyset$ for each n in N_1. Proceeding inductively, choose n_k in N_{k-1}, and x_k in U_{n_k}. Either each neighborhood of x_k meets all but finitely many of the sets U_n with n in N_{k-1}, or there exists an open neighborhood $V_k \subseteq U_{n_k}$ of x_k and an infinite subset N_k of N_{k-1} such that $V_k \cap U_n = \emptyset$ for each n in N_k. If the induction continues, we get an infinite disjoint family $\{V_n\}$ of nonempty sets. Since X is in $\mathfrak{P}$, some infinite subfamily of $\{V_n\}$, and hence some infinite subfamily of $\{U_{n_k}\}$ must satisfy (α).

Proof of Theorem 3.1. Since a product space is pseudocompact if and only if each of its countable subproducts is pseudocompact [5, page 370], a product space is in $\mathfrak{P}$ if and only if each of its countable subproducts is in $\mathfrak{P}$. Thus it suffices to consider a countable product, say $X = \prod_i X_i$, of members of $\mathfrak{P}$.

Let $\mathscr{D}$ be any infinite collection of nonempty open subsets of X; we may suppose that $\mathscr{D} = \{U_n\}$ where each $U_n = \prod_i U_n^i$ with every U_n^i open and, for each n, all but finitely many of them equal to X_i. Do so.

Using part (iv) of the lemma, choose infinite subsets N_i of N with $N_i \subseteq N_{i-1}$ such that $\{U_n^i : n \in N_i\}$ satisfies (α). Choose n_i in N_i and (using part (ii) of the lemma) assume that for $j < i$, n_j is in N_i. To see that $\{U_{n_i} : i \in N\}$ satisfies (α), let Φ be any filter consisting of infinite subsets of N and choose, inductively, x_k in X_k so that $(x_1, \ldots, x_k)$ is in the closure of $\bigcup_{j\in N'} \prod_{i=1}^{k} U_{n_j}^i$ for each N' in Φ. (This can be done by part (ii) of the lemma.) Then for x the point (x_k), x is in the closure of $\bigcup_{j\in N'} U_{n_j}$ for each N' in $\emptyset$.

In view of Theorem 3.1 and the difficulty one usually experiences in trying to sue Frolík's characterization of members of $\mathfrak{P}$ to determine if a given space is in $\mathfrak{P}$, it is reasonable to ask: Is $\mathfrak{P}$ the class of products of some conveniently describable subclass of $\mathfrak{P}$? An obvious candidate for the "conveniently describable subclass" is $\mathfrak{P}^*$; but as we shall see $\mathfrak{P}^*$ is itself closed under products. To show this we need the following result.

3.3. Lemma. *A space X is in $\mathfrak{P}^*$ if and only if for each collection $\{U_n : n \in N\}$ of nonempty open subsets of X, some compact subset of X meets U_n for infinitely many n.*

Proof. That spaces with this property are in $\mathfrak{P}^*$ is trivial, so suppose X is in $\mathfrak{P}^*$ and let $\{U_n : n \in N\}$ be any countable collection of nonempty open subsets of X. Choose open sets V_n with cl $V_n \subseteq U_n$ and, without loss of generality, suppose that $\bigcap_{n>m} V_n = \emptyset$ for each m. If for some m and each $n > m$, $\bigcap_{i=m}^{n} V_i \neq \emptyset$, then we may suppose that for some $n_1 > m$ $W_1 = \text{int}(V_m \setminus V_{n_1})$ is not empty (since otherwise a point in V_m would be in the closure of each V_n for $n \geqq m$ and would therefore be in infinitely many of the U_n) and similarly that for $n_1 < n_2 < n_3 \ldots$ $W_i = \text{int}(V_{n_{i-1}} \setminus V_{n_i})$ is nonempty. Since X has $\mathfrak{P}^*$, some compact subset of X meets infinitely many of the sets W_i and hence infinitely many of the U_n. On the other hand, if for each m there exists an $n > m$ such that $\bigcap_{i=m}^{n} V_i = \emptyset$, then there exists a smallest such n, say $n(m)$, and the sets $W_1 = \bigcap_{i=1}^{n(1)-1} V_i$, $W_2 = \bigcap_{i=n(1)}^{n(n(1))-1} V_i$, etc. are disjoint, so again there exists a compact subset of X which meets infinitely many of the W_n and hence infinitely many of the U_n.

3.4. Theorem. *The class $\mathfrak{P}^*$ is closed under arbitrary products.*

Proof. Since the defining property of $\mathfrak{P}^*$ need involve only countably many basic open sets, and hence only countably many coordinants, it suffices to consider the case $X = \prod_n X_n$ with each X_n in $\mathfrak{P}^*$. Let $\{U_m : m \in N\}$ be a countable family of nonempty open subsets of X, say $U_m = \prod_n U_n^m$ with each U_n^m open and, inductively, choose $N_n \subseteq N_{n-1}$ and compact $K_n \subseteq X_n$ such that for $m \in N_n$, $U_n^m \cap K_n \neq \emptyset$. Choose n_i in N_i and x_{ij} in $U_i^{n_j}$ and set $K_i^* = K_i \cup \{x_{ij} : j < i\}$. Then $K = \prod_i K_i^*$ is compact and meets each of the U_{n_i}, so X is in $\mathfrak{P}^*$.

We might mention that the (possibly improper) subclass of $\mathfrak{P}^*$ consisting of all spaces X for which $k_R X$ is pseudocompact is also closed under arbitrary products, as is the class of all pseudocompact k_R-spaces. The proofs of these results depend upon a deeper study of k_R-spaces than is appropriate to this context, and will be given elsewhere. Returning to the problem of representing $\mathfrak{P}$ as a class of products, we introduce a subclass of $\mathfrak{P}$ which is, at least formally, broader than $\mathfrak{P}^*$ and which may also be considered to be a pseudocompact analog of the spaces considered in section 1. Let $\mathfrak{P}^{**}$ denote the class of spaces X with the property: If $\mathscr{D}$ is an infinite family of disjoint nonempty subsets of X, then for some compact subset K of X and some infinite subfamily $\mathscr{D}'$ of $\mathscr{D}$, each neighborhood of K meets all but finitely many of the members of $\mathscr{D}'$. Our final two results show that $\mathfrak{P}^{**}$ is a proper subclass of $\mathfrak{P}$ and that $\mathfrak{P}^{**}$ is also closed under arbitrary products.

3.5. Theorem. *$\mathfrak{P}^* \subseteq \mathfrak{P}^{**} \subseteq \mathfrak{P}$ and each product of members of $\mathfrak{P}^{**}$ is in $\mathfrak{P}^{**}$.*

Proof. That $\mathfrak{P}^* \subseteq \mathfrak{P}^{**}$ is obvious. Also, $\mathfrak{P}^{**}$ is contained in $\mathfrak{P}$ since for X in $\mathfrak{P}^{**}$ and Y pseudocompact, $X \times Y$ is pseudocompact: If $\{U_n \times V_n\}$ is a collection of nonempty open subsets of $X \times Y$ with $\{U_n\}$ disjoint, if K is a compact subset of X each neighborhood of which meets all but finitely many of the U_n, and if y is any cluster point of $\{V_n\}$, then for some x in K, (x, y) is a cluster point of $\{U_n \times V_n\}$ (otherwise $K \times \{y\}$ can be covered by finitely many neighborhoods $W_i \times W_{i'}$ of (x_i, y) such that $(\bigcup_i W_i) \cap U_n = \emptyset$ whenever $(\bigcap_i W_i) \cap V_n \neq \emptyset$). Finally, that $\mathfrak{P}^{**}$ is closed under arbitrary products follows by a diagonalization argument similar to that in the proof of 3.1. (Generalizing Glicksberg's proof of the corresponding result for pseudocompact products, one sees that it suffices to consider countable products.)

We might mention that the special case of the result $\mathfrak{P}^{**} \subseteq \mathfrak{P}$ obtained by replacing K by a single point is Theorem 5 of [1]. (Actually, this result is stated for lightly compact spaces, which are spaces in which each disjoint family of nonempty open subsets has a cluster point, and with no separation assumptions. However, our Theorems 3.1 and 3.3, and their proofs, remain true if we drop all separation assumptions, provided pseudocompact is taken to mean lightly compact and, in 3.1, $\mathfrak{P}$ is replaced by the class of spaces which, except for complete regularity, satisfy Frolík's characterization of the completely regular members of $\mathfrak{P}$.)

3.6. Example. $\mathfrak{P}^{**} \neq \mathfrak{P}$.

We first note that $\beta N \setminus N$ contains a dense subset P of cardinality 2^c such that each compact subset of P is finite and $\beta N \setminus (N \cup P)$ is dense: Assuming the continuum hypothesis we could take P to be the set of P-points of $\beta N \setminus N$; otherwise let P_0 be a dense subset of $\beta N \setminus N$ of cardinality 2^c whose complement is also dense, index the collection of countable subsets of P_0 whose closures are contained in P_0 as $\{C_\alpha : : \alpha \in W\}$ where W is the least ordinal of its cardinality, and choose inductivelly points p_α and p'_α in P_0 such that p_α is an accumulation point of C_α which is not in $\bigcup_{\beta < \alpha} C_\beta \cup \{p_\beta, p'_\beta\}$ while p'_α is not in $\{p_\beta : \beta \leqq \alpha\}$ – this is always possible since card (cl C_α) $= 2^c$ while card $W \leqq 2^c$. The set $P = P_0 \setminus \{p_\alpha : \alpha \in W\}$ has the desired properties.

Now let I be the unit interval; let S be the set of sequences $\{s_n\}$ in I such that each s_n is rational or each s_n is irrational; and, for $s = \{s_n\} \in S$ let P_s be the copy in $\mathrm{cl}_{\beta(I \times N)}\{(s_n, n) : n \in N\}$ of P or $\beta N \setminus (N \cup P)$ according as the s_n are rational or irrational. Set $X = I \times N \cup \bigcup_{s \in S} P_s$ topologized as a subspace of $\beta(I \times N)$. Then X is in P (indeed, except that we have been more particular about our choice of P, X is Frolík's example of a space in $\mathfrak{P} \setminus \mathfrak{P}^*$).

To see that X is not in $\mathfrak{P}^{**}$, suppose K is a compact subset of $X \setminus (I \times N)$ each neighborhood of which meets all but finitely many of the open sets $I \times \{n\}$ and let $\pi^\beta : \beta(I \times N) \to \beta N$ be the extension of the projection $\pi : I \times N \to N \subseteq \beta N$. First suppose there exists a point x_0 in $P \setminus \pi^\beta(K)$; then there exists a closed neighborhood, U, of $\pi^\beta(K)$ which does not contain x_0. Let U' be the inverse, under π^β, of U; then U' is a closed neighborhood of K and for each rational sequence s, P_s is not contained in U'. It follows that U' cannot contain all but finitely many of the terms of any rational sequence, which contradicts our assumptions on K. Hence $P \subseteq \pi^\beta(K)$, so for each x in P there exists a sequence of rationals s such that the point in P_s corresponding to x is in K. But since there are only c rational sequences and $K \cap P_s$ is finite for each rational sequence s, while the cardinality of P is 2^c, this is impossible.

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