Czechoslovak Mathematical Journal

Hana Lovicarová

Periodic solutions of a weakly nonlinear wave equation in one dimension

Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 2, 324–342

Persistent URL: http://dml.cz/dmlcz/100899

Czechoslovak Mathematical Journal, 19 (94) 1969, Praha

PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION IN ONE DIMENSION

HANA LOVICAROVÁ, Praha

(Received April 18, 1968)

1. INTRODUCTION

In this paper we shall investigate the equation

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x) \tag{1.1}$$

on the domain $G = R \times (0, \pi)$ $(R = (-\infty, \infty))$ of the plane (t, x) with the boundary conditions

$$\lim_{(t,x)\to(t,0)} u(t, x) = 0 = \lim_{(t,x)\to(t,\pi)} u(t, x) . \tag{1.2}$$

We shall seek a solution of the problem (1.1), (1.2) 2π-periodic in t under the assumption that the function f is 2π-periodic in t.

Vejvoda in [1] gave some sufficient conditions for the existence of periodic solutions of the problem (1.1), (1.2). In [3] the existence of 2π-periodic solution of the problem (1.1), (1.2) is proved if f depends only on t, x, u and $f_u \leqq -\gamma < 0$. Further in this paper Rabinowitz proved, that the problem (1.1), (1.2) has a 2π-periodic solution if the right hand side of the equation (1.1) has the form $\varepsilon(\alpha u_t + g(t, x, u))$ where α is a constant.

In this paper in paragraph 2 the existence of 2π-periodic solution in the linear case is treated. In paragraph 3 some auxiliary theorems are introduced. In paragraph 4 the existence of 2π-periodic solution of the problem (1.1), (1.2) is proved under the assumptions $\partial f/\partial u_t \leqq -\gamma < 0$ on G_1 and $\sup_{G_1} \partial f/\partial u_x - \inf_{G_1} \partial f/\partial u_x < \gamma$, where $G_1 = G \times R \times R \times R$, and certain restriction on the growth of f. In paragraph 5 the existence and continuous dependence on ε of 2π-periodic solution, if $f = f(t, x, u, \varepsilon)$ and $f_u \leqq -\gamma < 0$, is proved under weaker assumptions on the smoothness of f than in paper [3].

We conclude the introduction with some notations. Let $D_i f$ denote the derivative of f with respect to i^{th} variable. Then we shall denote by C_k the Banach space of functions defined on G, 2π-periodic in t, for which $D_1^i D_2^j f$ $(i + j \leqq k,\ i, j \geqq 0)$ are continuous and bounded on G, with the norm:

$$f \in C_k : |f|_{C_k} = \|f\|_k = \sup_{i+j \leqq k} \sup_{(t,x) \in G} |D_1^i D_2^j f(t, x)| .$$

Let us note, that functions which belong to C_k have derivatives up to the order $k - 1$ continuous up to the boundary of G.

Let C^k denote the Banach space of 2π-periodic functions p (of one variable) which are continuous together with their derivatives up to the order k and for which $[p] = \int_0^{2\pi} p(y)\, dy = 0$. The norm in C^k is given: $|p|_{C^k} = |p|_k = \sup (|D^i p(y)|;\ 0 \leqq i \leqq k,\ y \in R)$.

The space of all linear operators mapping a Banach space B_1 into a Banach space B_2 will be denoted by $[B_1 \to B_2]$.

$R(A)$, $N(A)$ respectively denote a range and a null space respectively of the operator A.

2. THE LINEAR CASE

It is known (see e.g. [1]), that every classical solution of the homogeneous problem (1.1), (1.2) is 2π-periodic and has a form

$$u(t, x) = p(t + x) - p(t - x), \tag{2.1}$$

where p is 2π-periodic and continuous together with its second derivative.

For nonhomogeneous equation

$$u_{tt} - u_{xx} = f(t, x) \tag{2.2}$$

we shall derive necessary and sufficient condition for the existence of 2π-periodic solution, which fulfils the condition (1.2).

Let $u \in C_2$ be a solution of (2.2), (1.2). Integrating the equation (2.2) over the triangle $[(t, x), (t - x + \pi, \pi), (t + x - \pi, \pi)]$ and using the Green formula on the left hand side we obtain (as $u(t, \pi) = 0$):

$$u(t, x) = -\frac{1}{2} \int_{t+x-\pi}^{t-x+\pi} u_x(\tau, \pi)\, d\tau - \frac{1}{2} \int_x^{\pi} \int_{t+x-\xi}^{t-x+\xi} f(\tau, \xi)\, d\tau\, d\xi .$$

Since $u(t, 0) = 0$, we get

$$\int_0^{\pi} \int_{t-\xi}^{t+\xi} f(\tau, \xi)\, d\tau\, d\xi = - \int_{t-\pi}^{t+\pi} u_x(\tau, \pi)\, d\tau = \text{const}$$

because u_x is also 2π-periodic. Differentiating this relation with respect to t we get

$$\int_0^{\pi} [f(t + \xi, \xi) - f(t - \xi, \xi)]\, d\xi = 0 . \tag{2.3}$$

If the function f has continuous and bounded derivative $D_1 f$, the condition (2.3) is also sufficient for the existence of a solution $u \in C_2$ of the problem (2.2), (1.2). Indeed, if (2.3) holds, then

$$\int_0^\pi \int_{t-\xi}^{t+\xi} f(\tau, \xi)\, d\tau\, d\xi = \text{const} = k \tag{2.4}$$

and it is easily seen, that the function

$$u(t, x) = -\frac{1}{2}\int_x^\pi \int_{t+x-\xi}^{t-x+\xi} f(\tau, \xi)\, d\tau\, d\xi + \frac{1}{2}\frac{\pi - x}{\pi} k \tag{2.5}$$

is the saught solution.

Now let us prove, that the space C_0 can be written in the form of a direct summ $C_0 = N \oplus N^\perp$ where N is the set of such functions for which (2.3) is fulfilled (i.e. for which there exists a solution of the problem (2.2), (1.2)) and $N^\perp$ is the set of functions which have the form (2.1) (i.e. which are solutions of the homogeneous problem).

Let us define the operators Z and Q on the spaces C^0 and C_0 respectively:

$$p \in C^0 : Zp(t, x) = p(t + x) - p(t - x), \quad (t, x) \in G, \tag{2.6}$$

$$f \in C_0 : Q f(y) = \frac{1}{2\pi}\int_0^\pi (f(y - s, s) - f(y + s, s))\, ds, \quad y \in R. \tag{2.7}$$

Lemma 2.1. 1) $Z \in [C^k \to C_k]$ *for* $k \geqq 0$ *and*

$$p \in C^k \Rightarrow \|Zp\|_k \leqq 2|p|_k. \tag{2.8}$$

2) $Q \in [C_k \to C^k]$ *for* $k \geqq 0$ *and*

$$f \in C_k \Rightarrow |Qf|_k \leqq \|f\|_k. \tag{2.9}$$

3) $QZ = E$.

4) *The operator* $P_1 = ZQ$ *is a projector of* C_0 *on* $R(Z)$ *and for* $f \in C_k$ *it holds*

$$\|P_1 f\|_k \leqq 2\|f\|_k. \tag{2.10}$$

5) *The operator* $P_2 = E - P_1$ *is a projector of* C_0 *on* $N(Q)$ *and for* $f \in C_k$ *it holds*

$$\|P_2 f\|_k \leqq 3\|f\|_k. \tag{2.11}$$

Proof. It is obvious that 1) holds.

ad 2) Let $f \in C_0$. Evidently, Qf is continuous and 2π-periodic. Let us prove, that $[Qf] = 0$.

$$[Qf] = \int_0^{2\pi} Q f(y)\, dy = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^\pi (f(y - s, s) - f(y + s, s))\, ds\right) dy =$$

$$= \frac{1}{2\pi}\int_0^\pi \left(\int_0^{2\pi} f(y - s, s)\, dy - \int_0^{2\pi} f(y + s, s)\, dy\right) ds = 0$$

because of $\int_0^{2\pi} f(y - s, s)\,dy = \int_{-2s}^{2\pi-2s} f(y + s, s)\,dy = \int_0^{2\pi} f(y + s, s)\,dy$. For $f \in C_k$ it holds: $D^i Q f(y) = Q D_1^i f(y)$ and from here we get, that $Qf \in C^k$ and $|Qf|_k \leqq \|f\|_k$.

ad 3) Let $p \in C^0$. Then $Zp(t, x) = p(t + x) - p(t - x)$ and so

$$QZ\, p(y) = \frac{1}{2\pi} \int_0^\pi [(p(y) - p(y - 2s)) - (p(y + 2s) - p(y))]\,ds =$$

$$= p(y) - \frac{1}{2\pi} \int_0^\pi p(y - 2s)\,ds - \frac{1}{2\pi} \int_0^\pi p(y + 2s)\,ds = p(y)$$

because $\int_0^\pi p(y \pm 2s)\,ds = \frac{1}{2}[p] = 0$.

ad 4) $P_1^2 = (ZQ)(ZQ) = Z(QZ)\,Q = ZQ = P_1$ and so P_1 is a projector. For $f \in C_k$, $P_1 f \in C_k$ and further $\|P_1 f\|_k = \|Z(Qf)\|_k \leqq 2|Qf|_k \leqq 2\|f\|_k$. Obviously $R(P_1) \subset R(Z)$. On the other hand if $f \in R(Z)$, then there exists $p \in C^0$ such that $f = Zp$ and $P_1 f = (ZQ)\,Zp = Z(QZ)\,p = Zp = f$ and so $R(P_1) = R(Z)$.

ad 5) According to 4) P_2 is a projector and for $f \in C_k$ $\|P_2 f\|_k = \|(E - P_1) f\|_k \leqq \|f\|_k + \|P_1 f\|_k \leqq 3\|f\|_k$. Let us prove, that $R(P_2) = N(Q)$. $QP_2 = Q(E - ZQ) = Q - QZQ = 0$ and so $R(P_2) \subset N(Q)$. For $f \in N(Q)$: $P_2 f = (E - ZQ) f = f - Z(Qf) = f$ and so $R(P_2) = N(Q)$.

Let us denote $N = N(Q) = R(P_2)$, $N^\perp = R(Z) = R(P_1)$. Then by lemma 2.1 $C_0 = N \oplus N^\perp$. Further let N_k denote $N \cap C_k$. From lemma 2.1 it follows easily, that N_k is a closed subspace of C_k and P_2 is a linear bounded operator from C_k onto N_k.

Let us define the operators S' and S on the space N:

$$S'f(t, x) = \frac{1}{2} \frac{\pi - x}{\pi} \int_0^\pi \int_{t-\xi}^{t+\xi} f(\tau, \xi)\,d\tau\,d\xi\,, \quad (t, x) \in G\,, \tag{2.12}$$

$$Sf(t, x) = -\frac{1}{2} \int_x^\pi \int_{t+x-\xi}^{t-x+\xi} f(\tau, \xi)\,d\tau\,d\xi\,, \quad (t, x) \in G\,, \tag{2.13}$$

and let us prove the following

Lemma 2.2.

$$1)\ S' \in [N_k \to C_{k+1}]\,, \quad \|S'f\|_{k+1} \leqq \frac{\pi^2}{2} \|f\|_k\,. \tag{2.14}$$

$$2)\ S \in [N_k \to C_{k+1}]\,, \quad \|Sf\|_{k+1} \leqq \left(k + \frac{\pi^2}{2}\right) \|f\|_k\,. \tag{2.15}$$

Proof. ad 1) For $f \in N$ $\int_0^\pi \int_{t-\xi}^{t+\xi} f(\tau, \xi)\,d\tau\,d\xi = \text{const}$ and so only $S'f$ and $D_2 S'f$ are different from zero.

$$\|S'f\|_0 \leqq \tfrac{1}{2}\|f\|_0 \left(\int_0^\pi \int_{t-\xi}^{t+\xi} d\tau\,d\xi \right) = \frac{\pi^2}{2} \|f\|_0$$

$$\|D_2 S'f\|_0 \leqq \frac{1}{2\pi} \|f\|_0 \left(\int_0^\pi \int_{t-\xi}^{t+\xi} d\tau\,d\xi \right) = \frac{\pi}{2} \|f\|_0 \leqq \frac{\pi^2}{2} \|f\|_0\,.$$

From here follows that $\|S'f\|_1 \leqq (\pi^2/2)\,\|f\|_0$ and $\|S'f\|_{k+1} = \|S'f\|_1 \leqq (\pi^2/2)\,.$ $.\,\|f\|_0 \leqq (\pi^2/2)\,\|f\|_k$.

ad 2) From the form of the operator S it follows immediately that $\|Sf\|_0 \leqq (\pi^2/2)\,.$ $.\,\|f\|_0$. Differentiating the relation (2.13) we get

$$(2.16)\qquad D_1Sf(t, x) = \frac{1}{2}\int_x^\pi (f(t + x - s, s) - f(t - x + s, s))\,\mathrm{d}s\,,$$

$$(2.17)\qquad D_2Sf(t, x) = \frac{1}{2}\int_x^\pi (f(t + x - s, s) + f(t - x + s, s))\,\mathrm{d}s$$

and from here $\|D_iSf\|_0 \leqq \pi\|f\|_0 \leqq (\pi^2/2)\,\|f\|_0$ $(i = 1, 2)$.

We shall prove that for $k \geqq 1$ and $f \in N_k$ there hold the relations:

$$(2.18)\qquad D_1^{k+1}Sf = D_1SD_1^kf\,,$$

$$(2.19)\qquad D_1^kD_2Sf = D_2SD_1^kf\,,$$

$$(2.20)\qquad D_2^{k+1}Sf = -\tfrac{1}{2}\sum_{i=0}^{k-1}(1 + (-1)^i)\,D_2^{k-1-i}D_1^if + D_{b(k)}SD_1^kf\,,$$

where $b(k) = 1$ for odd k and $b(k) = 2$ for even k.

The first and the second relations are obvious, the third one will be proved by induction with respect to k.

For $k = 1$ we get by differentiating of the relation (2.17):

$$D_2^2Sf = -f + D_1SD_1f\,.$$

Let us suppose that the relation (2.20) holds for $k = n$. Then

$$D_2^{n+2}Sf = D_2(D_2^{n+1}Sf) = -\tfrac{1}{2}\sum_{i=0}^{n-1}(1 + (-1)^i)\,D_2^{n-i}D_1^if + D_2D_{b(n)}SD_1^nf$$

$b(n) = 1$ for odd n and then

$$D_2D_{b(n)}SD_1^nf = D_2SD_1^{n+1}f = -\tfrac{1}{2}(1 + (-1)^n)\,D_2^0D_1^nf + D_{b(n+1)}\,SD_1^{n+1}f$$

$b(n) = 2$ for even n and then

$$D_2D_{b(n)}SD_1^nf = D_2^2SD_1^nf = -D_1^nf + D_1SD_1^{n+1}f =$$
$$= -\tfrac{1}{2}(1 + (-1)^n)\,D_2^0D_1^nf + D_{b(n+1)}SD_1^{n+1}f\,.$$

In both cases we get

$$D_2^{n+2}Sf = -\tfrac{1}{2}\sum_{i=0}^{n}(1 + (-1)^i)\,D_2^{n-i}D_1^if + D_{b(n+1)}SD_1^{n+1}f\,.$$

The inductive step is performed.

Let $f \in N_k$. We shall estimate $\|D_1^i D_2^j Sf\|_0$ for $i + j \leqq k + 1$. If $j = 0$, $i \geqq 1$, then it holds

$$\|D_1^i Sf\|_0 = \|D_1 S D_1^{i-1} f\|_0 \leqq \pi \|D_1^{i-1} f\|_0 \leqq \frac{\pi^2}{2} \|f\|_k .$$

If $j > 0$, then $\|D_1^i D_2^j Sf\|_0 = \|D_2^j S D_1^i f\|_0 = \|-\frac{1}{2}\sum_{m=0}^{j-2}(1 + (-1)^m) D_2^{j-2-m} D_1^{i+m} f +$ $+ D_{b(j-1)} S D_1^{i+j-1} f\|_0 \leqq \sum_{m=0}^{j-2} \|D_2^{j-2-m} D_1^{i+m} f\|_0 + \pi \|D_1^{i+j-1} f\|_0 \leqq (k + \pi^2/2) \|f\|_k$ and from here we get that for $k \geqq 0$, $f \in N_k$ it holds

$$\|Sf\|_{k+1} \leqq \left(k + \frac{\pi^2}{2}\right) \|f\|_k .$$

Lemma is proved.

Let us define the operator A on N

$$f \in N : Af = P_2(S + S')f . \tag{2.21}$$

From the preceding it follows that for $k \geqq 0$

$$A \in [N_k \to N_{k+1}], \quad \|Af\|_{k+1} \leqq 3(k + \pi^2) \|f\|_k . \tag{2.22}$$

Remark 2.1. $R(P_1) = R(Z)$ is by (2.1) a class of solutions of a homogeneous equation (2.2) and so the function $u = Af = (S + S')f - P_1(S + S')f$ for $f \in N_1$ is by (2.5), (2.4) a classical solution of the equation (2.2).

3. AUXILIARY THEOREMS

Lemma 3.1. *Let r_i be positive numbers ($i = 0, 1, \ldots, k + 1$, k nonnegative integer) and let $M_1 = \{p \in C^k, |D^i p|_0 \leqq r_i,\ i = 0, 1, \ldots, k\}$, $M_2 = \{p \in C^{k+1}, |D^i p|_0 \leqq r_i,$ $i = 0, 1, \ldots, k + 1\}$ (hence $M_2 \subset M_1$). Let T be a continuous mapping of $M_1 \subset C^k$ into C^k, which maps M_2 into itself. Then there exists a fixed point of the operator T in M_1.*

Proof. The closure $\overline{M}_2$ of the set M_2 in the space C^k is a convex and compact subset of $M_1 \subset C^k$ and by the assumptions of the lemma T is a continuous mapping of $\overline{M}_2$ into itself. Hence, by the Schauder fixed point theorem T has a fixed point in $\overline{M}_2 \subset M_1$.

Lemma 3.2. *Let the operator I be given on C^0 by prescription*

$$p \in C^0 : Ip(y) = \int_0^y p(s)\, ds + \int_0^{2\pi} \frac{s}{2\pi}\, p(s)\, ds , \quad y \in R . \tag{3.1}$$

Then it holds

(3.2) 1) $D^{k+1}Ip = D^k p$, $(p \in C^k)$, $ID^{k+1}p = D^k p$, $(p \in C^{k+1})$ *for* $k \geqq 0$,

2) $I \in [C^k \to C^{k+1}]$, $|Ip|_{k+1} \leqq \frac{\pi}{2} |p|_k$.

Proof. By an easy calculation we can verify that 1) holds and that Ip is a unique primitive function of p for which $[Ip] = 0$ and then $Ip \in C^{k+1}$ for $p \in C^k$. To estimate the norm of Ip let us remark that we can add to Ip any function of the form $\int_0^{2\pi} f(y) \,.\, p(s)\,\mathrm{d}s = f(y)\,[p] = 0$.

$$|Ip(y)| = \left| \int_0^y p(s)\,\mathrm{d}s + \int_0^{2\pi} \frac{s}{2\pi}\, p(s)\,\mathrm{d}s - \int_0^{2\pi} \left(\frac{y}{2\pi} + \frac{1}{2} \right) p(s)\,\mathrm{d}s \right| =$$

$$= \left| \int_0^y \left(\frac{s-y}{2\pi} + \frac{1}{2} \right) p(s)\,\mathrm{d}s + \int_y^{2\pi} \left(\frac{s-y}{2\pi} - \frac{1}{2} \right) p(s)\,\mathrm{d}s \right| \leqq$$

$$\leqq |p|_0 \left(\int_0^y \left| \frac{s-y}{2\pi} + \frac{1}{2} \right| \mathrm{d}s + \int_y^{2\pi} \left| \frac{s-y}{2\pi} - \frac{1}{2} \right| \mathrm{d}s \right) = \frac{\pi}{2} |p|_0 \,.$$

From here and from 1) our assertion follows.

Lemma 3.3. *Let* $p \in C^0$, $J = (0, \pi)$, g *be a continuous and bounded function on* J, $-\beta \leqq g(s) \leqq -\gamma < 0$ $(s \in J)$. *Let us denote* $J_k^+ = \{s \in J,\, p(y) - p(y - 2ks) \geqq 0\}$, $J_k^- = J \setminus J_k^+$, k *a nonzero integer. Then it holds*

(3.3)

$$-\pi\gamma\, p(y) - \pi\beta \sup_{s \in J_k^+} (p(y) - p(y - 2ks)) \leqq \int_0^\pi g(s)\,(p(y) - p(y - 2ks))\,\mathrm{d}s \leqq$$

$$\leqq -\pi\gamma\, p(y) + \pi\beta \sup_{s \in J_k^-} (p(y - 2ks) - p(y)) \,.$$

Proof. Because of $\int_{J_k^+} p(y - 2ks)\,\mathrm{d}s = -\int_{J_k^-} p(y - 2ks)\,\mathrm{d}s$, we obtain

$$\int_J g(s)\,(p(y) - p(y - 2ks))\,\mathrm{d}s = \int_{J_k^+} + \int_{J_k^-} \leqq -\gamma \int_{J_k^+} (p(y) - p(y - 2ks))\,\mathrm{d}s +$$

$$+ \int_{J_k^-} |g(s)|\,(p(y - 2ks) - p(y))\,\mathrm{d}s \leqq -\gamma\, p(y)\, m(J_k^+) - \gamma \int_{J_k^-} p(y - 2ks)\,\mathrm{d}s +$$

$$+ \pi\beta \sup_{s \in J_k^-} (p(y - 2ks) - p(y)) \leqq -\gamma\, p(y)\, m(J_k^+) - \gamma \int_{J_k^-} p(y)\,\mathrm{d}s +$$

$$+ \pi\beta \sup_{s \in J_k^-} (p(y - 2ks) - p(y)) = -\pi\gamma\, p(y) + \pi\beta \sup_{s \in J_k^-} (p(y - 2ks) - p(y)) \,.$$

On the other hand

$$\int_{J_k^+} g(s)\,(p(y) - p(y - 2ks))\,ds + \int_{J_k^-} g(s)\,(p(y) - p(y - 2ks))\,ds \geqq$$

$$\geqq -\beta \int_{J_k^+} (p(y) - p(y - 2ks))\,ds - \gamma \int_{J_k^-} (p(y) - p(y - 2ks))\,ds \geqq$$

$$\geqq -\pi\beta \sup_{s \in J_k^+} (p(y) - p(y - 2ks)) - \gamma\, p(y)\, m(J_k^-) - \gamma \int_{J_k^+} p(y - 2ks)\,ds \geqq$$

$$\geqq -\pi\beta \sup_{s \in J_k^+} (p(y) - p(y - 2ks)) - \gamma\, p(y)\, m(J_k^-) - \gamma \int_{J_k^+} p(y)\,ds =$$

$$= -\pi\beta \sup_{s \in J_k^+} (p(y) - p(y - 2ks)) - \pi\gamma\, p(y)\,.$$

Lemma is proved.

Lemma 3.4. *Let $p \in C^0$, $J = (0, \pi)$, g be a continuous and bounded function on J, $a \leqq g(s) \leqq b$ $(s \in J)$, k a nonzero integer, $J_k^+ = \{s \in J,\ p(y) + p(y - 2ks) \geqq 0\}$, $J_k^- = J \setminus J_k^+$. Then*

(3.4)
$$\pi a\, p(y) + (b - a)\, p(y)\, m(J_k^-) - \frac{\pi}{2}(b - a)\,|p|_0 \leqq$$

$$\leqq \int_0^\pi g(s)\,(p(y) + p(y - 2ks))\,ds \leqq$$

$$\leqq \pi a\, p(y) + (b - a)\, p(y)\, m(J_k^+) + \frac{\pi}{2}(b - a)\,|p|_0\,.$$

Proof.

$$\int_J g(s)\,(p(y) + p(y - 2ks))\,ds = \int_{J_k^+} + \int_{J_k^-} \leqq b\, p(y)\, m(J_k^+) + b \int_{J_k^+} p(y - 2ks)\,ds +$$

$$+ a\, p(y)\, m(J_k^-) + a \int_{J_k^-} p(y - 2ks)\,ds = p(y)\,(bm(J_k^+) + a\, m(J_k^-)) +$$

$$+ (b - a) \int_{J_k^+} p(y - 2ks)\,ds \leqq \pi a\, p(y) + (b - a)\, p(y)\, m(J_k^+) + \frac{\pi}{2}(b - a)\,|p|_0\,.$$

On the other hand

$$\int_{J_k^+} \pi(s)\,(p(y) + p(y - 2ks))\,ds + \int_{J_k^-} g(s)\,(p(y) + p(y - 2ks))\,ds \geqq$$

$$\geqq a\, p(y)\, m(J_k^+) + a \int_{J_k^+} p(y - 2ks)\, ds + b\, p(y)\, m(J_k^-) + b \int_{J_k^-} p(y - 2ks)\, ds =$$

$$= p(y)\,(a\, m(J_k^+) + b\, m(J_k^-)) + (b - a) \int_{J_k^-} p(y - 2ks)\, ds \geqq$$

$$\geqq \pi a\, p(y) + (b - a)\, p(y)\, m(J_k^-) - \frac{\pi}{2}(b - a)\, |p|_0 \,.$$

Let us conclude this paragraph by some estimates of the norm of a composite function.

Let f be a function of $(n + 2)$ real variables and let $u_m \in C_k$ $(m = 1, \ldots, n)$. Let us denote $f[u_1, \ldots, u_n]$ the function defined on G by

$$f[u_1, \ldots, u_n]\,(t, x) = f(t, x, u_1(t, x), \ldots, u_n(t, x))\,. \tag{3.5}$$

Then it holds: Each derivative of the function $f[u_1, \ldots, u_n]$ of the order $l \leqq k$ has not more than $l!\,(n + 2)^l$ members each of them being estimated at the point (t, x) by

$$\sup_{0 \leqq |i| \leqq l} |D^i f(t, x, u_1(t, x), \ldots, u_n(t, x))|\, (\max (\sup_{1 \leqq m \leqq n}\, \sup_{i + j \leqq l} |D_1^i D_2^j u_m(t, x)|, 1))^l$$

where i denotes the vector $(i_1, i_2, \ldots, i_{n+2})$, i_m a nonnegative integers, $|i| = \sum_{m=1}^{n+2} i_m$ and D^i denotes the derivative $D_1^{i_1} D_2^{i_2} \ldots D_{n+2}^{i_{n+2}}$.

If f is such that for any $\varrho > 0$

$$F_f(k, \varrho) = \sup_{|i| \leqq k}\, \sup_{\substack{|\alpha_m| \leqq \varrho \\ (t,x) \in G}} |D^i f(t, x, \alpha_1, \ldots, \alpha_n)| < +\infty \tag{3.6}$$

then for any $u_m \in C_k$, $\|u_m\|_k \leqq r$, $\|u_m\|_0 \leqq r_0$ $(m = 1, \ldots, n,\ r \geqq 1)$ the function $f[u_1, \ldots, u_n]$ belongs to C_k and

$$\|f[u_1, \ldots, u_n]\|_k \leqq k!\,(n + 2)^k\, F_f(k, r_0)\, r^k\,. \tag{3.7}$$

Let us denote $K_f(k, r_0, r) = k!\,(n + 2)^k\, F_f(k, r_0)\, r^k$.

Let $u_m, v_m \in C_k$, $\|u_m\|_k \leqq r$, $\|v_m\|_k \leqq r$, $\|u_m\|_0 \leqq r_0$, $\|v_m\|_0 \leqq r_0$ $(m = 1, \ldots, n)$ and let $D^i f$ be continuous for $|i| \leqq k + 1$. Then from the mean-value theorem we obtain

$$f[u_1, \ldots, u_n] - f[v_1, \ldots, v_n] = \sum_{m=1}^{n} g_m(u_m - v_m)$$

where $g_m(t, x) = \int_0^1 D_{m+2} f[v_1, \ldots, v_{m-1}, v_m + \varrho(u_m - v_m), u_{m+1}, \ldots, u_n]\,(t, x)\, d\varrho$ Evidently the functions $g_m \in C_k$ and $\|g_m\|_k \leqq K_f(k + 1, r_0, r)$. For $i + j \leqq k$ we have

$$\|D_1^i D_2^j(f[u_1, \ldots, u_n] - f[v_1, \ldots, v_n])\|_0 =$$

$$= \left\| \sum_{l=0}^{i} \sum_{h=0}^{j} \binom{i}{l} \binom{j}{h} \sum_{m=1}^{n} D_1^{i-l} D_2^{j-h} g_m \, D_1^l D_2^h(u_m - v_m) \right\|_0 \leqq$$

$$\leqq 2^{i+j} K_f(k+1, r_0, r) \sum_{m=1}^{n} \|u_m - v_m\|_{i+j} \, .$$

Thus the following lemma holds:

Lemma 3.5. *Let $D^i f$ be continuous for $|i| \leqq k + 1$ and let the condition (3.5) be fulfilled. Then for $u_m, v_m \in C_k$, $\|u_m\|_0 \leqq r_0$, $\|v_m\|_0 \leqq r_0$, $\|u_m\|_k \leqq r$, $\|v_m\|_k \leqq r$ $(m = 1, \ldots, n)$ the following estimates hold*

(3.8) 1) $\|f[u_1, \ldots, u_n]\|_k \leqq K_f(k, r_0, r)$,

(3.9) 2) $\|f[u_1, \ldots, u_n] - f[v_1, \ldots, v_n]\|_k \leqq 2^k K_f(k+1, r_0, r) \sum_{m=1}^{n} \|u_m - v_m\|_k$,

where $K_f(k, r_0, r) = k! \, (n+2)^k \, F_f(k, r_0) \, r^k$, $F_f(k, r_0)$ is given by (3.6).

4. NONLINEAR EQUATION

Let us solve the problem (1.1), (1.2) under the assumptions:

1° $D^i f$ are continuous for $|i| \leqq k + 1$ and the assumption (3.6) is fulfilled.
2° There exist $\gamma > 0$, $r_0 > 0$ such that

(4.1) a) $D_4 f \leqq -\gamma < 0$ on $G_2 = G \times \langle -\pi r_0, \pi r_0 \rangle \times \langle -2r_0, 2r_0 \rangle \times \times \langle -2r_0, 2r_0 \rangle$,

(4.2) b) $d = \gamma r_0 - \sup \{|f(t, x, u, 0, w)|, (t, x) \in G, |u| \leqq \pi r_0, |w| \leqq 2r_0\} > 0$,

(4.3) c) $\sup_{G_2} D_5 f - \inf_{G_2} D_5 f - \gamma = -\alpha < 0$.

Let $f\{p, u\}$ denote the function

(4.4) $$f\{p, u\} = f[ZIp + u, Zp + D_1 u, Z_1 p + D_2 u] ,$$

where Z_1 is the operator defined by $Z_1 p(t, x) = p(t + x) + p(t - x)$.

We shall prove the existence of a 2π-periodic solution of the problem (1.1), (1.2) in the following way: First we shall prove that if ε is sufficiently small and $p \in C^{k-1}$ then there exists a function $a^\varepsilon(p) \in C_k$ which satisfies the equation $(D_1^2 - D_2^2)(ZIp + a^\varepsilon(p)) = \varepsilon P_2 f\{p, a^\varepsilon(p)\}$ and further we shall seek such p for which $P_2 f\{p, a^\varepsilon(p)\} = f\{p, a^\varepsilon(p)\}$.

Let r_i $(i = 0, \ldots, k)$ be positive numbers, $r = \pi \max(r_i, 0 \leqq i \leqq k) + 1$. Let us denote for $i = 0, \ldots, k$

$$A_i = \{p \in C^i, |D^j p|_0 \leqq r_j, j = 0, \ldots, i\}, \quad B_i = \{u \in N_{i+1}, \|u\|_{i+1} \leqq 1\} \tag{4.5}$$

(then $A_{i+1} \subset A_i$, $B_{i+1} \subset B_i$).

Lemma 4.1. *The equation*

$$u = \varepsilon A P_2 f\{p, u) \tag{4.6}$$

has for $p \in A_0$ *and* $\varepsilon < [54\, 2^k(k + \pi^2) K_f(k + 1, \pi r_0 + 1, r)]^{-1}$ *a unique solution* $a^\varepsilon(p) \in B_0$ *and further it holds*

$$1) \quad a^\varepsilon(p) \in B_i \quad \text{for} \quad p \in A_i \quad (i = 0, \ldots, k),$$

$$2) \quad \|a^\varepsilon(p)\|_{i+1} \leqq \varepsilon K_1 \quad (p \in A_i), \tag{4.7}$$

$$3) \quad \|a^\varepsilon(p) - a^\varepsilon(q)\|_{i+1} \leqq \varepsilon K_2 |p - q|_i \quad (p, q \in A_i), \tag{4.8}$$

where $K_1 = 9(k + \pi^2) K_f(k, \pi r_0 + 1, r)$, $K_2 = 18(k + \pi^2)(\pi + 4) 2^k K_f(k + 1, \pi r_0 + 1, r)$.

Proof. Let $p \in A_i$ $(0 \leqq i \leqq k)$. Then by (2.11) and (2.22) $\varepsilon A P_2 f\{p, u\}$ maps N_{i+1} into itself. Using lemma 3.4 we get for $u \in B_i$

$$\begin{aligned} \|\varepsilon A P_2 f\{p, u\}\|_{i+1} &\leqq \varepsilon 3(i + \pi^2)\, 3 \|f\{p, u\}\|_i \leqq \\ &\leqq \varepsilon 9(k + \pi^2) K_f(k, \pi r_0 + 1, r) < 1 \end{aligned} \tag{4.9}$$

and so $\varepsilon A P_2 f\{p, u\}$ maps B_i into itself. Further for $u \in B_i$, $v \in B_i$

$$\begin{aligned} \|\varepsilon A P_2 f\{p, u\} - \varepsilon A P_2 f\{p, v\}\|_{i+1} &\leqq \varepsilon 9(i + \pi^2) \|f\{p, u\} - f\{p, v\}\|_i \leqq \\ &\leqq \varepsilon 9(k + \pi^2) 2^k K_f(k + 1, \pi r_0 + 1, r) (\|u - v\|_i + \|D_1 u - D_1 v\|_i + \\ &+ \|D_2 u - D_2 v\|_i) \leqq \varepsilon 27(k + \pi^2) 2^k K_f(k + 1, \pi r_0 + 1, r) \|u - v\|_{i+1} \leqq \\ &\leqq \tfrac{1}{2} \|u - v\|_{i+1}. \end{aligned} \tag{4.10}$$

We get that $\varepsilon A P_2 f\{p, u\}$ is a contraction on B_0 for $p \in A_0$. Hence there exists a unique solution $a^\varepsilon(p) \in B_0$ of the equation (4.6). As for $p \in A_i$ the operator $\varepsilon A P_2 f\{p, u\}$ is also a contraction in B_i, there exists for $p \in A_i \subset A_0$ a solution of the equation (4.6) in $B_i \subset B_0$ and from the uniqueness in B_0 it follows that $a^\varepsilon(p) \in B_i$ for $p \in A_i$.

The assertion 2) follows immediately from (4.9).

The solution $a^\varepsilon(p)$ we can get by the method of successive approximations: $u_0 = a^\varepsilon(q)$, $u_{n+1} = \varepsilon A P_2 f\{p, u_n\}$. By the well known estimates, using (4.10), we get

$$\|a^\varepsilon(p) - a^\varepsilon(q)\|_{i+1} = \lim_{n\to\infty} \|u_n - u_0\|_{i+1} \leqq 2\|u_1 - u_0\|_{i+1} \leqq$$
$$\leqq 2\|\varepsilon A P_2 f\{p, a^\varepsilon(q)\} - \varepsilon A P_2 f\{q, a^\varepsilon(q)\}\|_{i+1} \leqq \varepsilon 18(i + \pi^2)\| f\{p, a^\varepsilon(q)\} -$$
$$- f\{q, a^\varepsilon(q)\}\|_i \leqq \varepsilon 18(k + \pi^2)\, 2^k K_f(k + 1, \pi r_0 + 1, r)\,(\|ZI(p - q)\|_i +$$
$$+ \|Z(p - q)\|_i + \|Z_1(p - q)\|_i) \leqq$$
$$\leqq \varepsilon 18(k + \pi^2)\,(4 + \pi)\, 2^k K_f(k + 1, \pi r_0 + 1, r)\, | p - q|_i\,.$$

Lemma is proved.

Now let us solve the equation

$$P_2 f\{p, a^\varepsilon(p)\} = f\{p, a^\varepsilon(p)\}\,, \tag{4.11}$$

where $a^\varepsilon(p)$ is defined in lemma 4.1.

We shall solve this equation with help of lemma 3.1. The role of the sets M_1, M_2 respectively will play the sets A_{k-1}, A_k respectively with r_0 fulfilling the assumption 2° and r_i which are given by the recurrent formula

$$r_{i+1} = \frac{1}{\alpha}\,(F_f(1, \pi r_0) + (i + 1)!\, 8^{i+1} 2 F_f(i + 1, \pi r_0)\,. \tag{4.12}$$
$$.\,[\max\,(r_0, \ldots, r_i)]^{i+1}) + 1\,.$$

Further let $r = \pi \max r_i + 1$, A_i and ε be as in lemma 4.1 and besides it $\varepsilon < \min\,(\alpha, d)\,(2^k 3 K_f(k + 1, \pi r_0 + 1, r)\, K_1)^{-1}$, where K_1 is given by (4.7).

The equation (4.11) is by 5) of lemma 2.1 equivalent to the equation

$$Q f\{p, a^\varepsilon(p)\} = 0\,. \tag{4.13}$$

Let T_1, T_2 denote the operators defined on A_0

$$T_1 p = Q f\{p, 0\}\,, \tag{4.14}$$

$$T_2 p = Q(f\{p, a^\varepsilon(p)\} - f\{p, 0\}) \tag{4.15}$$

and for $\delta > 0$

$$T_\delta p = (E + \delta T_1 + \delta T_2)\, p\,. \tag{4.16}$$

According to lemma 2.1 and lemma 4.1 the mappings T_1, T_2 map A_i into C^i. Further it is obvious that to solve the equation (4.13) means to find a fixed point of the operator T_δ for some $\delta > 0$.

Using (2.9), (3.9), (4.7) we get the estimate for the operator T_2

$$(4.17)\qquad |T_2p|_i = |Q(f\{p, a^\varepsilon(p)\} - f\{p, 0\})|_i \leqq \|f\{p, a^\varepsilon(p)\} - f\{p, 0\}\|_i \leqq \leqq 2^i 3K_f(i + 1, \pi r_0 + 1, r)\, \|a^\varepsilon(p)\|_{i+1} \leqq \varepsilon K_3\,,$$

where $K_3 = 2^k 3K_f(k + 1, \pi r_0 + 1, r)\, K_1$ (K_1 is given by (4.7)).

Let $\eta = \min\,[(d - \varepsilon K_3)\,(F_f(1, \pi r_0))^{-1}, r_0]$, $0 \leqq \delta \leqq \delta_0 \leqq \min\,(\eta(F_f(1, \pi r_0) + \varepsilon K_3)^{-1}, \gamma^{-1})$ and let us prove that T_δ maps A_0 into itself. If y is such that $|p(y)| \leqq \leqq r_0 - \eta$, then

$$|T_\delta\, p(y)| = \left|p(y) + \frac{\delta}{2\pi}\int_0^\pi [f\{p, 0\}\,(y - s, s) - f\{p, 0\}\,(y + s, s)]\,\mathrm{d}s + \delta T_2\, p(y)\right| \leqq \leqq r_0 - \eta + \delta F_f(0, \pi r_0) + \delta\varepsilon K_3 \leqq r_0\,.$$

If $r_0 - \eta \leqq p(y) \leqq r_0$, then from the same expression we obtain the estimate: $T_\delta\, p(y) \geqq r_0 - \eta - \delta F_f(0, \pi r_0) - \delta\varepsilon K_3 \geqq -r_0$. Using the mean-value theorem we get the operator T_δ in the form

$$T_\delta\, p(y) = p(y) + \frac{\delta}{2\pi}\int_0^\pi [g_1(y, s)\,(p(y) - p(y - 2s)) + g_2(y, s)\,(p(y) - p(y + 2s))]\,\mathrm{d}s + + \frac{\delta}{2\pi}\int_0^\pi (f[ZIp, 0, Z_1p]\,(y - s, s) - f[ZIp, 0, Z_1p]\,(y + s, s))\,\mathrm{d}s + \delta T_2\, p(y)\,,$$

where $g_m(y, s) = \int_0^1 D_4 f[ZIp, \varrho Zp, Z_1p]\,(y + (-1)^m s, s)\,\mathrm{d}\varrho$ $(m = 1, 2)$.

From this expression we get for $r_0 - \eta \leqq p(y) \leqq r_0$ by lemma 3.3 the estimate

$$T_\delta\, p(y) \leqq p(y) - \delta\gamma\, p(y) + \delta F_f(1, \pi r_0)\,\eta + + \delta \sup\,\{|f(t, x, u, 0, w)|, (t, x) \in G, |u| \leqq \pi r_0, |w| \leqq 2r_0\} + + \delta\varepsilon K_3 \leqq r_0 + \delta(-d + \eta F_f(1, \pi r_0) + \varepsilon K_3) \leqq r_0\,.$$

In a similar way (using the first inequality in lemma 3.3) we get for $-r_0 \leqq \leqq p(y) \leqq -r_0 + \eta$ that $-r_0 \leqq T_\delta\, p(y) \leqq r_0$ and thus $T_\delta p \in A_0$ for $p \in A_0$.

Let us assume that there exist such $\delta_j > 0$, $0 \leqq j \leqq i \leqq k - 1$, that for $0 < \delta \leqq \delta_j$ the operator T_δ maps the set A_j into itself and let us seek δ_{i+1} such that T_δ for $0 < < \delta \leqq \delta_{i+1}$ maps A_{i+1} into itself.

$$(4.18)\qquad D^{i+1} T_\delta\, p(y) = D^{i+1}\, p(y) + + \frac{\delta}{2\pi}\int_0^\pi [D_4 f\{p, 0\}(y - s, s)\,(D^{i+1}\, p(y) - D^{i+1} p(y - 2s)) + + D_4 f\{p, 0\}\,(y + s, s)\,(D^{i+1}\, p(y) - D^{i+1} p(y + 2s)) +$$

$$+ D_5 f\{p, 0\}(y - s, s)(D^{i+1} p(y) + D^{i+1} p(y - 2s)) -$$
$$- D_5 f\{p, 0\}(y + s, s)(D^{i+1} p(y) + D^{i+1} p(y + 2s))] \, ds +$$
$$+ \delta X_{i+1}(y) + \delta D^{i+1} T_2 \, p(y),$$

where X_{i+1} is the sum of at most $(i + 1)! \, 4^{i+1}$ members of the form

$$\frac{1}{2\pi} \int_0^\pi [D^n f\{p, 0\}(y - s, s) \, h(y - s, s) - D^n f\{p, 0\}(y + s, s) \, h(y + s, s)] \, ds,$$

$|\boldsymbol{n}| \leqq i + 1$ and h is the product of at most $i + 1$ members $D_1^j ZIp$, $D_1^k Zp$, $D_1^l Z_1 p$ $(1 \leqq j \leqq i + 1, 1 \leqq k \leqq i, 1 \leqq l \leqq i)$ and from here an estimate for X_{i+1} follows

(4.19) $\quad |X_{i+1}|_0 \leqq (i + 1)! \, 8^{i+1} 2F_f(i + 1, \pi r_0) [\max(r_0, \ldots, r_i)]^{i+1} \equiv c_{i+1}.$

Let us suppose that $p \in A_{i+1}$. Then for y for which $|D^{i+1} p(y)| \leqq r_{i+1} - 1$ we get by (4.18), (4.19)

(4.20) $\quad D^{i+1} T_\delta \, p(y) \leqq r_{i+1} - 1 + \delta(2F_f(1, \pi r_0) \, 2r_{i+1} + c_{i+1} + \varepsilon K_3) \leqq r_{i+1}$

if $0 < \delta < \delta_{i+1} = \min [\delta_i, (4F_f(1, \pi r_0) \, r_{i+1} + c_{i+1} + \varepsilon K_3)^{-1}]$.

If $r_{i+1} - 1 \leqq D^{i+1} \, p(y) \leqq r_{i+1}$, then from (4.18) we get $D^{i+1} T_\delta \, p(y) \geqq -r_{i+1}$ and further by lemma 3.3 and lemma 3.4

$$D^{i+1} T_\delta \, p(y) \leqq r_{i+1} + \delta(-\gamma r_{i+1} + F_f(1, \pi r_0) +$$
$$+ (\sup_{G_2} D_5 f\{p, 0\} - \inf_{G_2} D_5 f\{p, 0\}) \, r_{i+1} + c_{i+1} + \varepsilon K_3) \leqq$$
$$\leqq r_{i+1} + \delta(-\alpha r_{i+1} + F_f(1, \pi r_0) + c_{i+1} + \varepsilon K_3) \leqq r_{i+1}.$$

For $-r_{i+1} \leqq D^{i+1} \, p(y) \leqq -r_{i+1} + 1$ we proceed analogously and finally we get $|D^{i+1} T_\delta \, p(y)| \leqq r_{i+1}$ if $|D^{i+1} \, p(y)| \leqq r_{i+1}$.

Thus we have proved that for δ fulfilling (4.20) and r_{i+1} given by (4.12) the operator T_δ maps the set A_{i+1} into itself.

T_δ is a continuous mapping on C^k and from above it follows that it fulfils the assumptions of lemma 3.1 with $M_1 = A_{k-1}$ and $M_2 = A_k$. Thus there exists a fixed-point $p_0 \in A_{k-1}$ of the operator T_δ. This p_0 satisfies the equation (4.13) and hence

$$a^\varepsilon(p_0) = \varepsilon A f\{p_0, a^\varepsilon(p_0)\}.$$

From remark 2.1 it follows that the function $u_\varepsilon = ZIp_0 + a^\varepsilon(p_0)$ is for $k \geqq 2$ a classical solution of the problem (1.1), (1.2). We have proved the following theorem

Theorem 1. *Let f be defined on $G_1 = R \times (0, \pi) \times R \times R \times R$ and fulfil the following assumptions*:

1) *f has derivatives up to the order $k + 1$ and for $r > 0$* $\sup \sup_{|i| \leqq k+1} \{|D^i f(t, x, u, v, w)|,$ $(t, x) \in G, |u| \leqq r, |v| \leqq r, |w| \leqq r\} < +\infty$.

2) *There exist $r_0 > 0$ and $\gamma > 0$ such that*

a) $D_4 f \leqq -\gamma < 0$ *on* $G_2 = G \times \langle -\pi r_0, \pi r_0 \rangle \times \langle -2r_0, 2r_0 \rangle \times \langle -2r_0, 2r_0 \rangle$,

b) $\sup_{G_2} D_5 f - \inf_{G_2} D_5 f - \gamma = -\alpha < 0$,

c) $d = \gamma r_0 - \sup \{|f(t, x, u, 0, w)|, (t, x) \in G, |u| \leqq \pi r_0, |w| \leqq 2r_0\} > 0$.

Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ there exists the function $u_\varepsilon \in C_k$ which is a solution of the problem (1.1), (1.2).

5. ANOTHER NONLINEAR CASE

We shall solve the equation

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, \varepsilon) \tag{5.1}$$

with the boundary conditions (1.2). Let f be defined on $G_3 = R \times (0, \pi) \times R \times \times \langle 0, \varepsilon_0 \rangle$ and fulfil the following assumptions

1) $D_1^{i_1} D_2^{i_2} D_3^{i_3} f$, $i_1 + i_2 + i_3 \leqq k + 1$, are defined and continuous on G_3.

2) There exists $\gamma > 0$ such that

$$D_3 f \leqq -\gamma < 0 \quad \text{on} \quad G_4 = G \times \langle -2r_0 - 1, 2r_0 + 1 \rangle \times \langle 0, \varepsilon_0 \rangle, \tag{5.2}$$

where

$$r_0 > \frac{1}{\gamma} \sup \left(|f(t, x, 0, 0)|; (t, x) \in G\right).$$

$$\text{3)} \quad \sup_{\substack{|i| \leqq k+1 \\ i_4 = 0}} \sup_{G_4} |D^i f(t, x, u, \varepsilon)| < +\infty. \tag{5.3}$$

In this paragraph $f[u, \varepsilon](t, x) = f(t, x, u(t, x), \varepsilon)$.

As in the preceding case we shall seek a solution $a^\varepsilon(p)$ of the equation $u = \varepsilon A P_2 f[Zp + u, \varepsilon]$ and then we shall prove that there exists a function p such that $P_2 f[Zp + a^\varepsilon(p), \varepsilon] = f[Zp + a^\varepsilon(p), \varepsilon]$ with help of the implicit function theorem. We could proceed in the same way as in paragraph 4, but using the implicit function theorem we obtain immediately a continuous dependence of the solution on ε.

Similarly as in lemma 4.1 for $\varepsilon < \bar{\varepsilon} = \min \left([18(k + \pi^2) 2^k K_f(k + 1, 2r_0 + 1, r)]^{-1}, \varepsilon_0\right)$ and $p \in A_i = \{p \in C^i, |D^j p|_0 \leqq r_j, j = 0, \ldots, i\}$ the operator $\varepsilon A P_2 f[Zp + u, \varepsilon]$ is a contraction on $B_i = \{u \in N_{i+1}, |u|_{i+1} \leqq 1\}$, hence we get a unique solution $a^\varepsilon(p) \in B_i$ for $p \in A_i$ for which

$$\text{1)} \quad \|a^\varepsilon(p)\|_{i+1} \leqq \varepsilon K_1, \tag{5.4}$$

$$\text{2)} \quad \|a^\varepsilon(p) - a^\varepsilon(q)\|_{i+1} \leqq \varepsilon K_2 |p - q|_i, \tag{5.5}$$

where $K_1 = 9(k + \pi^2) K_f(k, 2r_0 + 1, r)$, $K_2 = 18(k + \pi^2) 2^{k+1} K_f(k + 1, 2r_0 + 1, r)$, $r = 2 \max_{0 \leqq i \leqq k} r_i + 1$.

Further we shall prove that $a^\varepsilon(p)$ is continuous for $(p, \varepsilon) \in A_k \times \langle 0, \bar{\varepsilon}\rangle$ and that there exists G-derivative $a^\varepsilon_p(p)$ continuous in (p, ε). As (5.5) holds, it suffices to prove that for a fixed p the function $a^\varepsilon(p)$ is continuous in ε. Let $\varepsilon_1, \varepsilon_2 \in \langle 0, \bar{\varepsilon}\rangle$. Then $a^{\varepsilon_1}(p)$ we get from $a^{\varepsilon_2}(p)$ by the method of succesive approximations: $u_0 = a^{\varepsilon_2}(p)$, $u_{n+1} = \varepsilon_1 A P_2 f[Zp + u_n, \varepsilon_1]$. Then

$$\|a^{\varepsilon_1}(p) - a^{\varepsilon_2}(p)\|_{k+1} = \lim_{n\to\infty} \|u_n - u_0\|_{k+1} \leqq 2\|u_1 - u_0\|_{k+1} \leqq$$
$$\leqq 18(k + \pi^2)\, \|\varepsilon_1 f[Zp + a^{\varepsilon_2}(p), \varepsilon_1] - \varepsilon_2 f[Zp + a^{\varepsilon_2}(p), \varepsilon_2]\|_k \leqq$$
$$\leqq 18(k + \pi^2)\, [(\varepsilon_1 - \varepsilon_2)\, \|f[Zp + a^{\varepsilon_2}(p), \varepsilon_1]\|_k +$$
$$+ \varepsilon_2 \|f[Zp + a^{\varepsilon_2}(p), \varepsilon_1] - f[Zp + a^{\varepsilon_2}(p), \varepsilon_2]\|_k] \leqq \omega(|\varepsilon_1 - \varepsilon_2|)\,,$$

where ω is a function on $\langle 0, \bar{\varepsilon}\rangle$, continuous in 0 and $\omega(0) = 0$, ω depends on f, k and r.

To prove the existence of $a^\varepsilon_p(p)$ let us note that the function $v^\varepsilon(p) = a^\varepsilon(p) + Zp$ satisfies the equation

$$v^\varepsilon(p) = Zp + \varepsilon A P_2 f[v^\varepsilon(p), \varepsilon]\,.$$

Then according to the known theorem (see e.g. [4]) there exists for ε sufficiently small $(\varepsilon < (\|A\|\ \|P_2\| \sup_{G_4} |D_3 f|)^{-1})$ a G-derivative $v^\varepsilon_p(p) = [E - \varepsilon R_\varepsilon(p)]^{-1} Zq$ and hence $a^\varepsilon(p)$ has a G-derivative

$$a^\varepsilon_p(p)\,(q) = ([E - \varepsilon R_\varepsilon(p)]^{-1} - E)\, Zq\,, \tag{5.6}$$

where

$$R_\varepsilon(p)\,(w) = A P_2 (D_3 f[v^\varepsilon(p), \varepsilon]\, w)\,.$$

It is obvious that this derivative is continuous in p and ε.

Let $\tilde{\varepsilon} \leqq \bar{\varepsilon}$ be such that all above assumptions are fulfilled for $\varepsilon < \tilde{\varepsilon}$. Let us denote

$$V(p, \varepsilon) = Q f[Zp + a^\varepsilon(p), \varepsilon] \tag{5.7}$$

and let us prove that the operator V fulfils the assumptions of the implicit function theorem.

The operator V maps $C^k \times \langle 0, \tilde{\varepsilon}\rangle$ into C^k. By lemma 3.1 we shall prove that the equation $V(p, 0) = 0$ has a unique solution $p_0 \in C^k$. As in the preceding paragraph we shall prove the existence of a fixed point of the operator $T_\delta p = p + \delta V(p, 0)$. Let $c = \sup\,(|f(t, x, 0, 0)|;\ (t, x) \in G)$ and $r_0 > c/\gamma$. Further let r_i $(i = 1, \ldots, k + 1)$ be given by recurrent formulas

$$r_i = \max\left(\frac{1}{\gamma} F_f(1, 2r_0) + \frac{1}{\gamma} 2^i (i + 1)!\, F_f(i, 2r_0)\, [\max\,(r_0, \ldots, r_{i-1})]^i, 1\right). \tag{5.8}$$

Let us denote $M_1 = \{p \in C^k, |D^i p|_0 \leqq r_i, i = 0, \ldots, k\}$, $M_2 = \{p \in C^{k+1}, |D^i p|_0 \leqq r_i, i = 0, \ldots, k + 1\}$. Let $p \in M_2$. We shall prove that also $T_\delta p \in M_2$.

If $0 < \eta < r_0$, then for y such that $|p(y)| \leqq r_0 - \eta$ we get $|T_\delta\, p(y)| \leqq r_0 - \eta + \delta F_f(0, 2r_0)$.

From the mean-value theorem we get the operator T_δ in the form

$$T_\delta\, p(y) = p(y) +$$

$$+ \frac{\delta}{2\pi}\int_0^\pi [g_1(y, s)(p(y) - p(y - 2s)) + g_2(y, s)(p(y) - p(y + 2s))]\, \mathrm{d}s +$$

$$+ \frac{\delta}{2\pi}\int_0^\pi [f(y - s, s, 0, 0) - f(y + s, s, 0, 0)]\, \mathrm{d}s\,,$$

where

$$g_m(y, s) = \int_0^1 D_3 f[\varrho Z p, 0]\,(y + (-1)^m s, s)\, \mathrm{d}\varrho\,.$$

Then by lemma 3.3 we get for $r_0 - \eta \leqq p(y) \leqq r_0$

$$T_\delta\, p(y) \leqq p(y) - \delta\gamma\, p(y) + \delta\eta F_f(1, 2r_0) + \delta c\,.$$

Obviously for such y

$$T_\delta\, p(y) \geqq r_0 - \eta - F_f(0, 2r_0)\,.$$

If $0 < \eta < \min\,[(\gamma r_0 - c)\,(F_f(1, 2r_0))^{-1}, r_0]$, $0 < \delta \leqq \delta_0 = \min\,[\eta(F_f(0, 2r_0))^{-1}, \gamma^{-1}]$, then $|T_\delta\, p(y)| \leqq r_0$. In the same way we can make the estimates if $-r_0 \leqq p(y) \leqq -r_0 + \eta$ and hence $|T_\delta p|_0 \leqq r_0$.

For $i \geqq 1$ we have

$$D^i T_\delta\, p(y) = D^i p(y) + \frac{\delta}{2\pi}\int_0^\pi [D_3 f[Zp, 0]\,(y - s, s)\,(D^i p(y) - D^i p(y - 2s)) +$$

$$+ D_3 f[Zp, 0]\,(y + s, s)\,(D^i p(y) - D^i p(y + 2s))]\, \mathrm{d}s + \delta\, X_i(y)\,,$$

where $|X_i(y)|$ is estimated by $c_i = 2^i(i + 2)!\, F_f(i, 2r_0)\,[\max\,(r_0, \ldots, r_{i-1})]^i$.

Now if we choose $\eta = 1$ and $0 < \delta \leqq \delta_i = \min\,(\delta_{i-1}, (2F_f(1, 2r_0)\, r_i + c_i)^{-1})$ we can prove similarly as above that $|D^i T_\delta p|_0 \leqq r_i$. Then the mapping T_δ fulfils the assumptions of lemma 3.1 and hence there exists a fixed point $p_0 \in M_1$ of the operator T_δ which is a solution of the equation $V(p, 0) = 0$.

This p_0 is unique in C^0, because if $p_1 \in C^0$ is another solution of the equation $V(p, 0) = 0$, then for $p' = p_0 - p_1$ it holds

$$0 = V(p_0, 0)\,(y) - V(p_1, 0)\,(y) =$$

$$= \frac{1}{2\pi}\int_0^\pi [g_1(y, s)\,(p'(y) - p'(y - 2s)) + g_2(y, s)\,(p'(y) - p(y + 2s))]\, \mathrm{d}s\,,$$

where

$$g_m(y, s) = \int_0^1 D_3 f[Zp_0 + \varrho(Zp_1 - Zp_0)] (y + (-1)^m s, s) \, d\varrho .$$

From this expression we get for $p' \neq$ const. and y_0 such that $p'(y_0) = \max (p'(y); y \in R)$, $V(p_0, 0)(y_0) - V(p_1, 0)(y_0) < 0$ which is a contradiction and hence $p' =$ const $= 0$ because $[p'] = 0$.

From above it follows that the operator $V(p, \varepsilon)$ is continuous in p and ε in a neighbourhood of $(p_0, 0)$ and that it has a G-derivative $V_p(p, \varepsilon)$ continuous in p and ε in the neighbourhood of $(p_0, 0)$. Further we must prove that the operator $H = V_p(p_0, 0)$ has an inverse operator H^{-1}. It is easily seen that H maps C^k into itself. We shall prove that H is an $1-1$ mapping. Let $p \in C^k$ be such that $Hp = 0$. Let $p(y_0) = \max (p(y); y \in R)$. If for some s $p(y_0 - 2s) < p(y_0)$ or $p(y_0 + 2s) < p(y_0)$, then

$$0 = \int_0^\pi [D_3 f[Zp_0, 0] (y - s, s) (p(y_0) - p(y_0 - 2s)) + \\ + D_3 f[Zp_0, 0] (y + s, s) (p(y_0) - p(y_0 + 2s))] \, ds < 0 .$$

This is a contradiction and hence $p = \text{const} = 0$ because $[p] = 0$.

Let us denote $g(y, s) = D_3 f[Zp_0, 0] (y - s, s) + D_3 f[Zp_0, 0] (y + s, s)$, $g_0(y) = \int_0^\pi g(y, s) \, ds$. Then we can write the operator H as a sum $H = H_1 + H_2$, where

$$H_1 \, p(y) = \frac{1}{2\pi} g_0(y) \, p(y) - \frac{1}{2\pi} \int_0^{2\pi} g_0(s) \, p(s) \, ds ,$$

$$H_2 \, p(y) = \frac{1}{2\pi} \int_0^{2\pi} g_0(s) \, p(s) \, ds - \frac{1}{2\pi} \int_0^\pi (D_3 f[Zp_0, 0] (y - s, s) \, p(y - 2s) + \\ + D_3 f[Zp_0, 0] (y + s, s) \, p(y + 2s)) \, ds .$$

Evidently H_1, H_2 are the operators from C^k into C^k, H_1 has on C^k a bounded H_1^{-1}

$$H_1^{-1} \, p(y) = \frac{1}{g_0(y)} \left(p(y) - \left(\int_0^{2\pi} \frac{1}{g_0(s)} \, ds \right)^{-1} \int_0^{2\pi} \frac{1}{g_0(s)} \, p(s) \, ds \right).$$

We shall prove that the operator H_2 is completely continuous. Let U be a bounded set of C^k. To prove that $H_2(U)$ is compact in C^k it suffices to prove that the derivatives of the order k of functions from $H_2(U)$ fulfil the assumptions of Arzela's theorem. It is obvious that they are uniformly bounded. Further (if $k \geqq 1$)

$$D^k H_2 \, p(y) = - \frac{1}{2\pi} \int_0^\pi [D_3 f[Zp_0, 0] (y - s, s) \, D^k p(y - 2s) + \\ + D_3 f[Zp_0, 0] (y + s, s) \, D^k p(y + 2s)] \, ds + X_k =$$

$$= \frac{1}{4\pi} \int_y^{y-2\pi} D_3 f[Zp_0, 0] \left(\frac{y + s}{2}, \frac{y - s}{2} \right) D^k p(s) \, ds -$$

$$- \frac{1}{4\pi} \int_y^{y+2\pi} D_3 f[Zp_0, 0] \left(\frac{s + y}{2}, \frac{s - y}{2} \right) D^k p(s) \, ds + X_k .$$

In X_k are only derivatives of p up to the order $k - 1$, so they are equicontinuous. Further it is easily seen that the first and second integrals are also equicontinuous with respect to $p \in U$. So the operator H_2 is completely continuous. As the operator $E + H_1^{-1}H_2 = H_1^{-1}H$ is also an $1-1$ operator and $H_1^{-1}H_2$ is a completely continuous operator, there exists by the well known theorem a linear bounded $(E + H_1^{-1}H_2)^{-1}$ on C^k and then there exists on C^k also the linear bounded $H^{-1} = (E + H_1^{-1}H_2)^{-1} H_1^{-1}$.

Now we have verified all assumptions of the implicit function theorem and hence the following theorem holds:

Theorem 2. *Let f be defined on $G_3 = R \times (0, \pi) \times R \times \langle 0, \varepsilon_0 \rangle$ and fulfil the following assumptions*:

1) $D_1^{i_1} D_2^{i_2} D_3^{i_3} f$, $i_1 + i_2 + i_3 \leqq k + 1$, are defined and continuous on G_3.
2) There exists $\gamma > 0$ such that

$$D_3 f \leqq -\gamma < 0 \quad \text{on} \quad G \times \langle -2r_0 - 1, 2r_0 + 1 \rangle \times \langle 0, \varepsilon_0 \rangle ,$$

where

$$r_0 > \gamma^{-1} \sup \left(|f(t, x, 0, 0)|; (t, x) \in G \right) .$$

3) $\sup_{\substack{|i| \leqq k+1 \\ i_4 = 0}} \sup \{ |D^i f(t, x, u, \varepsilon), (t, x) \in G, |u| \leqq 2r_0 + 1, \varepsilon \in \langle 0, \varepsilon_0 \rangle \} < +\infty$.

Let $p_0 \in C^k$ be a solution of the equation $Qf[Zp, 0] = 0$ (which is unique). Then there exists $\varepsilon^* > 0$ such that for $\varepsilon \in \langle 0, \varepsilon^* \rangle$ there exists a solution u_ε of the problem (5.1), (1.2) such that $u_0 = Zp_0$ and u_ε depends continuously on ε in the space C_k.

References

[1] *Vejvoda O.:* Periodic solutions of a linear and weakly nonlinear wave equation in one dimension, I., Czech. Math. J., Vol. *14*, 1964, pp. 341—382.

[2] *Hale J. K.:* Periodic solutions of a class of hyperbolic equations containing a small parameter, Arch. Rat. Mech. Anal., Vol. *23*, 1967, pp. 380—398.

[3] *Rabinowitz P. H.:* Periodic solutions of nonlinear hyperbolic partial differential equations, Comm. Pure Appl. Math., Vol. *20*, 1967, pp. 145—205.

[4] *Канторович Л. В., Акилов Г. П.:* Функциональный анализ в нормированных пространствах. Москва, Гос. издат. физико-матем. лит., 1959.

[5] *De Simon L., Torelli G.:* Soluzioni periodiche di equazioni a derivate parziali di tipo iperbolico non lineari, Rend. Sem. Mat. Univ. Padova 1968, vol *XL*, 380—401.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).