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SYSTEM OF LAYERS OF AN ORDERED SET

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INTRODUCTION

In this paper there is studied the system $\mathfrak{N}(G)$ of all subsets N of an ordered set G fulfilling the axioms: $(I\ \mathfrak{N})$ *for* $x, y \in N$, $x \neq y$ *there does not exist* $z \in G$, $z \leqq x$, $z \leqq y$; $(\mathrm{II}\ \mathfrak{N})$ N *is maximal with respect to the property described in* $(I\ \mathfrak{N})$. A set N with these properties is called a *layer of the ordered set* G.

We defined the ordering $\precsim$ on the system $\mathfrak{N}(G)$ as follows: For $N_1, N_2 \in \mathfrak{N}(G)$ we have $N_1 \precsim N_2$ if and only if to any element $n_2 \in N_2$ there exists at least one element $n_1 \in N_1$ such that $n_1 \geqq n_2$. A particular case of the system $\mathfrak{N}(G)$ is the ordered system of all decompositions on some set which can be identified with the ordered system of all equivalences on the same set. In [3] 2.7, it is shown that the ordered set $\mathscr{K}(Q)$ $(\mathscr{R}(Q))$ of classes of compactifications (relative compactifications) of a non-compact space Q, is a particular case of a system $\mathfrak{N}(G)$, too.

In the first section there are introduced the basic algebraic concepts and the notation which will be used in the following. In Section 2, there are studied the properties of a system $\mathfrak{N}(G)$ ordered by means of the relation $\precsim$ for an arbitrary ordered set G. In Section 3, there are studied these properties under a supposition that the set $G' = (o) \oplus G$ is an upper, or lower semilattice. Here o denotes a symbol different from the elements of the set G and $\oplus$ denotes Birkhoff ordinal sum. The Section 4 is devoted to the study of properties of an ordered system $\mathfrak{N}(G)$ under the assumption that G' is a distributive lattice. In 4.7, there are introduced sufficient and necessary conditions for $\mathfrak{N}(G)$ to be distributive or modular lattice, under an assumption that G' is a distributive lattice. I do not know a solution for the general case.

1. FUNDAMENTAL ALGEBRAIC CONCEPTS AND NOTATIONS

In the paper, there are used current concepts and theorems from the theory of ordered sets. Instead of the term "a partially ordered set" we shall use the term "an *ordered set*". For two ordered isomorphic sets X and Y we shall write $X \cong Y$.

A dually ordered set in respect to an ordered set X will be denoted by $\check{X}$. An ordered set $(X, \leqq)$ will be called *down-directed* if for $x, y \in X$ a $z \in X$ exists such that $z \leqq x$, $z \leqq y$.

Let $I \neq \emptyset$. The Cartesian product $X_\iota, \iota \in I$ will be denoted by $\mathfrak{P}X_\iota(\iota \in I)$. The symbol $\prod X_\iota(\iota \in I)$ will denote the cardinal product of ordered sets X_ι, $\iota \in I$.

Under a *lower (upper) semi-lattice* we understand an ordered set in which any pair of elements has an infimum (supremum). A *complete lower (upper) semi-lattice* is an ordered set in which any non-void subset has the infimum (supremum).

Let $(S, \leqq)$ be a semi-lattice (lower or upper) with the least element o. An *atom of the semi-lattice* S is an element $a \in S$ such that $a \neq o$ and $o < b \leqq a$, implies $b = a$ for each $b \in S$.

A semi-lattice S will be termed *atomic* if for any element $s \in S$, $s \neq o$ there exists at least one atom a of the semi-lattice S such that $a \leqq s$.

A semi-lattice S will be called *strongly atomic* if for $s_1, s_2 \in S$, $s_1 \neq o \neq s_2 \neq s_1$ there is $\emptyset \neq \{a \mid a \leqq s_1, a \in A\} \neq \{a \mid a \leqq s_2, a \in A\} \neq \emptyset$, where A is the set of all atoms of the semi-lattice S.

In this paper, G will stand for a non-empty set ordered by a relation $\leqq$. $x \lessdot y$ for $x, y \in G$ will denote that $x < y$ and $z = x$ or $z = y$ for $z \in G$, $x \leqq z \leqq y$. $\mathrm{A}(g)$, $g \in G$, denotes the set $\{t \mid t \in G, t \leqq g\}$. In section 3 and 4, $G' = (o) \oplus G$ holds, where o is a symbol different from all elements of the set G and $\oplus$ denotes Birkhoff's ordinal sum ([1]). Infima and suprema in G' will be, as usual, denoted by $\wedge, \bigwedge, \vee, \bigvee$.

2. SYSTEM OF LAYERS $\mathfrak{N}(G)$ FOR AN ORDERED SET G

Definition 2.1. A set $M \subseteq G$ has the *property* (h) *(in a set* G*)* if for $x \in M$, $y \in M$, $x \neq y$ we have $\mathrm{A}(x) \cap \mathrm{A}(y) = \emptyset$.

Definition 2.2. $\mathfrak{N}(G)$ is the system of all subsets N of G fulfilling the following axioms:

(I $\mathfrak{N}$) N has the property (h).

(II $\mathfrak{N}$) $N \cup (z)$ fails to have the property (h) for $z \in G - N$.

A subset N with the properties mentioned in the definition 2.2 is called a *layer of the ordered set* G.

2.1. *Let* $M \subseteq G$ *have the property* (h). *Then, there exists at least one layer* $N \in \mathfrak{N}(G)$ *such that* $M \subseteq N$.

Proof. Let us denote the system of all subsets $A \subseteq G$ possessing the property (h) and fulfilling the relation $A \supseteq M$ by $\mathfrak{A}$. If we order the system $\mathfrak{A}$ by means of inclusion, it follows from Zorn's lemma that there exists a maximal element $N \in \mathfrak{A}$. We have $N \in \mathfrak{N}(G)$ and $M \subseteq N$.

2.2. $\emptyset \notin \mathfrak{N}(G)$, $\mathfrak{N}(G) \neq \emptyset$.

Proof. From the axiom (II $\mathfrak{N}$) it follows that $\emptyset \notin \mathfrak{N}(G)$. From 2.1 we can get $\mathfrak{N}(G) \neq \emptyset$.

2.3. *A set $M \subseteqq G$ fulfils the axiom (II $\mathfrak{N}$) if and only if to every $g \in G$ there exist elements $m \in M$ and $g' \in G$ such that $g' \leqq g$, $g' \leqq m$.*

Proof. Let $M \subseteqq G$. If the given condition is satisfied, then evidently M fulfils the axiom (II $\mathfrak{N}$). If the set M fulfils the axiom (II $\mathfrak{N}$) and $g \in G - M$, then from the axiom (II $\mathfrak{N}$) there follows the existence of an element $g' \in G$ and $m \in M$ with the mentioned property. If $g \in M$, then we get the mentioned condition for $m = g' = g$.

Definition 2.3. For $N_1 \in \mathfrak{N}(G)$, $N_2 \in \mathfrak{N}(G)$ we put $N_1 \precsim N_2$, if for every $n_2 \in N_2$ there exists at least one element $n_1 \in N_1$ such that $n_1 \geqq n_2$.[1])

2.4. *The relation $\precsim$ is an ordering.*

Proof. Reflexivity and transitivity are evident. Let $N_1 \precsim N_2$, $N_2 \precsim N_1$ hold for $N_1, N_2 \in \mathfrak{N}(G)$. For $n_2 \in N_2$ there exists $n_1 \in N_1$ such that $n_1 \geqq n_2$. Furthermore, $n_2' \in N_2$ exists such that $n_1 \leqq n_2'$. From the axiom (I $\mathfrak{N}$) it then follows that $n_2 = n_1 = n_2'$. Thus $N_2 \subseteqq N_1$. Similarly it turns out that $N_1 \subseteqq N_2$.

2.5. *Let $N_1, N_2 \in \mathfrak{N}(G)$, $N_1 \precsim N_2$. Then, for any $n_2 \in N_2$ exactly one element $n_1 \in N_1$ exists such that $n_1 \geqq n_2$, and for any $n_1' \in N_1$ at least one element $n_2' \in N_2$ such that $n_1' \geqq n_2'$.*

Proof. The first part of the assertion follows from the axiom (I $\mathfrak{N}$). Let $n_1' \in N_1$. By 2.3, $n_2' \in N_2$ and $g' \in G$ exist such that $g' \leqq n_1'$, $g' \leqq n_2'$. Since $N_1 \precsim N_2$, so $n_1^* \in N_1$ exists such that $n_1^* \geqq n_2'$. From the axiom (I $\mathfrak{N}$) it follows at once that $n_1' = n_1^*$; from this the assertion follows.

2.6. *Let $N \in \mathfrak{N}(G)$, $M \subseteqq G$ have the property (h). Let for any element $n \in N$ at least one element $m \in M$ exist such that $n \leqq m$. Then $M \in \mathfrak{N}(G)$, $M \precsim N$.*

Proof. Let $g \in G$. By 2.3, $g' \in G$ and $n \in N$ exist such that $g \geqq g'$, $n \geqq g'$. According to the assumption there exists $m \in M$, $m \geqq n$. Consequently $m \geqq g'$, $g \geqq g'$ and by 2.3 we have $M \in \mathfrak{N}(G)$. Evidently $M \precsim N$.

Example 2.1. Let $P \neq \emptyset$. Let G be the set of all non-void subsets of the set P, ordered by means of inclusion. Then $\mathfrak{N}(G)$ represents the set of all decompositions on P ordered in such a way that the decomposition (P) is the least one.

[1]) The relation $\precsim$ can be introduced in the same way among all subsets of the set G. In this case, however, the relation need not be an ordering but is a quasiordering.

Example 2.2. Let $P \neq \emptyset$ be a topological T_1-space (or more generally Čech's B-space).[2]) Let G be the set of all non-void closed sets of the space P ordered by means of inclusion. Then $\mathfrak{N}(G)$ is the set of all closed decompositions on the space P, ordered in such a manner that the decomposition (P) is the least one.

Example 2.3. The ordered set $\mathscr{K}(Q)$ ($\mathscr{R}(Q)$) of classes of compactifications (relative compactifications) of a non-compact space Q and the set $\mathfrak{N}(\mathrm{I}(Q))$ are isomorphic, where $\mathrm{I}(Q)$ denotes the set of all proper filters on Q without a cluster point ([3], 2.7).

2.7. *$\mathfrak{N}(G)$ possesses the least element if and only if G is a cardinal sum of sets with the largest elements.*

The set N_0 of all these largest elements is then the least element in $\mathfrak{N}(G)$.

Proof. I. The above condition is equivalent to the condition that to any $g \in G$ there exists a maximal element $m \in G$ such that $m \geqq g$ and the set of all maximal elements of the set G has the property (h). This set of all maximal elements of the set G is then the set N_0.

II. Suppose that N'_0 is the least element in $\mathfrak{N}(G)$. For $g \in G$ there exists, by 2.1, $N_g \in \mathfrak{N}(G)$, $g \in N_g$. Since $N'_0 \precsim N_g$, then $m \in N'_0$ exists such that $m \geqq g$. If m is not maximal, then $m' \in G$ exists such that $m' > m$. According to 2.1 $N_{m'} \in \mathfrak{N}(G)$, $m' \in N_{m'}$. Since $N'_0 \precsim N_{m'}$, then $m'' \in N'_0$ exists such that $m'' \geqq m'$. From this it follows that $m'' > m$; $m, m'' \in N'_0$, which is a contradiction. Consequently, m is maximal. For that reason, for any $g \in G$ there exists a maximal element $m \in G$ such that $g \leqq m$, $m \in N'_0$. Hence, the above condition follows.

III. Suppose the above condition is satisfied. Then $N_0 \in \mathfrak{N}(G)$. For $N \in \mathfrak{N}(G)$, $n \in N$ there exists $m \in N_0$ such that $n \leqq m$. Thus $N_0 \precsim N$.

Thereby the assertion is proved.

2.8. *$\mathfrak{N}(G)$ has the largest element exactly if for any element $g \in G$ a minimal element m on the set G exists such that $m \leqq g$.*

The largest element of the system $\mathfrak{N}(G)$ is then the set of all minimal elements of the set G.

Proof. I. Suppose N_0 is the largest element in $\mathfrak{N}(G)$. For $g \in G$ there exists by 2.1 $N_g \in \mathfrak{N}(G)$, $g \in N_g$. Since $N_g \precsim N_0$, by 2.5 there exists at least one element $m \in N_0$ such that $g \geqq m$. If m is not minimal, then $m' \in G$, $m' < m$. According to 2.1, $N_{m'} \in \mathfrak{N}(G)$, $m' \in N_{m'}$. Since $N_{m'} \precsim N_0$, by 2.5 there exists at least one element $m'' \in N_0$ such that $m' \geqq m''$. Then $m'' < m$; $m'', m \in N_0$ which is a contradiction. m is therefore a minimal element of the set G.

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[2]) Čech's B-space is a topological space fulfilling the following axioms: $\overline{\emptyset} = \emptyset$, $(\overline{x}) = (x)$, $X \subseteq \overline{X}$, $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$ ([2]).

II. Let us suppose that the above condition is fulfilled. Let us denote N_0 the set of all minimal elements. Evidently, N_0 fulfills the axioms (I $\mathfrak{N}$) and (II $\mathfrak{N}$); consequently $N_0 \in \mathfrak{N}(G)$. Let $N \in \mathfrak{N}(G)$ and let $n_0 \in N_0 - N$. Since $(n_0) \cup N$ fails to have the property (h), $n \in N$ exists such that $n \geqq n_0$. Thus $N_0 \geqq N$.

Thus, the assertion is proved.

Let $I \neq \emptyset$ and for $\iota \in I$ let $N_\iota \in \mathfrak{N}(G)$. Put $\mathfrak{F}(N_\iota, \iota \in I) = \{N \mid N \in \mathfrak{N}(G),\ N \geqq N_\iota$ for each $\iota \in I\}$. Put $\mathrm{A}(s) = \bigcap \mathrm{A}(s(\iota))$ $(\iota \in I)$ for $s \in \mathfrak{P}N_\iota(\iota \in I)$. Denote $\mathfrak{S}N_\iota(\iota \in I) = \{s \mid s \in \mathfrak{P}N_\iota(\iota \in I),\ \mathrm{A}(s) \neq \emptyset\}$.

2.9. *Let $I \neq \emptyset$, $N_\iota \in \mathfrak{N}(G)$ for any element $\iota \in I$. Then $\mathfrak{F}(N_\iota, \iota \in I) \neq \emptyset$ if and only if for an arbitrary element $g \in G$, $s \in \mathfrak{S}_\iota N(\iota \in I)$ and $g' \in G$ exist such that $g' \in A(s_g)$ and $g' \leqq g$.*

If $\mathfrak{F}(N_\iota, \iota \in I) \neq \emptyset$, then $\mathfrak{F}(N_\iota, \iota \in I) \cong \prod \mathfrak{N}(\mathrm{A}(s))\ (s \in \mathfrak{S}N_\iota(\iota \in I))$.

Proof. For the sake of simplicity let us denote $\mathfrak{S}N_\iota(\iota \in I) = \mathfrak{S}$, $\mathfrak{F}(N_\iota, \iota \in I) = \mathfrak{F}$, and in the case $\mathfrak{S} \neq \emptyset$ let us denote $\prod \mathfrak{N}(\mathrm{A}(s))\ (s \in \mathfrak{S}N_\iota(\iota \in I)) = \prod$.

I. a) For $N \in \mathfrak{F}$, $n \in N$ there exists $s \in \mathfrak{S}$ such that $n \in \mathrm{A}(s)$. Actually $n_\iota \in N_\iota$ exists for any $\iota \in I$ such that $n \leqq n_\iota$ (because $N \geqq N_\iota$). Putting $s(\iota) = n_\iota$, we have $s \in \mathfrak{P}N_\iota(\iota \in I)$ and $n \in \mathrm{A}(s)$, and consequently $s \in \mathfrak{S}$.

b) For $s, s' \in \mathfrak{S}$, $s \neq s'$ we have $\mathrm{A}(s) \cap \mathrm{A}(s') = \emptyset$. Actually $\iota_0 \in I$ exists such that $s(\iota_0) \neq s'(\iota_0)$ and if an element $x \in \mathrm{A}(s) \cap \mathrm{A}(s')$ existed, then $x \leqq s(\iota_0)$, $x \leqq s'(\iota_0)$. This is a contradiction, because $s(\iota_0), s'(\iota_0) \in N_{\iota_0}$.

II. Assume that for an arbitrary element $g \in G$ there exists $s_g \in \mathfrak{S}$ and $g' \in G$ such that $g' \in \mathrm{A}(s_g)$ and $g' \leqq g$. Then $\mathfrak{S} \neq \emptyset$. Put $\varphi(f) = \bigcup f(s)\ (s \in \mathfrak{S})$ for $f \in \prod$. Then $\varphi(f) \subseteqq G$.

a) Let $f \in \prod$ and let $x, y \in \varphi(f)$, $x \neq y$. Then $s_x, s_y \in \mathfrak{S}$ exist such that $x \in f(s_x) \subseteqq \mathrm{A}(s_x)$, $y \in f(s_y) \subseteqq \mathrm{A}(s_y)$. If $z \in G$ exists such that $z \leqq x$, $z \leqq y$, then according to I.b $s_x = s_y$. Then we have $z \in \mathrm{A}(s_x)$ which is a contradiction because $f(s_x) \in \mathfrak{N}(\mathrm{A}(s_x))$ and $x, y \in f(s_x)$. Thus a set $\varphi(f)$ fulfils the axiom (I $\mathfrak{N}$).

b) Let there be $f \in \prod$ and $g \in G$. According to the assumption, $s_g \in \mathfrak{S}$ and $g' \in G$ exist such that $g' \in \mathrm{A}(s_g)$ and $g' \leqq g$. Since $g' \in \mathrm{A}(s_g)$ and $f(s_g) \in \mathfrak{N}(\mathrm{A}(s_g))$, so by 2.3 there exist $g'' \in \mathrm{A}(s_g)$ and $n \in f(s_g)$ such that $g'' \leqq g'$, $g'' \leqq n$. We have $g'' \in G$, $n \in \varphi(f)$, $g'' \leqq n$, $g'' \leqq g$ and according to 2.3 the set $\varphi(f)$ fulfils the axiom (II $\mathfrak{N}$).

c) Let $f \in \prod$. From II.a and II.b it follows that $\varphi(f) \in \mathfrak{N}(G)$. Let $n \in \varphi(f)$. Then there exists $s_n \in \mathfrak{S}$ such that $n \in f(s_n)$. Since $f(s_n) \subseteqq \mathrm{A}(s_n)$, then $n \leqq s_n(\iota)$ for any $\iota \in I$. Thus, $\varphi(f) \geqq N_\iota$ for any $\iota \in I$ and therefore $\varphi(f) \in \mathfrak{F}$.

d) Let $f, g \in \prod$, $f \neq g$. Then $s_0 \in \mathfrak{S}$ exists such that $f(s_0) \neq g(s_0)$. Since $f(s) \subseteqq \mathrm{A}(s)$, $g(s) \subseteqq \mathrm{A}(s)$ for any $s \in \mathfrak{S}$, it follows from I.b that $\varphi(f) \neq \varphi(g)$. Consequently φ is a one-to-one mapping of $\prod$ into the set $\mathfrak{F}$.

e) Let $N \in \mathfrak{F}$, $s \in \mathfrak{S}$. Let us put $f(s) = N \cap \mathrm{A}(s)$. The set $f(s)$ possesses evidently the property (h) in $\mathrm{A}(s)$. For $g \in \mathrm{A}(s)$ there exist by 2.3 $g' \in G$ and $n \in N$ such that $g' \leqq g$, $g' \leqq n$. We have $g' \in \mathrm{A}(s)$ and by I.a and I.b we have also $n \in \mathrm{A}(s)$. $f(s) \in \in \mathfrak{N}(\mathrm{A}(s))$ follows from 2.3. Thus $f \in \prod$ and since $\varphi(f) \subseteq N$, we have $\varphi(f) = N$ as a consequence of I.a.

f) Let $f, g \in \prod$, $f \lesssim g$. Then for any element $s \in \mathfrak{S}$ we have $f(s) \lesssim g(s)$; consequently $\varphi(f) \lesssim \varphi(g)$.

g) Let $N, N' \in \mathfrak{F}$, $N \lesssim N'$. According to II.e, $N \cap \mathrm{A}(s)$, $N' \cap \mathrm{A}(s) \in \mathfrak{N}(\mathrm{A}(s))$ for any element $s \in \mathfrak{S}$. For $n' \in N' \cap \mathrm{A}(s)$, $(s \in \mathfrak{S})$, $n \in N$ exists such that $n' \leqq n$. From I.a and I.b it follows that $n \in N \cap \mathrm{A}(s)$. Thus, for $s \in \mathfrak{S}$ we have $N \cap \mathrm{A}(s) \lesssim N' \cap \mathrm{A}(s)$. It follows from II.e that $\varphi^{-1}(N) \lesssim \varphi^{-1}(N')$.

φ is therefore an isomorphism between the sets $\prod$ and $\mathfrak{F}$.

III. Let $\mathfrak{F} \neq \emptyset$, $g \in G$. Then there exists $N \in \mathfrak{F}$. By 2.3 there exist $g' \in G$ and $n \in N$ such that $g' \leqq g$, $g' \leqq n$. According to I.a $s_g \in \mathfrak{S}$ exists such that $n \in \mathrm{A}(s_g)$. Since we have $g' \in \mathrm{A}(s_g)$, the mentioned condition is satisfied.

Thus the assertion is proved.

2.10. *Let $I \neq \emptyset$. A set $\{N_\iota, \iota \in I\}$, $N_\iota \in \mathfrak{N}(G)$ has a supremum if and only if*

(1) *for an arbitrary element $g \in G$ there exist $s_g \in \mathfrak{S}N_\iota(\iota \in I)$ and $g' \in G$ such that $g' \in \mathrm{A}(s_g)$ and $g' \leqq g$,*

(2) *for any element $s \in \mathfrak{S}N_\iota(\iota \in I)$ the set $\mathfrak{N}(\mathrm{A}(s))$ has the least element.*

Then $\sup N_\iota(\iota \in I) = \bigcup M(s)\ (s \in \mathfrak{S}N_\iota(\iota \in I))$, *where $M(s)$ is the least element of the set $\mathfrak{N}(\mathrm{A}(s))$ which is equal to the set of all maximal elements of the set* $\mathrm{A}(s)$.

Proof. I. Conditions (1) and (2) are equivalent to the statement saying that the set $\{N_\iota, \iota \in I\}$ has a supremum (by 2.9 and due to the fact that the cardinal product of ordered sets possesses the least element if and only if each of its factors has the least element).

II. Let there exist $\sup \{N_\iota, \iota \in I\}$. Then according to 2.9 the set $\prod \mathfrak{N}(\mathrm{A}(s))\ (s \in \in \mathfrak{S}N_\iota(\iota \in I))$ has the least element M. If φ is an isomorphism described in II, proof 2.9, then $\sup \{N_\iota, \iota \in I\} = \varphi(M) = \bigcup M(s)\ (s \in \mathfrak{S}N_\iota(\iota \in I))$. The set $M(s)$ is the least element of the set $\mathfrak{N}(\mathrm{A}(s))$ and by 2.7 it is equal to the set of all maximal elements of the set $\mathrm{A}(s)$.

For $N_1, N_2 \in \mathfrak{N}(G)$, $N_1 \gg N_2$ will denote that $N_1 \succ N_2$ and, for $N \in \mathfrak{N}(G)$, $N_1 \gtrsim \gtrsim N \gtrsim N_2$, we have either $N = N_1$ or $N = N_2$.

2.11. *Let $N_1 \in \mathfrak{N}(G)$. Then the following assertions are equivalnent:*

(A) *$N_2 \in \mathfrak{N}(G)$ and $N_1 \gg N_2$,*

(B) *$N_2 = (N_1 - \mathrm{A}(x_0)) \cup (x_0)$, where the element $x_0 \in G - N_1$ has these properties:*

(α) $\mathrm{A}(x_0) \cap N_1 \in \mathfrak{N}(\mathrm{A}(x_0))$,

(β) $\mathrm{A}(x) \cap N_1 \notin \mathfrak{N}(\mathrm{A}(x))$ for $x \in G - N_1$, $x < x_0$.

Proof. I. Let $y \in G$, $N_2 = (N_1 - A(y)) \cup (y)$, $g \in G - N_2$. We shall show that in this case $N_2 \cup (g)$ fails to have the property (h).

According to 2.3 there exist $g' \in G$ and $n \in N$ such that $g \geqq g'$, $n \geqq g'$. If $n \notin A(y)$, then $(g) \cup N_2$ fails to have the property (h). If $n \in A(y)$ then $g' \leqq y$; consequently $(g) \cup N_2$ also fails to have the property (h).

II. Let $N_2 \in \mathfrak{N}(G)$, $N_1 \gtrapprox N_2$, $y \in N_2$. Then $N = (N_1 - A(y)) \cup (y) \in \mathfrak{N}(G)$ and we have $N_1 \gtrapprox N \gtrapprox N_2$.

Actually, the set N fulfils, according to I, the axiom (II $\mathfrak{N}$). If $a, b \in N$, $a \neq b$ and $a \neq y \neq b$, then $a, b \in N_1$, and therefore $A(a) \cap A(b) = \emptyset$. If $a \in N$ and $a \neq y$, then $a \in N_1 - A(y)$ and there exists $a' \in N_2$ such that $a' \geqq a$. Evidently $a' \neq y$ and consequently $A(a') \cap A(y) = \emptyset$; from this it follows that $A(a) \cap A(y) = \emptyset$. Thus the set N fulfils the axiom (I $\mathfrak{N}$). Hence, $N \in \mathfrak{N}(G)$. Evidently $N_1 \gtrapprox N \gtrapprox N_2$.

III. Let $y \in G$, $A(y) \cap N_1 \in \mathfrak{N}(A(y))$, $N_2 = (N_1 - A(y)) \cup (y)$. Then $N_2 \in \mathfrak{N}(G)$ and we have $N_2 \lessapprox N_1$.

Inded, the set N_2 fulfils, by I, the axiom (II $\mathfrak{N}$). If $a, b \in N_2$, $a \neq b$ and $a \neq y \neq b$, then $a, b \in N_1$; thus $A(a) \cap A(b) = \emptyset$. If $a \in N$ and $a \neq y$, then $a \in N_1 - A(y)$. If there exists $c \in A(a) \cap A(y)$, then according to 2.3 there exist $d \in A(y)$ and $n \in$ $\in A(y) \cap N_1$ such that $d \leqq c$, $d \leqq n(A(y) \cap N_1 \in \mathfrak{N}(A(y)))$. Then $d \leqq a$, $d \leqq n$, $a, n \in N_1$, $a \neq n$ which is a contradiction. Consequently, N_2 fulfils the axiom (I $\mathfrak{N}$); thus $N_2 \in \mathfrak{N}(G)$. Evidently $N_2 \lessapprox N_1$.

IV. Let $N_2 = (N_1 - A(y)) \cup (y) \in \mathfrak{N}(G)$ where $y \in G$. We are going to show that then $A(y) \cap N_1 \in \mathfrak{N}(A(y))$.

The set $A(y) \cap N_1$ has the property (h) in G; consequently it possesses the property (h) even in the set $A(y)$. Thus the axiom (I $\mathfrak{N}$) is valid. For $g \in G$ there exists, by 2.3, $g' \in G$ and $n \in N_1$ such that $g' \leqq g$, $g' \leqq n$. If $g \in A(y)$ then $g' \in A(y)$ and, since $N_2 \in \mathfrak{N}(G)$, we have $n \in A(y)$. Then $n \in A(y) \cap N_1$ and from 2.3 it follows that the set $A(y) \cap N_1 \in \mathfrak{N}(A(y))$.

V. Let (A) hold. According to II, $N_2 = (N_1 - A(x_0)) \cup (x_0)$ where $x_0 \in N_2 - N_1$. According to IV, the element x_0 fulfils the condition (α). If $x \leqq x_0$, $x \notin N_1$ and $A(x) \cap$ $\cap N_1 \in \mathfrak{N}(A(x))$, then by III, $N = (N_1 - A(x)) \cup (x) \in \mathfrak{N}(G)$ and we have $N \prec N_1$. Since $N \gtrapprox N_2$, then $N = N_2$, and consequently $x = x_0$. Thus (B) holds.

VI. Let (B) be valid. By III we have $N_2 \in \mathfrak{N}(G)$ and $N_2 \prec N_1$. Let $N \in \mathfrak{N}(G)$, $N_2 \prec N \lessapprox N_1$. For $x \in N - A(x_0)$ we have $x \in N_1$. Suppose $x \in N \cap A(x_0)$. Then $x < x_0$. According to II, $N' = (N_1 - A(x)) \cup (x) \in \mathfrak{N}(G)$ and according to IV we have $A(x) \cap N_1 \in \mathfrak{N}(A(x))$. From the definition of the property (β) $x \in N_1$ follows. Thus $N \subset N_1$ and $N = N_1$ follows from the axiom (II $\mathfrak{N}$) which means that $N_2 \lll N_1$.

Hence, the assertion is proved.

Definition 2.4. $r(G) = \sup \operatorname{card} N(N \in \mathfrak{N}(G))$.

2.12. (a) $1 \leqq \mathrm{r}(G) \leqq \text{card } G$.

(b) *The following assertions are equivalent:*

(A) *G is down-directed,*

(B) $\mathrm{r}(G) = 1$,

(C) $G \cong \widetilde{\mathfrak{N}(G)}$.

Proof. The assertion (a) follows from 2.2. Evidently the assertion (A) implies (B) and (C) follows from the asertion (B).

Suppose that $G \cong \widetilde{\mathfrak{N}(G)}$ and denote φ the corresponding isomorphism G on $\widetilde{\mathfrak{N}(G)}$. Let $a, b \in G$, $a \neq b$ and let $g \in G$. By 2.3, $n_a \in \varphi(a)$, $g' \in G$ exist such that $g' \leqq n_a$, $g' \leqq g$. According to 2.3, $n_b \in \varphi(b)$, $g'' \in G$ exist such that $g'' \leqq g'$, $g'' \leqq n_b$. Let us put $s_g(a) = n_a$, $s_g(b) = n_b$. Then $s_g \in \varphi(a) \times \varphi(b)$, $g'' \in \mathrm{A}(s_g)$, $g'' \leqq g$. According to 2.9, $N \in \mathfrak{N}(G)$ exists such that $N \gtrsim \varphi(a)$, $N \gtrsim \varphi(b)$. Since φ is an isomorphism G on $\widetilde{\mathfrak{N}(G)}$, then $\varphi^{-1}(N) \in G$, $\varphi^{-1}(N) \leqq a$, $\varphi^{-1}(N) \leqq b$ are valid; consequently (A) holds.

3. SYSTEM OF LAYERS $\mathfrak{N}(G)$ FOR A SEMI-LATTICE G'

3.1. *Let G'* [3] *be a lower semi-lattice. Then $\mathfrak{N}(G)$ is an upper semi-lattice. For $N_1, N_2 \in \mathfrak{N}(G)$ we have* $\sup (N_1, N_2) = \bigcup(n_1 \wedge n_2) (n_1 \in N_1, n_2 \in N_2, n_1 \wedge n_2 > o)$.

Proof. Let $N_1, N_2 \in \mathfrak{N}(G)$. According to 2.3, $g' \in G$ and $n_1 \in N_1$ exist for an arbitrary element $g \in G$ such that $g' \leqq g$, $g' \leqq n_1$. By 2.3, $g'' \in G$ and $n_2 \in N_2$ exist such that $g'' \leqq g'$, $g'' \leqq n_2$. Let us put $s_g(1) = n_1$, $s_g(2) = n_2$. Then $s_g \in N_1 \times N_2$, $g'' \in \mathrm{A}(s_g)$, $g'' \leqq g$.

Let $s \in N_1 \times N_2$, $\mathrm{A}(s) \neq \emptyset$. Then $s(1) \wedge s(2)$ is the largest element of the set $\mathrm{A}(s)$ and, according to 2.7, the set $(s(1) \wedge s(2))$ is the least element of the set $\mathfrak{N}(\mathrm{A}(s))$.

From 2.10 the mentioned assertion follows.

3.2. *Let G' be an atomic complete lower semi-lattice. Then $\mathfrak{N}(G)$ is a complete upper semi-lattice. For $I \neq \emptyset$, $N_\iota \in \mathfrak{N}(G)$ $(\iota \in I)$ we have* $\sup N_\iota(\iota \in I) = \bigcup(\bigwedge n_\iota(\iota \in I)) (n_\iota \in N_\iota, \bigwedge n_\iota(\iota \in I) > o)$.

Proof. Let $I \neq \emptyset$, $N_\iota \in \mathfrak{N}(G)$ for $\iota \in I$. Let $g \in G$. Then an atom $a \in G'$ exists such that $a \leqq g$. Since a is a minimal element of the set G, then, by 2.3, for any $\iota \in I$ there exists $n_\iota \in N_\iota$ such that $n_\iota \geqq a$. Let us put $s_g(\iota) = n_\iota$. Then $s_g \in \mathfrak{S} N_\iota(\iota \in I)$ and we have $a \in \mathrm{A}(s_g)$, $a \leqq g$.

Let $s \in \mathfrak{S} N_\iota(\iota \in I)$, $n_\iota = s(\iota)$ for $\iota \in I$. Then the element $n = \bigwedge n_\iota(\iota \in I)$ is the largest element of the set $\mathrm{A}(s)$. According to 2.7 and 2.10 we have $\sup N_\iota(\iota \in I)$ and $\sup N_\iota(\iota \in I) = \bigcup(\bigwedge n_\iota(\iota \in I)) (n_\iota \in N_\iota, \bigwedge n_\iota(\iota \in I) > o)$.

[3] $G' = (o) \oplus G$, where o is some symbol different from all elements of the set G and $\oplus$ denotes Birkhoff's ordinal operation of addition (see [1]).

Remark. The assumption on atomicity cannot be omitted in the assumptions of the assertion 3.2. If, namely, $\mathfrak{N}(G)$ is a complete upper semi-lattice, then it possesses the largest element and from 2.8 it follows that G' is atomic.

Definition 3.1. Let $M \subseteqq G$. Let us put $a\varrho b$ for $a, b \in M$ if there exist an integer n and $x_i \in M$, $z_i \in G$ for $1 \leqq i \leqq n$ such that $z_i \leqq x_i$, $z_i \leqq x_{i+1}$, where $x_1 = a$, $x_{n+1} = b$. The relation ϱ is an equivalence. The decomposition on the set M corresponding to this equivalence will be called the ϱ-decomposition on the set M. If G is an upper semi-lattice and card $M < \aleph_0$, the set $\varrho(M)$ will stand for the set $\bigcup\{\bigvee m(m \in R)\}$ $(R \in \mathscr{R})$, where $\mathscr{R}$ stands for the system of all classes of the ϱ-decomposition on M. (For $M = \emptyset$ we have $\varrho(\emptyset) = \emptyset$). For an integer n we define recurrently $\varrho_n(M) = \varrho(\varrho_{n-1}(M))$, where $\varrho_0(M) = M$.

3.3. *Let G be an upper semi-lattice;* $N_1, N_2 \in \mathfrak{N}(G)$, card N_1 + card $N_2 < \aleph_0$. *Then there exists a non-negative integer l such that* $\varrho_l(N_1 \cup N_2) = \inf(N_1, N_2)$.

Proof. Put $M = N_1 \cup N_2$. In the case that the set $\varrho_k(M)$ fails to have the property (h) for a nonnegative integer k, then we have card $\varrho_k(M) >$ card $\varrho_{k+1}(M)$. Hence, from this it follows that a nonnegative integer l exists such that $\varrho_l(M)$ has the property (h). From 2.6 it follows that $\varrho_l(M) \in \mathfrak{N}(G)$, $\varrho_l(M) \lesssim N_1$, $\varrho_l(M) \lesssim N_2$.

Let $N \in \mathfrak{N}(G)$, $N \lesssim N_1$, $N \lesssim N_2$, $r \in \varrho_l(M)$. Let us put for a nonnegative integer n, $\varrho_n(M) \cap \mathrm{A}(r) = M_n$. For any $x \in M_n$(n nonnegative integer), $y \in N$ exists such that $x \leqq y$. For, if this were not the case, then the least nonnegative integer m existed such that this assertion would fails to hold. Since $N \lesssim N_1$, $N \lesssim N_2$, we have $m > 0$. Let $x \in M_m$. Then $x = \bigvee t(t \in T_x)$ where T_x is a class of the ϱ-decomposition of the set $\varrho_{m-1}(M)$ which is also a class of the ϱ-decomposition of the set M_{m-1}. According to the assumption, for any $t \in T_x$, there exists, notwithstanding, $y_t \in N$ such that $y_t \geqq t$. Evidently for $t, t' \in T_x$ we have $y_t = y_{t'}$. Thus, $y_t \geqq x$ for any $t \in T_x$ which is a contradiction.

Thus, $x_0 \in N$ exists such that $x_0 \geqq r$. Consequently $N \lesssim \varrho_l(M)$ from which $\varrho_l(M) = \inf(N_1, N_2)$ follows.

3.4. *Let $M_1, M_2 \subseteqq G$ have the property (h) in G and let T be a class of the ϱ-decomposition of the set $M_1 \cup M_2$. Then* card $T \leqq \mathrm{r}(G) + 1$.

Proof. Let $t_0 \in T$. For $t \in T$ there exists an integer $n > 0$ such that we have $t_i \in T$, $x_i \in G$ for $1 \leqq i \leqq n$ and $x_i \leqq t_{i-1}$, $x_i \leqq t_i$, where $t_n = t$. Let us denote $\mathrm{d}(t)$ such a least integer n for $t \neq t_0$, and for $t = t_0$ let us put $\mathrm{d}(t) = 0$.

Let $t \in T - (t_0)$. Then the set $\{t' \mid t' \in T, \mathrm{d}(t') = \mathrm{d}(t) - 1, g \in G$ exists such that $g \leqq t', g \leqq t\}$ is nonempty. Let us choose some element $\mathrm{a}(t)$ from this set. The set $\{g \mid g \in G, g \leqq \mathrm{a}(t), g \leqq t\}$ is nonempty and let us choose some element $\mathrm{b}(t)$ $(= \mathrm{b}_a(t))$ from this set and put $B = \mathrm{b}^1(T - (t_0))$ $(= \{\mathrm{b}(x) \mid x \in T - (t_0)\})$.

Put $\varphi(t_1, t_2) = 1$ for $t_1, t_2 \in M_1 \cup M_2$ if $t_1, t_2 \in M_1$ or $t_1, t_2 \in M_2$, $\varphi(t_1, t_2) = -1$ in the opposite case.

Since the sets M_1, M_2 have the property (h) in G we have $\{y \mid y \geqq \mathrm{b}(t)\} \cap T = \{\mathrm{a}(t), t\}$ for $t \in T - (t_0)$, and $\varphi(\mathrm{a}(t), t) = -1$. Hence, it follows that for $\mathrm{a}(t) \neq t_0$ we have $\varphi(a[\mathrm{a}(t)], t) = 1$; consequently $\mathrm{b}[\mathrm{a}(t)] \neq \mathrm{b}(t)$, because $\mathrm{a}[\mathrm{a}(t)] \neq t$. Thus, b is a one-to-one mapping of the set $T - (t_0)$ on B.

If there existed different elements $t_1, t_2 \in T - (t_0)$ and $g \in G$ such that $g \leqq \mathrm{b}(t_1)$, $g \leqq \mathrm{b}(t_2)$, then we should have $g \leqq t_1$, $g \leqq t_2$, $g \leqq \mathrm{a}(t_1)$, $g \leqq \mathrm{a}(t_2)$ from whence $\varphi(t_1, t_2) = -1$. Since $\varphi(\mathrm{a}(t_1), t_1) = \varphi(\mathrm{a}(t_2), t_2) = -1$, we have $\varphi(\mathrm{a}(t_2), t_1) = \varphi(\mathrm{a}(t_1), t_2) = 1$; from this $t_1 = \mathrm{a}(t_2)$, $t_2 = \mathrm{a}(t_1)$ follows. Thus $t_1 = \mathrm{a}[a(t_1)]$ which is a contradiction. Thus, the set B has the property (h) and consequently $\operatorname{card} B \leqq \mathrm{r}(G)$; from this, furthermore, the inequality $\operatorname{card} T \leqq \mathrm{r}(G) + 1$ follows.

4. SYSTEM OF LAYERS $\mathfrak{N}(G)$ FOR A DISTRIBUTIVE LATTICE G'

In this Section G' [3]) is a distributive lattice.

4.1. *Let $N \in \mathfrak{N}(G)$, $a, b \in N$. Then for $c \leqq a \vee b$, $c \in N$ we have either $c = a$ or $c = b$.*

Proof. If $a \neq c \neq b$, then $a \wedge c = b \wedge c = o$. Thus $c = c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) = o$, which is a contradiction.

4.2. *Let $N_1 \in \mathfrak{N}(G)$. Then, the following assertions are equivalent:*

(A) $N_2 \in \mathfrak{N}(G)$, $N_2 \ll N_1$,

(B) $N_2 = (N_1 - \{a, b\}) \cup (a \vee b)$ *where* $a, b \in N_1$, $a \neq b$ *or* $N_2 = (N_1 - (c)) \cup (d)$ *where* $d \in G$, $d \gg c$, $c \in N_1$ *and* $(c) \in \mathfrak{N}(\mathrm{A}(d))$.

Proof. I. Let (B) be valid. If $N_2 = (N_1 - (c)) \cup (d)$, where $d \in G$, $d \gg c$, $c \in N_1$, $(c) \in \mathfrak{N}(\mathrm{A}(d))$, then $\mathrm{A}(d) \cap N_1 = (c)$ and for $x \in G - N_1$, $x < d$ we have $\mathrm{A}(x) \cap N_1 = \emptyset$. According to 2.2 we have $\mathrm{A}(x) \cap N_1 \notin \mathfrak{N}(\mathrm{A}(x))$ and from 2.11 $N_2 \in \mathfrak{N}(G)$, $N_2 \ll N_1$ follows.

Let us suppose that $N_2 = (N_1 - \{a, b\}) \cup (x_0)$ where $a, b \in N_1$, $a \neq b$, $x_0 = a \vee b$. Since $a \neq b$, we have $x_0 \in G - N_1$. According to 4.1, $\mathrm{A}(x_0) \cap N_1 = \{a, b\}$. If there exists $c \in \mathrm{A}(x_0)$ such that the set $\{a, b, c\}$ has the property (h) in the set $\mathrm{A}(x_0)$, then it has the property (h) in the set G as well, and by 2.1, $N \in \mathfrak{N}(G)$, $N \supseteq \{a, b, c\}$ exists. According to 4.1 we have $c = a$ or $c = b$. Thus, $\mathrm{A}(x_0) \cap N_1 \in \mathfrak{N}(\mathrm{A}(x_0))$.

Let $x \in G - N_1$, $x < x_0$. Then we have $\mathrm{A}(x) \cap N_1 \subseteq \mathrm{A}(x_0) \cap N_1 = \{a, b\}$. Having $\mathrm{A}(x) \cap N_1 = \emptyset$, then, by 2.2, $\mathrm{A}(x) \cap N_1 \notin \mathfrak{N}(\mathrm{A}(x))$. If $\mathrm{A}(x) \cap N_1 = \{a, b\}$, then $x_0 = a \vee b \leqq x$ which is a contradiction with the supposition. If $\mathrm{A}(x) \cap N_1 = (a)$, then $x = x_0 \wedge x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = a \vee (b \wedge x)$. In the case of validity of $b \wedge x = o$ we would have $x = a \in N_1$, which is a contradiction with the assumption. Consequently $b \wedge x > o$. As $a \wedge b = o$, the set $\{a, b \wedge x\} \subseteq$

$\subseteqq A(x)$ is two-element and has the property (h) in $A(x)$. Thus, $(a) \notin \mathfrak{N}(A(x))$. If $A(x) \cap N_1 = (b)$, then it can be shown, in the same way, that $(b) \notin \mathfrak{N}(A(x))$. Consequently $A(x) \cap N_1 \notin \mathfrak{N}(A(x))$.

The assertion (A) follows from 2.11.

II. Let (A) hold. By 2.11 $x_0 \in G - N_1$ exists such that $N_2 = (N_1 - A(x_0)) \cup (x_0)$. If $A(x_0) \cap N_1 \supseteqq \{a, b\}$, where $a \neq b$, then according to I we have $N = (N_1 - \{a, b\}) \cup (a \vee b) \in \mathfrak{N}(G)$, $N \lessdot N_1$. Since $a \vee b \leqq x_0$, then $N \gtrsim N_2$. Thus $N_2 = N$.

If $A(x_0) \cap N_1 = (c)$, then by 2.11 $(c) \in \mathfrak{N}(A(x_0))$. For $x \in G$, $c < x < x_0$ we have according to 2.11, $(c) = A(x) \cap N_1 \notin \mathfrak{N}(A(x))$; thus $y \in A(x)$ exists such that $y \wedge c = o$. As $y \in A(x_0)$ is valid, too, $(c) \notin \mathfrak{N}(A(x_0))$ which is a contradiction. Thus $x_0 \gg c$.

Since $N_2 \lesssim N_1$, we have, according to 2.5, $A(x_0) \cap N_1 = \emptyset$, consequently (B) holds.

4.3. *For* $1 \leqq i \leqq 4$ *let be* $N_i \in \mathfrak{N}(G)$, $\sup(N_2, N_3) = N_1$, $\inf(N_2, N_3) = N_4$, $N_1 \gg N_2$. *Then* $N_3 \gg N_4$.

Proof. I. Let $N_2 = (N_1 - \{a, b\}) \cup (c)$ where $a, b \in N_1$, $a \neq b$, $c = a \vee b$. Since $N_1 \gtrsim N_3$, $d, e \in N_3$ exist such that $d \geqq a$, $e \geqq b$. If $d = e$, then $c \leqq d$, and consequently, $N_2 \gtrsim N_3$, which is impossible. Thus $d \neq e$.

Let us put $N = (N_3 - \{d, e\}) \cup (d \vee e)$. According to 4.2 we have $N \in \mathfrak{N}(G)$. $N \lessdot N_3$. We have $c \wedge (d \vee e) = (c \wedge d) \vee (c \wedge e) \geqq a \vee b = c$, thus $c \leqq d \vee e$, From this it follows that $N \lesssim N_2$.

II. Let $N_2 = (N_1 - (c)) \cup (d)$, where $d \in G$, $d \gg c$, $c \in N_1$ and $(c) \in \mathfrak{N}(A(d))$. Since $N_3 \lesssim N_1$, $e \in N_3$ exists such that $e \geqq c$. If $f \in A(d \vee e) f \wedge e = o$ exists, we have from $f \vee d \leqq d \vee e$, $f \vee d = (f \vee d) \wedge (d \vee e) = (f \wedge d) \vee d \vee (f \wedge e) \vee (d \wedge e) = d$; thus $f \leqq d$. Since $(c) \in \mathfrak{N}(A(d))$, there exists, according to 2.3, $g' \in A(d)$, $g' \neq o$ such that $g' \leqq c$, $g' \leqq f$. Then $g' \leqq e$ as well, which is a contradiction to $f \wedge e = o$. Consequently $(e) \in \mathfrak{N}(A(d \vee e))$.

Let $x \in G$, $e \leqq x \leqq d \vee e$. Then $c \leqq d \wedge e \leqq d \wedge x \leqq d \wedge (d \vee e) = d$. As $d \gg c$, we have either $c = d \wedge x$ or $d = d \wedge x$. If $c = d \wedge x$, then $e = c \vee e = (d \wedge x) \vee e = (e \vee d) \wedge (e \vee x) = (e \vee d) \wedge x = x$. If $d = d \wedge x$, we have $x \geqq d$; thus $x \geqq d \vee e$, from whence $x = d \vee e$ follows. If $e = d \vee e$, then $d \leqq e$ and consequently $N_2 \gtrsim N_3$. Then $\sup(N_2, N_3) = N_2 \neq N_1$ which is a contradiction. Thus $e < d \vee e$, and consequently $e \lessdot d \vee e$.

According to 4.2, $N = (N_3 - (e)) \cup (e \vee d) \in \mathfrak{N}(G)$, $N \lessdot N_3$. Evidently we have $N \lesssim N_2$, too.

III. By 4.2, I and II, $N \in \mathfrak{N}(G)$, $N \lessdot N_3$ and $N \lesssim N_2$ exists. From this $N \lesssim N_4$ follows and consequently $N = N_4$ or $N_4 = N_3$. If $N_4 = N_3$, then $N_2 \gtrsim N_3$, and consequently $N_1 = N_2$, which is impossible. Thus $N_4 \lessdot N_3$.

The assertion is proved.

4.4. *Let* $M = \{m \mid m \in G$, *m not being the largest element in* G, $m \wedge x > o$ *for every element* $x \in G\}$. *The following assertions are equivalent*:

(A) *For any set* $N \in \mathfrak{N}(G)$ *which is not the least element in* $\mathfrak{N}(G)$, *there exists* $N' \in \mathfrak{N}(G)$ *such that* $N' \ll N$.

(B) *For any element* $m \in M$, $m' \in G$ *exists such that* $m' \gg m$.

Proof. I. Let (A) hold and let $m \in M$. Then $(m) \in \mathfrak{N}(G)$ and, according to 2.7, (m) is not the least element in $\mathfrak{N}(G)$. For this reason $N' \in \mathfrak{N}(G)$, $N' \ll (m)$ exists. From 4.2 it follows that $N' = (m')$, where $m' \in G$, $m' \gg m$.

II. Let (B) hold and let $N \in \mathfrak{N}(G)$, N not being the least element in $\mathfrak{N}(G)$. If card $N \geqq 2$, then by 4.2, $N' \in \mathfrak{N}(G)$, $N' \ll N$ exist. If card $N = 1$, then $N = (m)$, where $m \in M$, according to 2.7. Thus $m' \in G$, $m' \gg m$ exists. From 4.2 it follows that $(m') \in \mathfrak{N}(G)$, $(m') \ll (m)$.

4.5. *For* $N_1, N_2 \in \mathfrak{N}(G)$, $\operatorname{card} N_1 + \operatorname{card} N_2 < \aleph_0$ *we have* $\varrho(N_1 \cup N_2) \in \mathfrak{N}(G)$, $\varrho(N_1 \cup N_2) = \inf(N_1, N_2)$. *We have* $\bigvee k(k \in K) \neq \bigvee k'(k' \in K')$ *for different classes* K, K' *of the* ϱ-*decomposition of the set* $N_1 \cup N_2$.

Proof. If K, K' are two different classes of the ϱ-decomposition of the set $N_1 \cup N_2$, then $\bigvee k(k \in K) \wedge \bigvee k'(k' \in K') = o$, because we have $k \wedge k' = o$ for $k \in K$, $k' \in K'$. Consequently, $\varrho(N_1 \cup N_2)$ has the property (h) in G and the first assertion follows from 3.3.

4.6. *Let* $N_i \in \mathfrak{N}(G)$ *for* $1 \leqq i \leqq 4$, $N_1 = \sup(N_2, N_3)$, $N_4 = \inf(N_2, N_3)$, $\operatorname{card} N_2 + \operatorname{card} N_3 < \aleph_0$. *Then we have* $v_1 = v_2 \wedge v_3$, $v_4 = v_2 \vee v_3$, *where* $v_i = \bigvee n(n \in N_i)$ *for* $1 \leqq i \leqq 4$.

Proof. From 3.1 $v_1 = v_2 \wedge v_3$ follows. Given $n_2 \in N_2$, $n_4 \in N_4$ exist such that $n_2 \leqq n_4$. From this it follows that $v_4 \geqq v_2$. In the same way, it turns out that $v_4 \geqq v_3$. From 2.6 it follows that $(v_2 \vee v_3) \in \mathfrak{N}(G)$ and $(v_2 \vee v_3) \leqq N_2$, $(v_2 \vee v_3) \leqq N_3$. Thus $(v_2 \vee v_3) \leqq N_4$ from which $v_2 \vee v_3 \geqq v_4$ follows. Consequently $v_4 = v_2 \vee v_3$.

4.7. $\mathfrak{N}(G)$ *is a distributive lattice if and only if* $\mathrm{r}(G) \leqq 2$; $\mathfrak{N}(G)$ *is a modular lattice if and only if* $\mathrm{r}(G) \leqq 3$.

Proof. I. Let $\mathrm{r}(G) \leqq 2$. According to 3.1 and 3.3, $\mathfrak{N}(G)$ is a lattice. Let $A, B, C \in \mathfrak{N}(G)$, $S_1 = \sup\{A, \inf(B, C)\}$, $S_2 = \inf\{\sup(A, B), \sup(A, C)\}$. For proving that $\mathfrak{N}(G)$ is a distributive lattice it is sufficient to show that $S_1 \geqq S_2$.

According to 4.5 we have $S_2 = \varrho(\sup(A, B) \cup \sup(A, C))$. Let $n \in S_1$. Then, by 3.1, we have $n = a \wedge d$, where $a \in A$, $d \in \inf(B, C)$. According to 4.5, $d = \bigvee t(t \in T)$, where T is a class of the ϱ-decomposition of the set $B \cup C$. By 3.4 we have card $T \leqq 3$. From 2.3 it follows that $T \cap B \neq \emptyset \neq T \cap C$. Thus we can assume that $T = \{b_1, b_2, c\}$, where $b_1, b_2 \in B$ and $c \in C$. Consequently $n = (a \wedge b_1) \vee (a \wedge b_2) \vee (a \wedge c) > o$.

a) Let $a \wedge b_1 = a \wedge b_2 = o$. Then $n = a \wedge c \in \sup(A, C)$, according to 3.1. Consequently, $n' \in S_2$ exists such that $n' \geqq n$.

b) Let us assume that $a \wedge b_1 > o$.

α) Let $a \wedge c = o$. If $a \wedge b_1 \geqq a \wedge b_2$, then $n = a \wedge b_1 \in \sup(A, B)$ according to 3.1. Consequently, $n' \in S_2$ exists such that $n' \geqq n$. If $a \wedge b_1$ non $\geqq a \wedge b_2$, then $a \wedge b_2 > o$ and $(a \wedge b_1) \wedge (a \wedge b_2) = o$, because $b_1 \neq b_2$ and consequently, $b_1 \wedge b_2 = o$. By 2.3, $a' \in A$ exists such that $a' \wedge c > o$. Hence it follows that the set $\{a \wedge b_1, a \wedge b_2, a' \wedge c\} \subseteqq G$ is a three element set and possesses the property (h) in G which is a contradiction.

β) Let $a \wedge c > o$. Since b_1 and c belong to the same class $T = \{b_1, b_2, c\}$ of the ϱ-decomposition of the set $B \cup C$ and $b_1 = b_2$ or $b_1 \wedge b_2 = o$, we have $b_1 \wedge c > o$. If $a \wedge b_1 \wedge c = o$, then according to 2,3, there exists $a' \in A$ such that $a' \wedge b_1 \wedge c > > o$. Evidently $a \neq a'$, thus $a \wedge a' = o$. Then the set $\{a \wedge b_1, a \wedge c, a' \wedge c\} \subseteqq G$ is a three element one and has the property (h) in G which is a contradiction. Thus $a \wedge b_1 \wedge c > o$. From this it follows that $a \wedge b_1$, $a \wedge c$ belong to the same class of the ϱ-decomposition of the set $\sup(A, B) \cup \sup(A, C)$, because by 3.1 we have $a \wedge b_1 \in \sup(A, B)$, $a \wedge c \in \sup(A, C)$. Consequently, $n' \in S_2$ exists such that $n' \geqq a \wedge b_1$, $n' \geqq a \wedge c$.

If $a \wedge b_2 > o$, then it can be shown in the same way that $n'' \in S_2$ exists such that $n'' \geqq a \wedge b_2$, $n'' \geqq a \wedge c$. Since $n' \geqq a \wedge c$, $n'' \geqq a \wedge c$, we have $n' = n''$; thus $n \leqq n'$.

If $a \wedge b_2 = o$, then $n = (a \wedge b_1) \vee (a \wedge c) \leqq n'$.

In this way it has been shown that $S_1 \geqq S_2$.

II. Let $\mathrm{r}(G) \leqq 3$. Let us assume that $N_i \in \mathfrak{N}(G)$, $1 \leqq i \leqq 5$ exist such that $N_1 \succ N_3 \succ N_4 \succ N_5$, $\sup(N_2, N_3) = \sup(N_2, N_4) = N_1$, $\inf(N_2, N_3) = \inf(N_2, N_4) = = N_5$. Let us put $N_i = \{n_1^i, n_2^i, n_3^i\}$, where $n_1^i, n_2^i, n_3^i \in G$, $1 \leqq i \leqq 5$. We can suppose that $n_j^1 \leqq n_j^2 \leqq n_j^5$, $n_j^1 \leqq n_j^3 \leqq n_j^4 \leqq n_j^5$ for $1 \leqq j \leqq 3$. Let us put $\sigma_i = \bigvee_{j=1}^{3} n_j^i$ for $1 \leqq i \leqq 5$. According to 4.6 we have $\sigma_2 \wedge \sigma_3 = \sigma_2 \wedge \sigma_4 = \sigma_1$, $\sigma_2 \vee \sigma_3 = \sigma_2 \vee \sigma_4 = \sigma_5$. Furthermore $\sigma_4 = \sigma_4 \wedge \sigma_5 = \sigma_4 \wedge (\sigma_2 \vee \sigma_3) = (\sigma_4 \wedge \wedge \sigma_2) \vee (\sigma_4 \wedge \sigma_3) = \sigma_1 \vee \sigma_3 = \sigma_3$. Thus $\sigma_4 = \sigma_3$.

Let $\operatorname{card} N_4 \geqq 2$. Since $N_3 \succ N_4$, we can suppose that $n_1^3 < n_1^4$, $n_1^4 \neq n_2^4$. If $n_1^4 \neq n_3^4$, we have $n_1^3 = n_1^4 \wedge (n_1^3 \vee n_2^4 \vee n_3^4) = n_1^4 \wedge \sigma_4 = n_1^4$, which is a contradiction. Thus $n_1^4 = n_3^4$. From this $n_1^5 = n_3^5$ follows. If $n_1^3 = n_3^3$, then $n_1^3 = n_1^4 \wedge (n_1^3 \vee \vee n_3^3 \vee n_2^4) = n_1^4 \wedge \sigma_4 = n_1^4$, which is a contradiction. Consequently $n_1^3 \neq n_3^3$. According to 3.1, $n_1^1 = n_1^2 \wedge n_1^4$ and $n_3^1 = n_3^2 \wedge n_3^4$. If it were $n_1^2 = n_3^2$, then $n_1^1 = = n_3^2 \wedge n_3^4 = n_3^1$, which is impossible because $n_1^1 \leqq n_1^3$, $n_3^1 \leqq n_3^3$ and $n_1^3 \neq n_3^3$. Consequently $n_1^2 \neq n_3^2$. Since $n_1^4 = n_3^4 \neq n_2^4$, we get n_1^3 non $\geqq n_2^1$, n_3^3 non $\geqq n_2^1$ and $n_1^3 \wedge n_3^3 = n_3^3 \wedge n_2^3 = o$. According to 3.1, $n_k^2 \wedge n_l^3 = o$ or $n_k^2 \wedge n_l^3 \in N_1$ for $1 \leqq k, l \leqq 3$. Thus, $n_1^3 \wedge n_3^3 = n_1^3 \wedge n_2^3 = n_2^3 \wedge n_3^3 = n_1^2 \wedge n_3^2 = n_1^3 \wedge n_3^2 = n_3^3 \wedge \wedge n_1^2 = o$ and $n_2^3 \wedge n_i^2 > o$ holds if and only if $n_i^2 = n_2^2$ $(i = 1, 2, 3)$, and $n_j^3 \wedge$

$\wedge\ n_2^2 > o$ if and only if $n_2^2 = n_j^2$ ($j = 1, 3$). From this it is easy to see that the sets $\{n_1^2, n_1^3\}$ and $\{n_3^2, n_3^3\}$ lie in different classes T_1 and T_3 of the ϱ-decomposition of the set $N_2 \cup N_3$. According to 4.5, $n_1^5 = \bigvee t(t \in T_1) \neq \bigvee t(t \in T_3) = n_3^5$, which is a contradiction.

Consequently, card $N_4 = 1$, so that $n_1^4 = n_2^4 = n_3^4 = \sigma_4$, $n_1^5 = n_2^5 = n_3^5 = \sigma_5$. Since $\sigma_4 = \sigma_3$, we have card $N_3 \geqq 2$. We can suppose that $n_2^3 \neq n_1^3 \neq n_3^3$. Then $n_2^1 \neq n_1^1 \neq n_3^1$ is valid. According to 3.1, $n_1^2 \wedge \sigma_4 = n_1^1$, $n_2^2 \wedge \sigma_4 = n_2^1$ and $n_3^2 \wedge \sigma_4 = n_3^1$. Thus $n_2^2 \neq n_1^2 \neq n_3^2$. Consequently $n_k^2 \wedge n_l^3 = o$ or $n_k^2 \wedge n_l^3 \in N_1$ holds for $1 \leqq k$, $1 \leqq 3$ by 3.1. From this it is easy to see that the sets $\{n_1^2, n_1^3\}$ and $\{n_2^2, n_2^3\}$ lie in different classes T_1 and T_2 of the ϱ-decomposition of the set $N_2 \cup N_3$. According to 4.5, $n_1^5 = \bigvee t(t \in T_1) \neq \bigvee t(t \in T_2) = n_2^5$ which is a contradiction.

From this, from 3.1 and 3.3, it follows that $\mathfrak{N}(G)$ is a modular lattice.

III. Let $r(G) \geqq 3$. Then there exists a set $N_1 \in \mathfrak{N}(G)$ which contains mutually different elements a, b, c. Let us put $N_2 = (N_1 - \{a, b\}) \cup (a \vee b)$, $N_3 = (N_1 - \{a, c\}) \cup (a \vee c)$, $N_4 = (N_1 - \{b, c\}) \cup (b \vee c)$, $N_5 = (N_4 - \{a, b \vee c\}) \cup (a \vee b \vee c)$. By 4.2, $N_i \in \mathfrak{N}(G)$ for $2 \leqq i \leqq 5$. Evidently, $N_5 \precsim N_2$, $N_5 \precsim N_3$. Let $N \precsim N_2$, $N \precsim N_3$. Then $n_1, n_2 \in N$ exist such that $n_1 \geqq a \vee b$, $n_2 \geqq a \vee c$. Since $n_1 \geqq a$, $n_2 \geqq a$, we have $n_1 = n_2$, from which $n_1 \geqq a \vee b \vee c$ follows; consequently, $N \precsim N_5$ which means that $N_5 = \inf(N_2, N_3)$. From 3.1 we get that $\sup(N_2, N_4) = \sup(N_3, N_4) = N_1$. Since $N_4 \succsim N_5$, we have $\sup\{N_4, \inf(N_2, N_3)\} = N_4$ whereas $\inf\{\sup(N_2, N_4), \sup(N_3, N_4)\} = N_1$. Consequently, $\mathfrak{N}(G)$ fails to be a distributive lattice.

IV. Let $r(G) \geqq 4$. Then there exists a set $N_1 \in \mathfrak{N}(G)$ which contains mutually different elements a, b, c, d. Let us put $N_2 = (N_1 - \{a, b\}) \cup (a \vee b)$, $N_3 = (N_2 - \{c, d\}) \cup (c \vee d)$, $N_4 = (N_1 - \{a, c\}) \cup (a \vee c)$, $N_5 = (N_4 - \{b, d\}) \cup (b \vee d)$, $N_6 = (N_3 - \{a \vee b, c \vee d\}) \cup (a \vee b \vee c \vee d) = (N_1 - \{a, b, c, d\}) \cup (a \vee b \vee c \vee d)$. By 4.2, $N_i \in \mathfrak{N}(G)$ for $2 \leqq i \leqq 6$, and evidently, $N_6 \precsim N_2$, $N_6 \precsim N_5$. Let $N \precsim N_2$, $N \precsim N_5$. Then $n_1, n_2, n_3 \in N$ exist such that $n_1 \geqq a \vee b$, $n_2 \geqq a \vee c$, $n_3 \geqq b \vee d$. Since $n_1 \geqq a$, $n_1 \geqq b$, $n_2 \geqq a$, $n_3 \geqq b$, we have $n_1 = n_2 = n_3$, from whence $n_1 \geqq a \vee b \vee c \vee d$ follows; consequently, $N \precsim N_6$, which means that $N_6 = \inf(N_2, N_5)$. According to 3.1, $\sup(N_3, N_5) = N_1$. We have $N_1 \succsim N_2 \succ N_3 \succsim N_6$. As $\sup\{N_3, \inf(N_2, N_5)\} = N_3 \neq N_2 = \inf\{\sup(N_3, N_5), N_2\}$, $\mathfrak{N}(G)$ is not a modular lattice.

Thus the assertion is proved.

Remark. If G is the set of all non-void subsets of a set $P \neq \emptyset$ ordered by means of inclusion (see example 2.1), then the assertion 4.7 is known and can be proved without difficulties.

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