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PRIME IDEALS AND MAXIMAL IDEALS IN SEMIGROUPS

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The relations between maximal and prime ideals in commutative rings are well known. If R is a ring, denote by N_0 the set of all nilpotent elements $\in R$. We recall the following results: a) In any commutative ring the intersection of all the prime ideals is N_0. b) In any commutative ring with identity element any maximal ideal is prime. c) If R is a commutative ring with identity element satisfying the descending chain condition every prime ideal of R (and different from R) is maximal.

Note explicitly that in a ring without an identity element maximal ideals need not be prime and prime ideals need not be maximal. [For these results see e.g. [9] and [2], where some notions concerning semigroups are involved.]

In this note we shall study the intersection of all prime ideals and the intersection of all maximal ideals in a (non-commutative) semigroup S. In particular we give a rather general necessary and sufficient condition in order that the set of all maximal ideals coincides with the set of all prime ideals. In all the paper we purposely avoid any chain condition.

There are some reasons to have in mind the following analogy:

rings with an identity element $\leftrightarrow$ semigroups satisfying $S^2 = S$,

rings without an identity element $\leftrightarrow$ semigroups in which $S^2 \neq S$.

We cannot await a full analogy since e.g. the result c) mentioned above does not hold even in the case of a finite commutative semigroup containing an identity element (see Example 4 below).

I

Ideal denotes always a two-sided ideal.

Definition. A non-empty ideal Q of a semigroup S is said *to be prime* if $AB \subset Q$ implies that $A \subset Q$ or $B \subset Q$, A, B being ideals of S.

Remark. There is an analogous definition: An ideal Q is completely prime if $a \,.\, b \in$

$\in Q$ implies that $a \in Q$ or $b \in Q$, a, b being elements $\in S$. An ideal which is completely prime is prime. But the converse need not be true. These concepts coincide if S is commutative. In this paper we consider prime ideals in the sense of our definition.[1]) Prime ideals in the case of a compact semigroup have been thoroughly studied in [5].

Example 1. The semigroup S itself is always a prime ideal of S. But S need not have prime ideals $\neq S$. Let e.g. $S = \{0, a, a^2, \ldots, a^{m-1}\}$, $m \geqq 2$, be a semigroup with zero in which $a^m = 0$. Any ideal $\neq S$ is of the form $I_\varrho = \{a^\varrho, \ldots, a^{m-1}, 0\}$, $2 \leqq \varrho \leqq \leqq m$. Since we have $I_\varrho \subsetneqq I_{\varrho-1}$ and $I_{\varrho-1} I_{\varrho-1} \subset I_\varrho$, I_ϱ is not a prime ideal of S.

Definition. An ideal M of S is called *maximal* if $M \subsetneqq S$ and there does not exist an ideal M_1 of S such that $M \subsetneqq M_1 \subsetneqq S$.

There are known some results concerning the existence of maximal ideals. (See [6], [7] and in the compact case [1], [4].) We shall not deal explicitly with these questions.

Example 2. The following example shows that a prime ideal need not be necessarily embeddable in a maximal ideal.

Denote by T_1 the multiplicative semigroup of numbers x satisfying $0 \leqq x < 1$. Adjoin an element a and consider the set $S = T_1 \cup \{a\}$. Define in S a commutative multiplication $\circ$ by

$$x \circ y = \begin{cases} xy, & \text{if } x \in T_1, \quad y \in T_1, \\ 0, & \text{if } x \in T_1, \quad y = a, \\ a, & \text{if } x = a, \quad y = a. \end{cases}$$

Then S is a semigroup and $S^2 = S$. S contains a unique maximal ideal, namely T_1. The set $I = \{0, a\}$ is a prime ideal of S. If $T_\alpha = \{x \mid 0 \leqq x < \alpha < 1\}$, then $\{0, a, T_\alpha\}$ is an ideal containing I, but clearly there does not exist a maximal ideal of S containing I.

In the following when speaking about maximal ideals we suppose, of course, that maximal ideals exist.

Theorem 1. *If S is a semigroup with $S = S^2$, then every maximal ideal of S is a prime ideal of S.*

Proof. Let M be a maximal ideal of S. Denote $S - M = P$. We first prove $P \subset \subset P^2$. We have

$$S = (M \cup P)^2 = M^2 \cup MP \cup PM \cup P^2 \subset M \cup P^2 .$$

Since $M \cap P = \emptyset$, we have $P \subset P^2$.

[1]) (Added May, 1968.) In a recent paper R. Fulp [11] is treating some problems analogous to ours by using the notion prime ideal in the sense of a completely prime ideal.

Let now A, B be two ideals of S, none of them contained in M such that $AB \subset M$. Since $A \not\subset M$ and M is maximal, we have $A \cup M = S$, hence $P \subset A$. By the same argument $P \subset B$. Hence $P^2 \subset AB$, the more $P \subset AB$. This contradicts $AB \subset M$.

Remark. If $S^2 \neq S$, then Theorem 1 does not hold. For, let $a \in S - S^2$. Then $M = S - \{a\}$ is a maximal ideal of S and it is certainly not prime, since $S^2 \subset M$ while $S \not\subset M$.

But we can prove the following:

Theorem 1a. *If M is a maximal ideal of a semigroup S such that $S - M$ contains either more than one element, or an idempotent, then M is a prime ideal of S.*

Proof. We shall use the following well known fact: If M is a maximal ideal of S, then the difference semigroup S/M is simple and if $S - M$ contains more than one element, then S/M cannot be nilpotent. Write again $S = M \cup P$, $M \cap P = \emptyset$. Let A, B be two ideals of S none of them contained in M such that $AB \subset M$. We again have $A \cup M = B \cup M = S$, hence $P \subset A$, $P \subset B$ and $P^2 \subset AB$; therefore $P^2 \subset M$. This would imply that S/M is nilpotent, which is impossible in both cases considered in the statement of our Theorem.

Example 3. The following example serves to clarify the situation. Let T be the multiplicative semigroup of numbers $\{x \mid 0 \leqq x \leqq \frac{1}{2}\}$ and G commutative group. Define in $S = T \cup G$ a multiplication $\circ$ by $x \circ y = 0$ if $x \in T$, $y \in G$, while the products in T and G remain the old ones. Then S is a semigroup with $S^2 \neq S$. Here T is a maximal ideal which is prime. There is an infinity of further maximal ideals, namely the sets $M_s = S - \{s\}$, where s is any element with $\frac{1}{4} < s \leqq \frac{1}{2}$, none of them being a prime ideal of S.

Example 4. The converse of Theorem 1 need not be true even in the case that S is finite and commutative and it contains an identity element. Let S be the set of all residue classes mod 6. We write $S = \{a_0, a_1, \ldots, a_5\}$, where $a_i a_k = a_l$ and l is defined by $l \equiv ik \pmod 6$. S contains a unique maximal ideal $M = \{a_0, a_2, a_3, a_4\}$. It is a prime ideal. But there are two further prime ideals $Q_1 = \{a_0, a_3\}$ and $Q_2 = \{a_0, a_2, a_4\}$, which are not maximal ideals.

The intersection of a finite number of ideals of a semigroup is not empty. [For if Q_1, Q_2 are ideals, we have $Q_1 Q_2 \subset Q_1 \cap Q_2$.] But there are semigroups in which the intersection of all the prime ideals is empty.

Example 5. Let S be the set of all integers $\geqq 2$, the multiplication being the ordinary multiplication of numbers. The sets $Q_p = \{p, 2p, 3p, \ldots\}$ (p = prime) are prime ideals of S and clearly the intersection $\bigcap_p Q_p$ (where p runs trough all primes) is empty. [Note that any union of the type $\bigcup_{p \in \Lambda} Q_p$ (Λ a subset of the set of all primes) is a prime ideal of S.]

In contradistinction to this we shall see in Theorem 2 that the intersection of all maximal ideals of any semigroup is always non-empty. [In our example the maximal ideals are the sets $M_p = S - \{p\}$ and we have $\bigcap_p M_p = S^2$.]

We intend to clarify under which conditions prime ideals are maximal ideals. To this end we first prove the following crucial theorem:

Theorem 2. *Let $\{M_\alpha \mid \alpha \in A\}$ be the set of all different maximal ideals of a semigroup S. Suppose* card $A \geqq 2$ *and denote $P_\alpha = S - M_\alpha$ and $M^* = \bigcap_\alpha M_\alpha$. We then have*:

a) *$P_\alpha \cap P_\beta = \emptyset$ for $\alpha \neq \beta$.*
b) *$S = [\bigcup_{\alpha \in A} P_\alpha] \cup M^*$.*
c) *For every $\nu \neq \alpha$ we have $P_\alpha \subset M_\nu$.*
d) *If I is an ideal of S and $I \cap P_\alpha \neq \emptyset$, then $P_\alpha \subset I$.*
e) *For $\alpha \neq \beta$ we have $P_\alpha P_\beta \subset M^*$, so that M^* is not empty.*

Remark. The case card $A = 1$ is trivial.

Proof. a) For $\alpha \neq \beta$ we have $M_\alpha \cup M_\beta = S$. Hence

$$P_\alpha \cap P_\beta = (S - M_\alpha) \cap (S - M_\beta) = S - (M_\alpha \cup M_\beta) = \emptyset\,.$$

b) We have

$$M^* = \bigcap_{\alpha \in A} M_\alpha = \bigcap_{\alpha \in A} (S - P_\alpha) = S - \bigcup_{\alpha \in A} P_\alpha\,.$$

Hence

$$S = [\bigcup_{\alpha \in A} P_\alpha] \cup M^*\,.$$

c) For $\alpha \neq \nu$ we have

$$P_\alpha = S \cap P_\alpha = (M_\nu \cup P_\nu) \cap P_\alpha = M_\nu \cap P_\alpha\,.$$

Hence $P_\alpha \subset M_\nu$.

d) If $I \cap P_\alpha \neq \emptyset$, the set $M_\alpha \cup I$ is an ideal of S which is larger than M_α, hence $M_\alpha \cup I = S$. Since $M_\alpha \cap P_\alpha = \emptyset$, we have $P_\alpha \subset I$.

e) Suppose for an indirect proof that there is a couple $u_\alpha \in P_\alpha$, $u_\beta \in P_\beta$ such that $u_\alpha u_\beta = u_\gamma$ is not contained in M^*. With respect to b) there is P_γ such that $u_\gamma \in P_\gamma$. Suppose first that $P_\gamma \neq P_\alpha$. Then $P_\alpha \subset S - P_\gamma = M_\gamma$ and $P_\alpha P_\beta \subset M_\gamma P_\beta \subset M_\gamma$, hence $u_\gamma \in M_\gamma$, which is a contradiction to $u_\gamma \in P_\gamma = S - M_\gamma$. Suppose next $P_\gamma = P_\alpha$. Then $P_\beta \subset S - P_\alpha = M_\alpha$ and $P_\alpha P_\beta \subset P_\alpha M_\alpha \subset M_\alpha$, hence $u_\gamma \in M_\alpha = S - P_\alpha$, which is a contradiction to $u_\gamma \in P_\alpha$.

Theorem 3. *Let S be a semigroup containing maximal ideals and let M^* be the* *intersection of all maximal ideals of S. Then every prime ideal of S containing M^* and different from S is a maximal ideal of S.*

Proof. Let Q be a prime ideal of S containing M^* and $Q \neq S$. We use the notations of Theorem 2. By d) we have

$$Q = S - [\bigcup_{\nu \in H} P_\nu] = \bigcap_{\nu \in H} (S - P_\nu) = \bigcap_{\nu \in H} M_\nu ,$$

where $H \subset \Lambda$ and H is not empty.

If card $H = 1$, we have $Q = M_\nu$, i.e. Q is a maximal ideal of S and our Theorem is proved.

We shall show that card $H \geqq 2$ cannot take place. Suppose for an indirect proof card $H \geqq 2$. Let $\beta \in H$ and denote $M' = \bigcap_{\substack{\nu \in H \\ \nu \neq \beta}} M_\nu$. We then have $Q = M' \cap M_\beta$ and $M'M_\beta \subset M' \cap M_\beta = Q$. Since Q is prime, we have either $M' \subset Q$ or $M_\beta \subset Q$.

a) The first possibility $M' \subset Q$ together with $Q \subset M'$ implies $Q = M'$. Further $M' = Q = M' \cap M_\beta$ implies $M' \subset M_\beta$. By Theorem 2c we have $P_\beta \subset \bigcap_{\nu \in H, \nu \neq \beta} M_\nu = M'$. Hence $P_\beta \subset M_\beta$, a contradiction with $P_\beta \cap M_\beta = \emptyset$.

b) The second possibility $M_\beta \subset Q$ together with $Q \subset M_\beta$ implies $Q = M_\beta$. Now $Q = M_\beta = M' \cap M_\beta$ would imply $M_\beta \subset M'$. Since M_β is maximal and $M' \subsetneqq S$ we have $M' = M_\beta$ .The relation $P_\beta \subset M' = M_\beta$ constitutes an apparent contradiction. This completes the proof of our Theorem.

Let now be $\mathfrak{G} = \{Q_\alpha \mid \alpha \in \Lambda_1\}$ the set of all prime ideals of S and different from S and $Q^* = \bigcap_{\alpha \in \Lambda_1} Q_\alpha$. Let $\mathfrak{M} = \{M_\alpha \mid \alpha \in \Lambda\}$ be the set of all maximal ideals of S and (as above) $M^* = \bigcap_{\alpha \in \Lambda} M_\alpha$.

If S satisfies $S^2 = S$ and $\mathfrak{M} \neq \emptyset$, $Q^* \neq \emptyset$ Theorem 1 implies $Q^* \subset M^*$.

Remark. Examples 1 and 5 are giving semigroups in which $\mathfrak{M} \neq \emptyset$ while $\mathfrak{G} = \emptyset$. On the other hand the multiplicative semigroup of real numbers x with $0 \leqq x < 1$ is a semigroup in which $\mathfrak{M} = \emptyset$ while $Q^* = \{0\} \neq \emptyset$.

Theorem 4. *Let S be a semigroup containing at least one maximal ideal. A prime ideal $Q \neq S$ is a maximal ideal of S if and only if $M^* \subset Q$.*

Proof. If Q is a maximal ideal, we clearly have $M^* \subset Q$. If conversely $M^* \subset Q$, then by Theorem 3 Q is a maximal ideal of S.

This implies:

Theorem 5. *Let S be a semigroup containing at least one maximal ideal. Then every prime ideal of S (and $\neq S$) is a maximal ideal of S if and only if $M^* \subset Q^*$.*

If moreover $S = S^2$, we have $Q^* \subset M^*$. Hence:

Theorem 6. *Let S be a semigroup with $S = S^2$, containing at least one maximal ideal. Then $\mathfrak{G} = \mathfrak{M}$ if and only if $Q^* = M^*$.*

II

There are some pertinent questions concerning the set Q^*.

We have seen that Q^* may be empty. Suppose that Q^* is not empty and *it does not contain an idempotent*. The following examples show that in this case S may have but need not have a kernel K (= minimal two-sided ideal).

Example 6. Let H be a simple semigroup without idempotents. (It is known that such semigroups exist. See e.g. [8], p. 229.) Adjoin an identity element e and consider the semigroup $S = H \cup \{e\}$. Here Q^* is not empty, it does not contain idempotents and the kernel K exists since clearly $Q^* = K = M^* = H$.

Example 7. Let S be the multiplicative semigroup of numbers $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$. There is a unique prime ideal different from S namely $Q^* = \{\frac{1}{2}, \frac{1}{4}, \dots\}$. Q^* does not contain an idempotent and, S does not contain a kernel.

The case when Q^* *contains idempotents* is more interesting.

Theorem 7. *If Q^* contains an idempotent, then S has a kernel K and Q^* contains exactly those idempotents which are contained in K.*

Proof. Suppose that Q^* contains an idempotent e. We first show that every ideal of S contains e. Suppose for an indirect proof that there is at least one ideal of S which does not contain e, so that the set $\mathfrak{N}_1$ of all ideals which do not contain e is non-empty. Denote by Q_1 a maximal element $\in \mathfrak{N}_1$ (maximal in the sense of Zorn). We claim that Q_1 is a prime ideal. If it were not so, there would be possible to find two ideals A, B such that $A \not\subset Q_1$, $B \not\subset Q_1$, but $AB \subset Q_1$. Since $Q_1 \subsetneqq Q_1 \cup A$ and $Q_1 \cup A$ is an ideal we would have $e \in Q_1 \cup A$, hence $e \in A$. Analogously $e \in B$. But then $e = e \,.\, e \in AB \subset Q_1$, which is a contradiction. Now since Q_1 is a prime ideal, we have $Q^* \subset Q_1$, hence $e \in Q_1$ contrary to the choice of Q_1. We conclude that $\mathfrak{N}_1$ is empty, i.e. every ideal of S contains e. Hence every ideal of S contains $\{e, eS, Se, SeS\} = SeS$. This implies that S has a kernel K which is clearly equal to the set SeS.

We have just seen that $K = Se_1S$ holds for any idempotent $e_1 \in Q^*$. Since $e_1 \in \in Se_1S$, we have $e_1 \in K$, so that there does not exist an idempotent $\in Q^*$ which is not contained in K. This proves Theorem 7.

Example 1 shows that $\mathfrak{G}$ may be empty even if S has a kernel. For the following considerations it is advantageous to consider the intersection of all prime ideals including S itself. Denote this set by $\bar{Q}^*$. If S has a kernel, the intersection of all ideals is non-empty and since S has at least one prime ideal, namely S itself, $\bar{Q}^*$ is then certainly non-empty.

Suppose that S has a kernel K. An element $x \in S$ is called K-potent if there is an integer $\varrho > 0$ such that $x^{\varrho} \in K$. An ideal I is called K-potent if there is an integer $\tau > 0$ such that $I^{\tau} = K$. We denote by

N_0 the set of all K-potent elements $\in S$;

N_1 the largest ideal contained in N_0;

N_2 the union of all K-potent ideals of S.

Clearly:

$$K \subset N_2 \subset N_1 \subset N_0 .$$

(Note that N_2 need not be itself K-potent.)

The problem arises what can be said about the sets Q^*, $\bar{Q}^*$ and M^* in connection with the sets just introduced. [The following considerations are related to those in [3] and [10].]

Theorem 8. *For any semigroup having a kernel we have* $N_2 \subset \bar{Q}^* \subset N_1$.

Proof. a) Let $x \in N_2$. Then x is contained in a K-potent ideal of S, say I. The smallest ideal containing x is $I_1 = x \cup Sx \cup xS \cup SxS$, hence $I_1 \subset I$. There exists therefore an integer $\varrho \geqq 1$ such that $I_1^{\varrho} = I^{\varrho} = K$. If Q_0 is any prime ideal of S, then $K = I_1^{\varrho} \subset Q_0$ implies $I_1 \subset Q_0$, hence $x \in Q_0$ and therefore $N_2 \subset \bar{Q}^*$.

b) To prove $\bar{Q}^* \subset N_1$ it is sufficient to prove $\bar{Q}^* \subset N_0$. For $\bar{Q}^*$ is an ideal and N_1 is the largest ideal contained in N_0 so that $\bar{Q}^* \subset N_0$ implies $\bar{Q}^* \subset N_1$.

If $S = N_0$ there is nothing to prove. Suppose therefore that $S - N_0 \neq \emptyset$. To prove $\bar{Q}^* \subset N_0$ it is sufficient to show that to any $z \in S - N_0$ there is a prime ideal Q_z such that $z \notin Q_z$. Consider to this end the set $\mathfrak{N}$ of all ideals of S which do not meet the cyclic semigroup $\{z, z^2, z^3, \dots\}$. Since z is not K-potent, K belongs to $\mathfrak{N}$ and $\mathfrak{N}$ is not empty. Denote by Q_z a maximal element $\in \mathfrak{N}$. We claim that Q_z is a prime ideal. Suppose for an indirect proof that there are two ideals $A \not\subset Q_z$, $B \not\subset Q_z$ such that $AB \subset Q_z$. Since $Q_z \subsetneqq Q_z \cup A$ and $Q_z \cup A$ is an ideal, there is a power z^n such that $z^n \in A$ and analogously there is a power z^m such that $z^m \in B$. But then we would have $z^{m+n} \in AB \subset Q_z$, contrary to the choice of Q_z. This proves our Theorem.

Remark. In general $N_2 \subset Q^* \subset N_1$ need not hold. In the semigroup of the Example 1 we have $K = \{0\}$ and since there are no prime ideals different from S, we have $\mathfrak{G} = \emptyset$. [On the other hand we have, of course, $N_2 = \bar{Q}^* = N_1$.]

We finally prove an analogous result for the set M^*.

Theorem 9. *If S is a semigroup with kernel satisfying $S^2 = S$, then if $\mathfrak{M} \neq \emptyset$, we have $N_1 \subset M^*$.*

Proof. Let M be any maximal ideal of S. We show that $N_1 \subset M$. If this were not the case we would have $N_1 \cup M = S$. Since M is maximal, S/M is a simple semigroup. Since $K \subset M$ every element $\in S/M$ is nilpotent. Hence $S - M$ contains a unique element and we may write $S = M \cup \{z\}$, where $z^2 \in M$. But then

$$S^2 = [M \cup \{z\}]^2 = M^2 \cup Mz \cup zM \cup z^2 \subset M \subsetneqq S.$$

This contradicts $S^2 = S$.

Remark. Theorem 9 does not hold if $S^2 \neq S$. Consider again the Example 1. In this case there is a unique maximal ideal $M = \{a^2, a^3, \ldots, a^{m-1}, 0\}$ while $N_1 = S$, so that $N_1 \subset M$ does not hold.

Corollary. *If S is a semigroup with kernel K satisfying $S^2 = S$ and $\mathfrak{M} \neq \emptyset$, we have:*

$$K \subset N_2 \subset \bar{Q}^* \subset N_1 \subset N_0 \cap M^*.$$

References

[1] *W. M. Faucett, R. J. Koch* and *K. Numakura:* Complements of maximal ideals in compact semigroups, Duke Math. J. *22* (1955), 655–661.

[2] *L. Lesieur - R. Croisot:* Algèbre noethérienne non commutative, Mémorial des Sciences Mathématiques, Paris, 1963.

[3] *Jiang Luh:* On the concept of radical of semigroups having kernel, Portugaliae Mathematica *19* (1960), 189–198.

[4] *R. J. Koch* and *A. D. Wallace:* Maximal ideals in compact semigroups, Duke Math. J. *21* (1954), 681–685.

[5] *K. Numakura:* Prime ideals and idempotents in compact semigroups, Duke Math. J. *24* (1957), 671–680.

[6] *Št. Schwarz:* O maksimal'nych idealach v teorii polugrupp I, II, Czechoslovak Math. J. *3* (78) (1953), 139–153, 365–383.

[7] *Št. Schwarz:* Maximálne ideály a štruktúra pologrúp, Mat. fyz. časopis SAV *3* (1953), 17–39.

[8] *Št. Schwarz:* Subsemigroups of simple semigroups, Czechoslovak Math. J. *13* (88) (1963), 226–239.

[9] *O. Zariski - P. Samuel:* Commutative algebra, vol. I, Princeton, 1958.

[10] *R. Šulka:* O nil'potentnych elementach, idealach i radikalach polugruppy, Mat. fyz. časopis SAV *13* (1963), 209–222.

[11] *R. Fulp:* Generalized semigroup kernels, Pacific J. Math. *24* (1968), 93–101.

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