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VALUATIONS IN GROUPS AND RINGS

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1. In [1], the author considered the valuations of groups and rings as mappings of these systems into semilattices and lattices respectively. As a matter of fact, this idea of valuation introduced by him, generalises the concept of the term used previously. There, the author established a connection between the valuations of groups and rings and the homomorphisms of the lattice $L(G)$ of subgroups of G and the lattice $L(R)$ of ideals of R, into the valuation semilattice and lattice respectively. Here, the results of [1] have been strengthened and as such, the said connection can be given in a more explicit form.

2. A mapping $N : G \to P$, of an additive group G into an upper semilattice P, is called *valuation*, if and only if, $N(a + b) \subseteq N(a) \cup N(b)$.

The valuation N is called *symmetric*, if $N(a) = N(-a)$, for all $a \in G$. Let the lattice of subgroups of G be denoted by $L(G)$. Let $V(G)$ be the set of all symmetric valuations of G into P and $HL(G)$, the set of all homomorphisms of the upper semilattice of the lattice $L(G)$ into P.

Theorem 1. *Let G be an additive abelian group and P a complete upper semilattice. Then the sets $V(G)$ and $HL(G)$ have the same power.*

Proof. Let $N \in V(G)$. If for any subgroup $G_1 \subseteq G$, we put $n(G_1) = \bigcup_{a \in G_1} N(a)$, then as it has been shown in [1], $n \in HL(G)$.

On the other hand, if $n \in HL(G)$, then by defining $N'(a) = n(\{a\})$, where $\{a\}$ is the cyclic group generated by $a \in G$, it has been shown in [1] that $N' \in V(G)$.

We shall show that this correspondence between $V(G)$ and $HL(G)$ is biunique.

Since N is a symmetric valuation into the complete upper semilattice P, we have, $N'(a) = n(\{a\}) = \bigcup_{k=-\infty}^{\infty} N(ka) \subseteq N(a)$. On the other hand, $N(a) \subseteq \bigcup_{k=-\infty}^{\infty} N(ka) = n(\{a\}) = N'(a)$. Hence $N = N'$.

Let now, $n \in HL(G)$, $N(a) = n(\{a\})$, and $n'(H) = \bigcup_{a \in H} N(a)$, where H is any subgroup of G. Then $n'(H) = \bigcup_{a \in H} N(a) = \bigcup_{a \in H} n(\{a\}) = n(\bigcup_{a \in H} \{a\}) = n(H)$. That is, $n' = n$. Hence the theorem.

3. Let R be a ring with 1. A mapping $N : R \to L$, of the ring R into the lattice L, is called *valuation*, if and only if,

i) $N(ab) \subseteq N(a) \cap N(b)$,

ii) $N(a + b) \subseteq N(a) \cup N(b)$.

As shown in [1], N is always symmetric. Let $L(R)$ be the lattice of ideals of R, $V(R)$ be the set of all valuations of R into L and $HL(R)$, the set of all homomorphisms of the upper semilattice of $L(R)$ into the upper semilattice of L.

Theorem 2. *Le R be a commutative ring with* 1 *and L be a complete lattice. Then the two sets $V(R)$ and $HL(R)$ have the same power.*

Proof. Let $N \in V(R)$. If we define $n(J) = \bigcup_{a \in J} N(a)$, where J is any ideal of R, then as shown in [1], $n \in HL(R)$.

Conversely, if $n \in HL(R)$, then by putting $N'(a) = n((a))$, where (a) is the principal ideal generated by $a \in R$, it has been proved in [1], that $N' \in V(R)$.

We shall now show that this correspondence between $V(R)$ and $HL(R)$ is biunique.

We have $N'(a) = n((a)) = \bigcup_{\varrho \in R} N(a\varrho) \subseteq N(a)$.

On the other hand, $N(a) \subseteq \bigcup_{\varrho \in R} N(a\varrho) = n((a)) = N'(a)$, since R contains 1.

Consequently, $N = N'$.

Further, let $n \in HL(R)$, $N(a) = n((a))$, $n'(J) = \bigcup_{a \in J} N(a)$, where J is any ideal in R.

Then $n'(J) = \bigcup_{a \in J} N(a) = \bigcup_{a \in J} n((a)) = n(\bigcup_{a \in J} (a)) = n(J)$.

That is, $n' = n$.

Hence the theorem.

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