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ON THE WELL DIMENSION OF ORDERED SETS

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1. INTRODUCTION

1.1. Notation. If G is a set then card G denotes the cardinality of G. If G is a linearly ordered set then $\bar{G}$ denotes the order type of G. A set G will be called non-trivial if card $G \geqq 2$; in the whole paper, all sets are assumed to be non-trivial and all types of ordered, resp. linearly ordered sets are assumed to be types of non-trivial sets. The identity of ordered sets will be denoted $=$, the isomorphism $\cong$. A linearly ordered set will be called a chain, a set in which every two distinct elements are incomparable will be called an antichain. For the operations with ordered sets we shall use the BIRKHOFF's notation ([1] or [2]) so that $G + H$, $G . H$, G^H denotes the cardinal sum, product and power whereas $G \oplus H$, $G \circ H$, ${}^H G$ denotes corresponding ordinal operations.

1.2. Lexicographic sum. Let H be an ordered set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. *Lexicographic sum* $\sum_{\alpha \in H} G_\alpha$ ([3]) is a set of all ordered pairs $[\alpha, x]$, where $\alpha \in H$, $x \in G_\alpha$, ordered in the following way: $[\alpha_1, x_1] \leqq [\alpha_2, x_2]$ if and only if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$, $x_1 \leqq x_2$. It is well known that this operation is a generalization of the Birkhoff's ordinal sum, cardinal sum and ordinal product for, if we choose $H = \{0, 1 \mid 0 < 1\}$ as a two-point chain, then $\sum_{\alpha \in H} G_\alpha$ is isomorphic with $G_0 \oplus G_1$; if we choose $H = \{0, 1 \mid 0 \parallel 1\}$ as a two-point antichain then $\sum_{\alpha \in H} G_\alpha$ is isomorphic with $G_0 + G_1$ and if we choose $G_\alpha = G$ for every $\alpha \in H$ then $\sum_{\alpha \in H} G_\alpha$ is identical with $H \circ G$.

1.3. Cardinal product. Let H be a set, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. *Cardinal product* $\prod_{\alpha \in H} G_\alpha$ is a set of all functions f defined on H and such that $f(\alpha) \in G_\alpha$, for every $\alpha \in H$, ordered in the following way: $f \leqq g$ if and only if $f(\alpha) \leqq g(\alpha)$ for every $\alpha \in H$. This operation is a generalization of the Birkhoff's cardinal product for, if we choose $H = \{0, 1\}$ as two-point set, then $\prod_{\alpha \in H} G_\alpha$ is isomorphic with $G_0 . G_1$.

For this reason, if $H = \{0, 1, \ldots, n\}$ is a finite set, we denote $\prod_{\alpha \in H} G_\alpha$ conventionally $G_0 . G_1 \ldots G_n$. If $G_\alpha = G$ for every $\alpha \in H$ then $\prod_{\alpha \in H} G_\alpha$ is identical with G^H in the case that H is ordered as an antichain.

1.4. Linear extension. Let a set of orders $\{\leqq_\alpha \mid \alpha \in H\}$ be given on the set G. If we assume these orders to be subsets of the cartesian square G^2 we can apply various set-theoretical operations to them. Especially it is easy to see that the intersection $\bigcap_{\alpha \in H} \leqq_\alpha = \leqq$ is again an order on G. This order is defined in the following way: $x \leqq y \Leftrightarrow x \leqq_\alpha y$ for every $\alpha \in H$. If $\leqq$ is an order on G and if $\precsim$ is a linear order on G such that $\leqq \subseteq \precsim$ (i.e. $x, y \in G$, $x \leqq y \Rightarrow x \precsim y$) we say that $\precsim$ is a *linear extension* of $\leqq$. In [11] E. SZPILRAJN has proved that any order $\leqq$ on G has at least one linear extension $\precsim$. He has proved the stronger result: Let $\leqq$ be an order on G and let x, y be elements of G such that $x \parallel y$. Then there exist two linear extensions $\precsim_1$, $\precsim_2$ of $\leqq$ such that $x \precsim_1 y$, $y \precsim_2 x$. From this it follows that the intersection of all linear extensions of $\leqq$ is $\leqq$.

1.5. Dimension. Let G be a set, let $\leqq$ be an order on G. From the Szpilrajn's theorem it follows, on G there exist systems of linear orders intersection of which is $\leqq$. Such systems are called *realizers* of $\leqq$ and if $\{\precsim_\alpha \mid \alpha \in H\}$ is a realizer of $\leqq$ we say that the orders $\precsim_\alpha$ *realize* $\leqq$. B. DUSHNIK and E. W. MILLER ([4]) call the *dimension* of the set G and denote dim G the smallest cardinality of the system of linear orders on G, which realizes $\leqq$. A linear extension of an ordered set G can be also defined as a one-one isotone mapping of G into a chain H. From this there follows that the dimension of G can be defined as the minimum of cardinalities of systems $\{f_\varkappa \mid \varkappa \in K\}$ (where $f_\varkappa$ is a one-one isotone mapping of G into a chain $L_\varkappa$ for every $\varkappa \in K$) such that $x, y \in G$, $x \leqq y \Leftrightarrow f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$. If every chain $L_\varkappa$ has the same order type α and if there exists at least one system $\{f_\varkappa \mid \varkappa \in K\}$ where $f_\varkappa$ is a one-one isotone mapping of G into $L_\varkappa$ with the property $x, y \in G$, $x \leqq y \Leftrightarrow f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$, then the minimum of cardinalities of such systems is called *α-dimension* of G and denoted α-dim G (H. KOMM [7]). Let G be an ordered set, L a chain of type α. In [9] there is proved that there exists a system $\{f_\varkappa \mid \varkappa \in K\}$ where $f_\varkappa$ is an isotone (not necessarily one-one isotone) mapping of G into L such that $x, y \in G$, $x \leqq y \Leftrightarrow f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$. The minimum of cardinalities of such systems is called *α-pseudodimension* of G and denoted α-pdim G. Properties of the characteristics dim G, α-dim G, α-pdim G are studied in [4], [5], [6], [7], [8], [9], [10].

2. WELL REALIZER AND PSEUDOREALIZER

2.1. Definition. Let G be an ordered set. We say that G satisfies the descending chain condition if $x_0, x_1, \ldots, x_n, \ldots \in G$, $x_0 \geqq x_1 \geqq \ldots \geqq x_n \geqq \ldots$ implies the existence of a positive integer n_0 such that $x_{n_0} = x_{n_0+1} = \ldots$

2.2. Definition. Let G be an ordered set, let H be a well-ordered set. A one-one isotone mapping φ of G into H is called a well extension of G.

2.3. Theorem. *Let G be an ordered set. Then G has a well extension if and only if G satisfies the descending chain condition.*

Proof. The necessity of this condition is clear. We shall prove its sufficiency. Hence let G – ordered by the relation $\leqq$ – satisfy the descending chain condition. Let G_0 be the set of all minimal elements in G (the mentioned assumption guarantees the existence of minimal elements in G). Assume that we have defined all sets G_α for every ordinal number $\alpha < \alpha_0$. Then let G_{α_0} denote the set of all minimal elements in $G - \bigcup_{\alpha<\alpha_0} G_\alpha$ (if $G - \bigcup_{\alpha<\alpha_0} G_\alpha$ is non-empty then it satisfies the descending chain condition so that the existence of minimal elements in $G - \bigcup_{\alpha<\alpha_0} G_\alpha$ is guaranteed). Then there exists the smallest ordinal number β such that $G_\beta = \emptyset$ for, if card $G \leqq \aleph_i$, then clearly $G_{\omega_{i+1}} = \emptyset$. Then it holds: $G = \bigcup_{\alpha<\beta} G_\alpha$ where the sets G_α are mutually disjoint and every G_α is an antichain with respect to $\leqq$. Choose any well ordering of G_α for every $\alpha < \beta$ and put $H = \sum_{\alpha<\beta} G_\alpha$. H as a lexicographic sum of well-ordered sets over a well-ordered set is a well-ordered set. Define a mapping φ of G onto H in the following way: $x \in G$, $x \in G_\alpha \Rightarrow \varphi(x) = [\alpha, x]$. φ is clearly a one-one mapping of G onto H. We shall show that φ is isotone. Let $x, y \in G$, $x \leqq y$. Then there exist ordinal numbers $\alpha_1 < \beta$, $\alpha_2 < \beta$ such that $x \in G_{\alpha_1}$, $y \in G_{\alpha_2}$. If it were $\alpha_1 > \alpha_2$ then x would be a minimal element in $G - \bigcup_{\alpha<\alpha_1} G_\alpha$ and $y \in \bigcup_{\alpha<\alpha_1} G_\alpha$ so that $x > y$ or $x \parallel y$ and this is a contradiction. Therefore $\alpha_1 \leqq \alpha_2$ and from this $\varphi(x) = [\alpha_1, x] \leqq [\alpha_2, y] = \varphi(y)$. Hence φ is a well extension of G.

2.4. Definition. Let G be an ordered set, let $\{L_\varkappa \mid \varkappa \in K\}$ be a system of well-ordered sets, let $f_\varkappa$ be a one-one isotone mapping of G into $L_\varkappa$. If $x, y \in G \Rightarrow x \leqq y$ if and only if $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$ then we say that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well realizer of the set G.

2.5. Theorem. *An ordered set G has a well realizer if and only if G satisfies the descending chain condition.*

Proof. The necessity of the mentioned condition follows from 2.3., for every $f_\varkappa$ is a well extension of G. We shall prove its sufficiency. Hence let G satisfy the descending chain condition. If G does not contain any incomparable elements then G is a well-ordered set so that $\{G, g\}$ is a well realizer of G when g is an identical mapping of G onto itself. In the opposite case it suffices to show that for any two incomparable elements $x_1, x_2 \in G$ there exist well-ordered sets L_1, L_2 and one-one isotone mappings f_1, resp. f_2 of G into L_1, resp. L_2 such that $f_1(x_1) < f_1(x_2)$, $f_2(x_1) > f_2(x_2)$. Hence let $x_1, x_2 \in G$, $x_1 \parallel x_2$. Put $G^1 = \{x \mid x \in G, x \leqq x_1\}$, $G^2 = G - G^1$. Both G^1 and G^2

satisfy the descending chain condition, hence according to 2.3. there exist well-ordered sets L^1, L^2 and one-one isotone mappings f^1, resp. f^2 of G^1 into L^1, resp. of G^2 into L^2. Put $L_1 = L^1 \oplus L^2$ and $f_1(x) = f^i(x)$ for $x \in G^i$ $(i = 1, 2)$. Then L_1 is clearly a well-ordered set and f_1 is a one-one isotone mapping of G into L_1 such that $f_1(x_1) < f_1(x_2)$; analogously we can construct a well-ordered set L_2 and a one-one isotone mapping f_2 of G into L_2 such that $f_2(x_1) > f_2(x_2)$.

2.6. Definition. Let G be an ordered set, let $\{L_\varkappa \mid \varkappa \in K\}$ be a system of well-ordered sets, let $f_\varkappa$ be a mapping of G into $L_\varkappa$. If $x, y \in G \Rightarrow x \leqq y$ if and only if $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$ then we say that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well-pseudorealizer of the set G.

2.7. Theorem. *Any ordered set G has a well pseudorealizer.*

Proof. Let G be an ordered set. By K_1 denote the set of all ordered pairs $[x, y]$ where $x, y \in G$, $x < y$, by K_2 the set of all ordered pairs $[x, y]$ where $x, y \in G$, $x \parallel y$. Put $K = K_1 \cup K_2$ and for every $\varkappa \in K$ let $L_\varkappa$ be a two-point chain, i.e. $L_\varkappa = \{0, 1 \mid 0 < 1\}$. Define a mapping $f_\varkappa$ of G into $L_\varkappa$ for every $\varkappa = [x, y]$ in the following way: $f_\varkappa(t) = 0$ if and only if $t \leqq x$. It is easy to see that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of G.

2.8. Theorem. *Let G be an ordered set, let K be a set and $L_\varkappa$ a well-ordered set for every $\varkappa \in K$. Then the following statements are equivalent:*

(A) $G \cong G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$.

(B) *For every $\varkappa \in K$ there exists a mapping $f_\varkappa$ of G into $L_\varkappa$ such that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of G.*

Proof. 1. Assume that (A) holds and let φ be an isomorphism of G onto $G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$. For every $x \in G$ and every $\varkappa \in K$ put $\Phi(x, \varkappa) = [\varphi(x)](\varkappa)$. Then Φ is a mapping of the set $G \times K$ into the set $\bigcup_{\varkappa \in K} L_\varkappa$ with the property $\Phi(x, \varkappa_0) \in L_{\varkappa_0}$. $\Phi(x, \varkappa_0)$ is therefore a mapping of G into $L_{\varkappa_0}$. Put $\Phi(x, \varkappa_0) = f_{\varkappa_0}(x)$. We shall show that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of G. Hence let $x, y \in G$, $x \leqq y$. Then $\varphi(x) \leqq \varphi(y)$ so that $[\varphi(x)](\varkappa) \leqq [\varphi(y)](\varkappa)$ for every $\varkappa \in K$. From this it follows $\Phi(x, \varkappa) \leqq \Phi(y, \varkappa)$ for every $\varkappa \in K$ and hence $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$. Suppose, on the contrary, that $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$. Then $\Phi(x, \varkappa) \leqq \Phi(y, \varkappa)$ for every $\varkappa \in K$, i.e. $[\varphi(x)](\varkappa) \leqq [\varphi(y)](\varkappa)$ for every $\varkappa \in K$ so that $\varphi(x) \leqq \varphi(y)$. As φ is an isomorphism, this implies $x \leqq y$. $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is therefore a well pseudorealizer of G and (B) holds.

2. Assume that (B) holds. Put $\Phi(x, \varkappa) = f_\varkappa(x)$ for every $x \in G$ and every $\varkappa \in K$. Then Φ is a mapping of the set $G \times K$ into the set $\bigcup_{\varkappa \in K} L_\varkappa$ with the property $\Phi(x_0, \varkappa) \in L_\varkappa$. Form the cardinal product $\prod_{\varkappa \in K} L_\varkappa$ and put $\Phi(x_0, \varkappa) = [\varphi(x_0)](\varkappa)$. Then φ is

a mapping of G onto a certain subset $G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$ and we shall show that φ is an isomorphism. Let $x, y \in G$, $x \leqq y$. As $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of G, we have $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$ so that $\Phi(x, \varkappa) \leqq \Phi(y, \varkappa)$ for every $\varkappa \in K$. From this $[\varphi(x)](\varkappa) \leqq [\varphi(y)](\varkappa)$ for every $\varkappa \in K$ and therefore $\varphi(x) \leqq \varphi(y)$. Suppose, on the contrary, that $\varphi(x) \leqq \varphi(y)$. Then $[\varphi(x)](\varkappa) \leqq [\varphi(y)](\varkappa)$ for every $\varkappa \in K$ so that $\Phi(x, \varkappa) \leqq \Phi(y, \varkappa)$ for every $\varkappa \in K$ and hence $f_\varkappa(x) \leqq f_\varkappa(y)$ for every $\varkappa \in K$. As $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of G, this implies $x \leqq y$. Finally it is easy to see that φ is a one-one mapping. φ is therefore an isomorphism and (A) holds.

2.9. Corollary. *Let G be an ordered set, let K be a set. Then the following statements are equivalent:*

(A) *There exists a well-ordered set L such that $G \cong G' \subseteqq L^K$.*

(B) *For every $\varkappa \in K$ there exists a well ordered set $L_\varkappa$ and a mapping $f_\varkappa$ of G into $L_\varkappa$ such that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of G.*

Proof. 1. Assume that (A) is true. Then (B) holds, according to 2.8., if we put $L_\varkappa = L$ for every $\varkappa \in K$.

2. Let (B) be true. Then according to 2.8. we have $G \cong G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$. Let L be such a well-ordered set that $L_\varkappa \cong L'_\varkappa \subseteqq L$ for every $\varkappa \in K$. The set L can be constructed for instance in the following way: choose any well ordering of the set K and put $L = \sum_{\varkappa \in K} L_\varkappa$. Then $\prod_{\varkappa \in K} L_\varkappa \cong \prod_{\varkappa \in K} L'_\varkappa \subseteqq L^K$. If φ is an isomorphism of $\prod_{\varkappa \in K} L_\varkappa$ onto $\prod_{\varkappa \in K} L'_\varkappa$ we have $G \cong G' \cong \varphi(G') = G'' \subseteqq \prod_{\varkappa \in K} L'_\varkappa \subseteqq L^K$ so that $G \cong G'' \subseteqq L^K$ and (A) holds.

2.10. Theorem. *Let G be an ordered set satisfying the descending chain condition, let K be a set. Then the following statements are equivalent:*

(A) *For every $\varkappa \in K$ there exists a well-ordered set $S_\varkappa$ such that $G \cong G' \subseteqq \prod_{\varkappa \in K} S_\varkappa$.*

(B) *For every $\varkappa \in K$ there exists a well-ordered set $T_\varkappa$ and a one-one isotone mapping $f_\varkappa$ of G into $T_\varkappa$ such that $\{T_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well realizer of G.*

Proof. 1. Assume that (A) holds. Let φ be an isomorphism of G onto $G' \subseteqq \prod_{\varkappa \in k} S_\varkappa$. Denote – similarly as in 2.8. – $[\varphi(x)](\varkappa_0) = g_{\varkappa_0}(x)$. Then $g_\varkappa$ is an isotone mapping of G into $S_\varkappa$ for every $\varkappa \in K$. Put $R_\varkappa = g_\varkappa(G)$ for every $\varkappa \in K$. Then $R_\varkappa \subseteqq S_\varkappa$ so that $R_\varkappa$ is a well-ordered set and $g_\varkappa$ is an isotone mapping of G onto $R_\varkappa$ for every $\varkappa \in K$. Now for every $\varkappa \in K$ and every $y \in R_\varkappa$ we have $g_\varkappa^{-1}(y) \subseteqq G$ so that $g_\varkappa^{-1}(y)$ satisfies the descending chain condition. Hence according to 2.3. there exists a well-ordered set $T_y^\varkappa$ and a one-one isotone mapping $f_y^\varkappa$ of the set $g_\varkappa^{-1}(y)$ into $T_y^\varkappa$. Put $T_\varkappa = \sum_{y \in R_\varkappa} T_y^\varkappa$. $T_\varkappa$ as a lexicographic sum of well-ordered sets over a well-ordered set is a well-ordered set. Define the mapping $f_\varkappa$ of G into $T_\varkappa$ in the following way: $f_\varkappa(x) =$

$= [g_\varkappa(x), f^\varkappa_{g_\varkappa(x)}(x)]$. It is easy to see that $f_\varkappa$ is a one-one mapping of G into $T_\varkappa$ for every $\varkappa \in K$. We shall show that $\{T_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well realizer of G. Let $x_1, x_2 \in G$, $x_1 \leqq x_2$. Then $\varphi(x_1) \leqq \varphi(x_2)$ so that $[\varphi(x_1)](\varkappa) \leqq [\varphi(x_2)](\varkappa)$ for every $\varkappa \in K$. From this there follows that $g_\varkappa(x_1) \leqq g_\varkappa(x_2)$ for every $\varkappa \in K$. Choose any $\varkappa_0 \in K$. If $g_{\varkappa_0}(x_1) < g_{\varkappa_0}(x_2)$ then $[g_{\varkappa_0}(x_1), f^{\varkappa_0}_{g_{\varkappa_0}(x_1)}(x_1)] < [g_{\varkappa_0}(x_2), f^{\varkappa_0}_{g_{\varkappa_0}(x_2)}(x_2)]$ in $\sum_{y \in R_\varkappa} T^\varkappa_y$ so that $f_{\varkappa_0}(x_1) < f_{\varkappa_0}(x_2)$. If $g_{\varkappa_0}(x_1) = g_{\varkappa_0}(x_2)$ then $x_1 \in g^{-1}_{\varkappa_0}[g_{\varkappa_0}(x_1)]$, $x_2 \in g^{-1}_{\varkappa_0}[g_{\varkappa_0}(x_1)]$ $(= g^{-1}_{\varkappa_0}[g_{\varkappa_0}(x_2)])$ so that $f^{\varkappa_0}_{g_{\varkappa_0}(x_1)}(x_1) \leqq f^{\varkappa_0}_{g_{\varkappa_0}(x_1)}(x_2) = f^{\varkappa_0}_{g_{\varkappa_0}(x_2)}(x_2)$ and hence $[g_{\varkappa_0}(x_1), f^{\varkappa_0}_{g_{\varkappa_0}(x_1)}(x_1)] \leqq [g_{\varkappa_0}(x_2), f^{\varkappa_0}_{g_{\varkappa_0}(x_2)}(x_2)]$ i.e. $f_{\varkappa_0}(x_1) \leqq f_{\varkappa_0}(x_2)$. Therefore $f_\varkappa(x_1) \leqq f_\varkappa(x_2)$ for every $\varkappa \in K$. Suppose, on the contrary, that $f_\varkappa(x_1) \leqq f_\varkappa(x_2)$ for every $\varkappa \in K$. Then $[g_\varkappa(x_1), f^\varkappa_{g_\varkappa(x_1)}(x_1)] \leqq [g_\varkappa(x_2), f^\varkappa_{g_\varkappa(x_2)}(x_2)]$ for every $\varkappa \in K$ and hence $g_\varkappa(x_1) \leqq g_\varkappa(x_2)$ for every $\varkappa \in K$. From this it follows that $[\varphi(x_1)](\varkappa) \leqq [\varphi(x_2)](\varkappa)$ for every $\varkappa \in K$, i.e. $\varphi(x_1) \leqq \varphi(x_2)$. As φ is an isomorphism, this implies $x_1 \leqq x_2$. Hence $\{T_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is really a well realizer of G and (B) holds.

2. Assume that (B) holds. Then $\{T_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is also a well pseudorealizer of G and (A) holds according to 2.8. if we put $S_\varkappa = T_\varkappa$ for every $\varkappa \in K$.

2.11. Corollary. *Let G be an ordered set satisfying the descending chain condition, let K be a set. Then the following statements are equivalent:*

(A) *There exists a well-ordered set L such that $G \cong G' \subseteq L^K$.*

(B) *For every $\varkappa \in K$ there exists a well-ordered set $L_\varkappa$ and a one-one isotone mapping $f_\varkappa$ of G into $L_\varkappa$ such that $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well realizer of G.*

Proof can be made similarly as proof of 2.9.

3. WELL DIMENSION

3.1. Definition. Let G be an ordered set satisfying the descending chain condition. We put $\operatorname{wdim} G = \min(\operatorname{card} K \mid \{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well realizer of $G)$; this cardinality will be called a well dimension of G.

3.2. Theorem. *Let G be an ordered set satisfying the descending chain condition, let $m > 0$ be a cardinality. Then the following statements are equivalent:*

(A) $\operatorname{wdim} G \leqq m$.

(B) *There exists a set K with $\operatorname{card} K = m$ and for every $\varkappa \in K$ a well-ordered set $L_\varkappa$ such that $G \cong G' \subseteq \prod_{\varkappa \in K} L_\varkappa$.*

Proof follows from 2.10.

3.3. Theorem. *Let G be an ordered set satisfying the descending chain condition, let $m > 0$ be a cardinality. Then the following statements are equivalent:*

(A) $\operatorname{wdim} G \leqq m$.

(B) *There exists a set K with* card $K = m$ *and a well-ordered set L such that* $G \cong G' \subseteqq L^K$.

Proof follows from 2.11.

3.4. Theorem. *Let G be an ordered set satisfying the descending chain condition. Then* wdim $G \leqq$ card G; *if G is finite and* card $G \geqq 4$ *then even* wdim $G \leqq$ $\leqq [\frac{1}{2}$ card $G]$.

Proof. If G is finite then clearly wdim G = dim G so that according to [5] wdim G = dim $G \leqq [\frac{1}{2}$ card $G]$ for card $G \geqq 4$. If G is infinite then card G = = card $(G \times G)$ and the assertion follows from the proof of 2.5.

3.5. Theorem. *Let G be an ordered set satisfying the descending chain condition and let* card $G \leqq \aleph_\alpha$. *Then* wdim $G = \omega_{\alpha+1}$ – dim $G = \omega_{\alpha+1}$-pdim G.

Proof. Clearly wdim $G \leqq \omega_{\alpha+1}$-dim G. Assume that wdim $G = m$ and let $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ be a well realizer of G of cardinality m. For every $\varkappa \in K$ put $M_\varkappa =$ $= f_\varkappa(G)$; then $\{M_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is also a well realizer of G and card $M_\varkappa \leqq \aleph_\alpha$ for every $\varkappa \in K$. From this $\overline{M}_\varkappa < \omega_{\alpha+1}$ for every $\varkappa \in K$ so that $\{M_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is an $\omega_{\alpha+1}$-realizer of G and hence $\omega_{\alpha+1}$-dim $G \leqq m$. Therefore $\omega_{\alpha+1}$-dim $G = m =$ wdim G. Further $\omega_{\alpha+1}$-pdim $G \leqq \omega_{\alpha+1}$-dim G = wdim G; on the other hand, if $\omega_{\alpha+1}$-pdim $G =$ $= n$, then according to [9] $G \cong G' \subseteqq L^K$ where L is a chain of type $\omega_{\alpha+1}$, K an anti-chain of cardinality n. From this it follows, according to 3.3., wdim $G \leqq n$ so that also wdim $G = \omega_{\alpha+1}$-pdim G.

B. Dushnik and E. W. Miller ([4]) and also H. Komm ([7]) have proved that to every cardinal number $m > 0$ there exists an ordered set G such that dim $G = m$. We shall prove an analogical theorem for the well dimension.

3.6. Theorem. *For any cardinal number $m > 0$ there exists an ordered set G satisfying the descending chain condition such that* wdim $G = m$.

Proof.[1]) Let M be a set with card $M = m$. Put $a_x = \{x\}$, $c_x = M - \{x\}$ for any $x \in M$ and denote $G = \{a_x, c_x \mid x \in M\}$ where G is ordered by the set inclusion. It is clear that G satisfies the descending chain condition. In [4] there is proved dim $G = m$; we shall prove that also wdim $G = m$. As dim $G \leqq$ wdim G, for any ordered set G satisfying the descending chain condition it is sufficient to prove wdim $G \leqq m$. If $m < \aleph_0$ then card $G < \aleph_0$ so that wdim G = dim $G = m$ for wdim G = dim G for any finite ordered set G. If $m \geqq \aleph_0$ then card $G = m$ so that wdim $G \leqq m$ according to 3.4. Therefore in both cases wdim $G = m$.

The fact that wdim G = dim G holds for any finite ordered set G leads us to the question whether it may be posible that wdim G = dim G holds for any ordered set G

[1]) The proof is accomplished, in a quite similar way, as that of Theorem 4.1. in [4].

satisfying the descending chain condition. The following example shows that this is not true.

3.7. Example. *Let G be an infinite antichain. Then* $\dim G < \operatorname{wdim} G$.

Proof. There is $\dim G = 2$. Assume that $\operatorname{wdim} G = 2$. Then there exists a well realizer $\{L_i, f_i \mid i = 1, 2\}$ of the set G of cardinality 2. Hence there is necessarily $x, y \in G, f_1(x) < f_1(y) \Rightarrow f_2(x) > f_2(y)$ i.e. the set $f_2(G) \subseteqq L_2$ is dual to $f_1(G) \subseteqq L_1$. As G is infinite, $f_1(G)$ contains a chain of type ω. From this it follows that $f_2(G) \subseteqq L_2$ contains a chain of type ω^* which is a contradiction.

3.8. Lemma. *Let H, $G_\alpha(\alpha \in H)$ be ordered sets satisfying the descending chain condition. Then $\sum_{\alpha \in H} G_\alpha$ satisfies the descending chain condition.*

Proof. Let $[\alpha_i, x_i] \in \sum_{\alpha \in H} G_\alpha$ $(i = 0, 1, 2, \ldots)$ and assume that $[\alpha_0, x_0] \geqq [\alpha_1, x_1] \geqq \geqq \ldots \geqq [\alpha_n, x_n] \geqq \ldots$ Then $\alpha_0 \geqq \alpha_1 \geqq \ldots \geqq \alpha_n \geqq \ldots$ and hence there exists a non-negative integer n_1 such that $\alpha_{n_1} = \alpha_{n_1+1} = \alpha_{n_1+2} = \ldots$ From this it follows $x_{n_1} \geqq \geqq x_{n_1+1} \geqq \ldots \geqq x_{n_1+k} \geqq \ldots$ and $x_{n_1+k} \in G_{\alpha_{n_1}}$ for every $k = 0, 1, 2, \ldots$ so that there exists k_1 such that $x_{n_1+k_1} = x_{n_1+k_1+1} = x_{n_1+k_1+2} = \ldots$ Therefore if we put $n_1 + k_1 = n_0$ we have $[\alpha_{n_0}, x_{n_0}] = [\alpha_{n_0+1}, x_{n_0+1}] = [\alpha_{n_0+2}, x_{n_0+2}] = \ldots$

3.9. Corollary. *Let G, H be ordered sets satisfying the descending chain condition. Then $G \oplus H$, $G + H$, $G \circ H$ satisfy the descending chain condition.*

3.10. Corollary. *Let G be an ordered set satisfying the descending chain condition, let H be a finite chain. Then ${}^H G$ satisfies the descending chain condition.*

Proof. If $\operatorname{card} H = n$ then ${}^H G \cong G_1 \circ G_2 \circ \ldots \circ G_n$ where $G_i \cong G$ $(i = 1, 2, \ldots, n)$ so that the statement follows from 3.9.

3.11. Theorem. *Let H, $G_\alpha(\alpha \in H)$ be ordered sets satisfying the descending chain condition. Then* $\operatorname{wdim} \sum_{\alpha \in H} G_\alpha = \sup \{\operatorname{wdim} H, \operatorname{wdim} G_\alpha(\alpha \in H)\}$.[2])

Proof. Denote $\sup \{\operatorname{wdim} H, \operatorname{wdim} G_\alpha(\alpha \in H)\} = m$. Let K be a set with $\operatorname{card} K = m$, let $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ be a well realizer of H, let $\{P^\alpha_\varkappa, g^\alpha_\varkappa \mid \varkappa \in K\}$ be a well realizer of G_α for every $\alpha \in H$. We can assume $L_\varkappa = f_\varkappa(H)$ for every $\varkappa \in K$ (in the other case we shall consider the set $f_\varkappa(H) \subseteqq L_\varkappa$ instead of $L_\varkappa$) and also $P^\alpha_\varkappa = g^\alpha_\varkappa(G_\alpha)$ for every $\varkappa \in K$ and every $\alpha \in H$. Put $S_{\varkappa\varrho} = \sum_{y \in L_\varkappa} P^{f_\varkappa^{-1}(y)}_\varrho(y)$ for any two elements $\varkappa, \varrho \in K$. $S_{\varkappa\varrho}$, as a lexicographic sum of well-ordered sets over a well-ordered set, is a well-ordered set for any $\varkappa \in K$, $\varrho \in K$. Define the mapping $h_{\varkappa\varrho}$ of $\sum_{\alpha \in H} G_\alpha$ into $S_{\varkappa\varrho}$ in the following way:

[2]) See Theorem 1 in [8].

$h_{\varkappa\varrho}([\alpha, x]) = [f_\varkappa(\alpha), g_\varrho^\alpha(x)]$. Put further $T_\varkappa = S_{\varkappa\varkappa}$, $r_\varkappa = h_{\varkappa\varkappa}$. We shall show that $\{T_\varkappa, r_\varkappa \mid \varkappa \in K\}$ is a well realizer of $\sum_{\alpha\in H} G_\alpha$. Let $[\alpha_1, x_1] \in \sum_{\alpha\in H} G_\alpha$, $[\alpha_2, x_2] \in \sum_{\alpha\in H} G_\alpha$, $[\alpha_1, x_1] \leqq [\alpha_2, x_2]$. Then either $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$, $x_1 \leqq x_2$. In the first case we have $f_\varkappa(\alpha_1) < f_\varkappa(\alpha_2)$ for every $\varkappa \in K$ so that $h_{\varkappa\varrho}([\alpha_1, x_1]) = [f_\varkappa(\alpha_1), g_\varrho^{\alpha_1}(x_1)] < [f_\varkappa(\alpha_2), g_\varrho^{\alpha_2}(x_2)] = h_{\varkappa\varrho}([\alpha_2, x_2])$ for any $\varkappa \in K$, $\varrho \in K$. In the second case there is $g_\varrho^{\alpha_1}(x_1) \leqq g_\varrho^{\alpha_1}(x_2)$ for every $\varrho \in K$ so that $h_{\varkappa\varrho}([\alpha_1, x_1]) = [f_\varkappa(\alpha_1), g_\varrho^{\alpha_1}(x_1)] \leqq [f_\varkappa(\alpha_1), g_\varrho^{\alpha_1}(x_2)] = h_{\varkappa\varrho}([\alpha_1, x_2]) = h_{\varkappa\varrho}([\alpha_2, x_2])$ for any $\varkappa \in K$, $\varrho \in K$. We have proved that even every $h_{\varkappa\varrho}$ is an isotone mapping. Further it is clear that every $h_{\varkappa\varrho}$ is a one-one mapping because every $f_\varkappa$ and every g_ϱ^α is a one-one mapping. Assume now that $[\alpha_1, x_1] \in \sum_{\alpha\in H} G_\alpha$, $[\alpha_2, x_2] \in \sum_{\alpha\in H} G_\alpha$ and that $r_\varkappa([\alpha_1, x_1]) \leqq r_\varkappa([\alpha_2, x_2])$ for every $\varkappa \in K$. Then $h_{\varkappa\varkappa}([\alpha_1, x_1]) = [f_\varkappa(\alpha_1), g_\varkappa^{\alpha_1}(x_1)] \leqq [f_\varkappa(\alpha_2), g_\varkappa^{\alpha_2}(x_2)] = h_{\varkappa\varkappa}([\alpha_2, x_2])$ for every $\varkappa \in K$. From this it follows $f_\varkappa(\alpha_1) \leqq f_\varkappa(\alpha_2)$ for every $\varkappa \in K$ which implies $\alpha_1 \leqq \alpha_2$ because $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is a well realizer of H. If $f_\varkappa(\alpha_1) < f_\varkappa(\alpha_2)$ for at least one (and thus for every) $\varkappa \in K$ we have $\alpha_1 < \alpha_2$ and hence $[\alpha_1, x_1] < [\alpha_2, x_2]$ in $\sum_{\alpha\in H} G_\alpha$. In the opposite case $f_\varkappa(\alpha_1) = f_\varkappa(\alpha_2)$ and therefore $\alpha_1 = \alpha_2$. Therefore in this case $g_\varkappa^{\alpha_1}(x_1) \leqq g_\varkappa^{\alpha_1}(x_2)$ for every $\varkappa \in K$. As $\{P_\varkappa^{\alpha_1}, g_\varkappa^{\alpha_1} \mid \varkappa \in K\}$ is a well realizer of G_{α_1} this implies $x_1 \leqq x_2$ and hence $[\alpha_1, x_1] \leqq [\alpha_1, x_2] = [\alpha_2, x_2]$. Thus $\{T_\varkappa, r_\varkappa \mid \varkappa \in K\}$ is really a well realizer of $\sum_{\alpha\in H} G_\alpha$ so that $\operatorname{wdim} \sum_{\alpha\in H} G_\alpha \leqq m$. On the other hand the set $\sum_{\alpha\in H} G_\alpha$ contains subsets H', $G'_\alpha(\alpha \in H)$ isomorphic with H, $G_\alpha(\alpha \in H)$: $H' = \{[\alpha, x_\alpha] \mid \alpha \in H$, $x_\alpha \in G_\alpha$ is any constantly chosen element$\}$, $G'_\alpha = \{[\alpha, x] \mid x \in G_\alpha$, $\alpha \in H$ is constant$\}$. From this it follows $\operatorname{wdim} H = \operatorname{wdim} H' \leqq \operatorname{wdim} \sum_{\alpha\in H} G_\alpha$, $\operatorname{wdim} G_\alpha = \operatorname{wdim} G'_\alpha \leqq \operatorname{wdim} \sum_{\alpha\in H} G_\alpha$ for every $\alpha \in H$ so that $\sup \{\operatorname{wdim} H, \operatorname{wdim} G_\alpha(\alpha \in H)\} = m \leqq \operatorname{wdim} \sum_{\alpha\in H} G_\alpha$ and altogether $\operatorname{wdim} \sum_{\alpha\in H} G_\alpha = m = \sup \{\operatorname{wdim} H, \operatorname{wdim} G_\alpha(\alpha \in H)\}$.

3.12. Corollary. *Let G, H be ordered sets satisfying the descending chain condition. Then* $\operatorname{wdim} (G \oplus H) = \max \{\operatorname{wdim} G, \operatorname{wdim} H\}$, $\operatorname{wdim} (G + H) = \max \{2, \operatorname{wdim} G, \operatorname{wdim} H\}$, $\operatorname{wdim} (G \circ H) = \max \{\operatorname{wdim} G, \operatorname{wdim} H\}$.

3.13. Corollary. *Let G be an ordered set satisfying the descending chain condition, let H be a finite chain. Then* $\operatorname{wdim} {}^H G = \operatorname{wdim} G$.

Proof. If H is a chain with $\operatorname{card} H = 2$ then according to 3.12. $\operatorname{wdim} {}^H G = \operatorname{wdim} (G \circ G) = \operatorname{wdim} G$. Now the statement follows by induction.

3.14. Lemma. *Let $G_1, G_2, \ldots, G_n$ be ordered sets satisfying the descending chain condition. Then $G_1 . G_2 \ldots G_n$ satisfies the descending chain condition.*

Proof. Let $[x_1^i, x_2^i, \ldots, x_n^i] \in G_1 . G_2 \ldots G_n$ for $i = 0, 1, 2, \ldots$ and let $[x_1^0, x_2^0, \ldots$

$\ldots, x_n^0] \geqq [x_1^1, x_2^1, \ldots, x_n^1] \geqq \ldots \geqq [x_1^m, x_2^m, \ldots, x_n^m] \geqq \ldots$ Then $x_1^0 \geqq x_1^1 \geqq \ldots \geqq$ $\geqq x_1^m \geqq \ldots, x_2^0 \geqq x_2^1 \geqq \ldots \geqq x_2^m \geqq \ldots, \ldots, x_n^0 \geqq x_n^1 \geqq \ldots \geqq x_n^m \geqq \ldots$ From this it follows that for every $i = 1, 2, \ldots, n$ there exists a non-negative integer m_i such that $x_i^{m_i} = x_i^{m_i+1} = \ldots$ Put $m_0 = \max\{m_1, m_2, \ldots, m_n\}$. Then $[x_1^{m_0}, x_2^{m_0}, \ldots, x_n^{m_0}] =$ $= [x_1^{m_0+1}, x_2^{m_0+1}, \ldots, x_n^{m_0+1}] = \ldots$

3.15. Corollary. *Let G be an ordered set satisfying the descending chain condition, let H be a finite antichain. Then G^H satisfies the descending chain condition.*

3.16. Corollary. *Let G be an ordered set satisfying the descending chain condition, let H be a finite ordered set. Then G^H satisfies the descending chain condition.*

Proof. Let $\bar{H}$ be the set H ordered as an antichain. Then $G^H \subseteqq G^{\bar{H}}$. $G^{\bar{H}}$ satisfies the descending chain condition according to 3.15., hence G^H also satisfies the descending chain condition.

3.17. Theorem. *Let G, H be ordered sets satisfying the descending chain condition. Then* $\text{wdim}(G . H) \leqq \text{wdim}\, G + \text{wdim}\, H$.

Proof. Denote $\text{wdim}\, G = m$, $\text{wdim}\, H = n$. According to 3.2. there exists a set K_1 with $\text{card}\, K_1 = m$ and for every $\varkappa \in K_1$ a well-ordered set $L_\varkappa$ such that $G \cong G' \subseteqq \subseteqq \prod_{\varkappa \in K_1} L_\varkappa$ and similarly there exists a set K_2 with $\text{card}\, K_2 = n$ and for every $\varkappa \in K_2$ a well-ordered set $L_\varkappa$ such that $H \cong H' \subseteqq \prod_{\varkappa \in K_2} L_\varkappa$. Assume that K_1, K_2 are disjoint and put $K = K_1 \cup K_2$. Then $\text{card}\, K = m + n$ and $G . H \cong G' . H' \subseteqq (\prod_{\varkappa \in K_1} L_\varkappa) . (\prod_{\varkappa \in K_2} L_\varkappa) \cong \prod_{\varkappa \in K} L_\varkappa$. From this there follows according to 3.2. $\text{wdim}(G . H) \leqq m + n = \text{wdim}\, G + \text{wdim}\, H$.

3.18. Note. The inequality $\leqq$ in 3.17 cannot be substituted by $=$. If, for example G, H are finite non-trivial antichains it is $\text{wdim}\, G = 2 = \text{wdim}\, H$ and as $G . H$ is also a finite non-trivial antichain we have $\text{wdim}(G . H) = 2 < \text{wdim}\, G + \text{wdim}\, H$. On the other hand, if G, H are non-trivial well-ordered sets, there is $\text{wdim}\, G = 1 = \text{wdim}\, H$ and – as it will be shown in 3.22. – $\text{wdim}(G . H) = 2 = \text{wdim}\, G + \text{wdim}\, H$.

3.19. Corollary. *Let $G_1, G_2, \ldots, G_n$ be ordered sets satisfying the descending chain condition. Then* $\text{wdim}(G_1 . G_2 \ldots G_n) \leqq \text{wdim}\, G_1 + \text{wdim}\, G_2 + \ldots + \text{wdim}\, G_n$.

Proof follows from 3.17. by induction.

3.20. Corollary. *Let G be an ordered set satisfying the descending chain condition, let H be a finite antichain. Then wdim $G \leqq$ card H . wdim G.*

3.21. Corollary. *Let G be an ordered set satisfying the descending chain condition, let H be a finite ordered set. Then* $\operatorname{wdim} G^H \leqq \operatorname{card} H \,.\, \operatorname{wdim} G$.

Proof. If $\bar{H}$ is the set H ordered as an antichain then $G^H \subseteqq G^{\bar{H}}$ and hence $\operatorname{wdim} G^H \leqq \leqq \operatorname{wdim} G^{\bar{H}} \leqq \operatorname{card} \bar{H} \,.\, \operatorname{wdim} G = \operatorname{card} H \,.\, \operatorname{wdim} G$.

3.22. Theorem. *Let $G_1, G_2, \ldots, G_n$ be well-ordered sets. Then* $\operatorname{wdim} (G_1 \,.\, G_2 \ldots \ldots G_n) = n$.

Proof. As $\operatorname{wdim} G_i = 1$ for $i = 1, 2, \ldots, n$ we have $\operatorname{wdim} (G_1 \,.\, G_2 \ldots G_n) \leqq n$ according to 3.19. Assume $\operatorname{wdim} (G_1 \,.\, G_2 \ldots G_n) = m < n$ and let $\{L_k, f_k \mid k = = 1, 2, \ldots, m\}$ be a well realizer of $G_1 \,.\, G_2 \ldots G_n$ of cardinality m. Choose for any $i = 1, 2, \ldots, n$ two elements $x_i, y_i \in G_i$ such that $x_i < y_i$ and denote $a_i = [x_1, x_2, \ldots \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n]$, $c_i = [y_1, y_2, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n]$. Then $a_i \in G_1 \,. \,. \, G_2 \ldots G_n$, $c_i \in G_1 \,.\, G_2 \ldots G_n$ for $i = 1, 2, \ldots, n$, $a_i < c_j$ for $i \neq j$, $a_i \parallel c_i$. Thus, there exists at least one $k_0 (1 \leqq k_0 \leqq m)$ such that $f_{k_0}(c_i) < f_{k_0}(a_i)$ and at the same time $f_{k_0}(c_j) < f_{k_0}(a_j)$ where $i \neq j$. As $a_i < c_j$ and $a_j < c_i$ in $G_1 \,.\, G_2 \ldots G_n$ we obtain $f_{k_0}(c_i) < f_{k_0}(a_i) < f_{k_0}(c_j) < f_{k_0}(a_j) < f_{k_0}(c_i)$, which is impossible. Hence $\operatorname{wdim} (G_1 \,.\, G_2 \ldots G_n) = n$.

3.23. Corollary. *Let L be a well-ordered set, let K be a finite antichain. Then* $\operatorname{wdim} L^K = \operatorname{card} K$.

4. WELL PSEUDODIMENSION

4.1. Definition. Let G be an ordered set. We put $\operatorname{wpdim} G = \min (\operatorname{card} K \mid \{L_\varkappa, f_\varkappa \mid \mid \varkappa \in K\}$ is a well pseudorealizer of G); this cardinality will be called a well pseudodimension of G.

4.2. Theorem. *Let G be an ordered set, let $m > 0$ be a cardinality. Then the following statements are equivalent:*

(A) $\operatorname{wpdim} G \leqq m$.

(B) *There exists a set K with* $\operatorname{card} K = m$ *and for every $\varkappa \in K$ a well-ordered set $L_\varkappa$ such that* $G \cong G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$.

Proof follows from 2.8.

4.3. Theorem. *Let G be an ordered set, let $m > 0$ be a cardinality. Then the following statements are equivalent:*

(A) $\operatorname{wpdim} G \leqq m$.

(B) *There exists a set K with* $\operatorname{card} K = m$ *and a well-ordered set L such that* $G \cong G' \subseteqq L^K$.

Proof follows from 2.9.

4.4. Theorem. *Let G be an ordered set. Then* wpdim $G \leqq$ card G; *if G is finite and* card $G \geqq 4$ *then* wpdim $G \leqq [\frac{1}{2}$ card $G]$.

Proof. If G is finite then clearly wpdim $G =$ wdim $G =$ dim G so that wpdim $G \leqq$ $\leqq [\frac{1}{2}$ card $G]$ for card $G \geqq 4$, according to [5]. If G is infinite then card $(G \times G) =$ $=$ card G and the statement follows from the proof of 2.7.

4.5. Theorem. *Let G be an ordered set and let* card $G \leqq \aleph_\alpha$. *Then* wpdim $G =$ $= \omega_{\alpha+1}$ − pdim G.

Proof. We have clearly wpdim $G \leqq \omega_{\alpha+1}$ − pdim G. Assume that wpdim $G = m$ and let $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ be a well pseudorealizer of G of cardinality m. Put $M_\varkappa =$ $= f_\varkappa(G)$ for any $\varkappa \in K$; then $\{M_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is also a well pseudorealizer of G and there is card $M_\varkappa \leqq \aleph_\alpha$ so that $\overline{M}_\varkappa < \omega_{\alpha+1}$ for every $\varkappa \in K$. $\{M_\varkappa, f_\varkappa \mid \varkappa \in K\}$ is therefore an $\omega_{\alpha+1}$ − pseudorealizer of G of cardinality m so that $\omega_{\alpha+1}$ − pdim $G \leqq m$. Hence $\omega_{\alpha+1}$ − pdim $G = m =$ wpdim G.

4.6. Theorem. *Let G be an ordered set satisfying the descending chain condition. Then* wpdim $G =$ wdim G.

Proof. We have clearly wpdim $G \leqq$ wdim G. Assume that wpdim $G = m$. Then according to 4.2. there exists a set K with card $K = m$ and for every $\varkappa \in K$ a well-ordered set $L_\varkappa$ such that $G \cong G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$. From this it follows according to 3.2. wdim $G \leqq m$ and hence wdim $G = m =$ wpdim G.

From 4.6. and 3.6. we obtain immediately

4.7. Theorem. *For any cardinal number $m > 0$ there exists an ordered set G such that* wpdim $G = m$.

4.8. Theorem. *Let H be an ordered set satisfying the descending chain condition, let $\{G_\alpha \mid \alpha \in H\}$ be a system of ordered sets. Then* wpdim $\sum_{\alpha \in H} G_\alpha =$ sup $\{$wdim H, wpdim $G_\alpha(\alpha \in H)\}$.

Proof. Put sup $\{$wdim H, wpdim $G_\alpha(\alpha \in H)\} = m$. Then there exists a well realizer $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ of the set H of cardinality m; further let $\{P_\varkappa^\alpha, g_\varkappa^\alpha \mid \varkappa \in K\}$ be a well pseudorealizer of the set G_α of cardinality m for every $\alpha \in H$. Now define the well-ordered sets $S_{\varkappa\varrho}$ and mappings $h_{\varkappa\varrho}$ of the set $\sum_{\alpha \in H} G_\alpha$ into $S_{\varkappa\varrho}$ for every $\varkappa \in K$, $\varrho \in K$, in the same way as in the proof of 3.11. and put $T_\varkappa = S_{\varkappa\varkappa}$, $r_\varkappa = h_{\varkappa\varkappa}$. We shall show that $\{T_\varkappa, r_\varkappa \mid \varkappa \in K\}$ is a well pseudorealizer of $\sum_{\alpha \in H} G_\alpha$. Let $[\alpha_1, x_1] \in \sum_{\alpha \in H} G_\alpha$, $[\alpha_2, x_2] \in \sum_{\alpha \in H} G_\alpha$, $[\alpha_1, x_1] \leqq [\alpha_2, x_2]$. Then either $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$, $x_1 \leqq x_2$. In the first case there is $f_\varkappa(\alpha_1) < f_\varkappa(\alpha_2)$ for every $\varkappa \in K$, $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ being a well realizer of H. Hence $[f_\varkappa(\alpha_1), g_\varrho^{\alpha_1}(x_1)] < [f_\varkappa(\alpha_2), g_\varrho^{2\alpha}(x_2)]$ for any $\varkappa \in K$, $\varrho \in K$, i.e. $h_{\varkappa\varrho}([\alpha_1, x_1]) < h_{\varkappa\varrho}([\alpha_2, x_2])$ for any $\varkappa \in K$, $\varrho \in K$. In the second case there is $g_\varrho^{\alpha_1}(x_1) \leqq \leqq g_\varrho^{\alpha_1}(x_2)$ for every $\varrho \in K$ so that $h_{\varkappa\varrho}([\alpha_1, x_1]) = [f_\varkappa(\alpha_1), g_\varrho^{\alpha_1}(x_1)] \leqq [f_\varkappa(\alpha_1), g_\varrho^{\alpha_1}(x_2)] = = h_{\varkappa\varrho}([\alpha_1, x_2]) = h_{\varkappa\varrho}([\alpha_2, x_2])$ for every $\varkappa \in K$, $\varrho \in K$. We have proved that even every $h_{\varkappa\varrho}$ is isotone. Now assume that $r_\varkappa([\alpha_1, x_1]) = h_{\varkappa\varkappa}([\alpha_1, x_1]) = [f_\varkappa(\alpha_1), g_\varkappa^{\alpha_1}(x_1)] \leqq [f_\varkappa(\alpha_2), g_\varkappa^{\alpha_2}(x_2)] = h_{\varkappa\varkappa}([\alpha_2, x_2]) = r_\varkappa([\alpha_2, x_2])$ for every $\varkappa \in K$. Then $f_\varkappa(\alpha_1) \leqq f_\varkappa(\alpha_2)$ for every $\varkappa \in K$ and hence $\alpha_1 \leqq \alpha_2$. If $f_\varkappa(\alpha_1) < f_\varkappa(\alpha_2)$ for at least one $\varkappa \in K$ we have $\alpha_1 < \alpha_2$ and therefore $[\alpha_1, x_1] < [\alpha_2, x_2]$ in $\sum_{\alpha \in H} G_\alpha$. In the opposite case there is $f_\varkappa(\alpha_1) = f_\varkappa(\alpha_2)$ for every $\varkappa \in K$ so that $\alpha_1 = \alpha_2$ and hence $g_\varkappa^{\alpha_1}(x_1) \leqq g_\varkappa^{\alpha_1}(x_2)$ for every $\varkappa \in K$. This implies $x_1 \leqq x_2$ in $G_{\alpha_1} = G_{\alpha_2}$ so that again $[\alpha_1, x_1] \leqq [\alpha_1, x_2] = [\alpha_2, x_2]$ in $\sum_{\alpha \in H} G_\alpha$. Hence $\{T_\varkappa, r_\varkappa \mid \varkappa \in K\}$ is really a well pseudorealizer of $\sum_{\alpha \in H} G_\alpha$ so that $\operatorname{wpdim} \sum_{\alpha \in H} G_\alpha \leqq m$. Analogously like in 3.11. we can easily prove that $\operatorname{wpdim} \sum_{\alpha \in H} G_\alpha \geqq m$ so that $\operatorname{wpdim} \sum_{\alpha \in H} G_\alpha = m = \sup \{\operatorname{wdim} H, \operatorname{wpdim} G_\alpha (\alpha \in H)\}$.

4.9. Corollary. *Let G, H be ordered sets. Then* $\operatorname{wpdim} (G \oplus H) = \max \{\operatorname{wpdim} G, \operatorname{wpdim} H\}$, $\operatorname{wpdim} (G + H) = \max \{2, \operatorname{wpdim} G, \operatorname{wpdim} H\}$.

4.10. Theorem. *Let H be a set, let G_α be an ordered set for every $\alpha \in H$. Then* $\operatorname{wpdim} \prod_{\alpha \in H} G_\alpha \leqq \sum_{\alpha \in H} \operatorname{wpdim} G_\alpha$.

Proof. Denote $\operatorname{wpdim} G_\alpha = m_\alpha$ for every $\alpha \in H$. According to 4.2. there exists a set K_α with $\operatorname{card} K_\alpha = m_\alpha$ and for every $\varkappa \in K_\alpha$ a well-ordered set $L_\varkappa$ such that $G_\alpha \cong G'_\alpha \subseteqq \prod_{\varkappa \in K_\alpha} L_\varkappa$. Assume that the sets K_α are disjoint and put $K = \bigcup_{\alpha \in H} K_\alpha$. Then $\operatorname{card} K = \sum_{\alpha \in H} m_\alpha = \sum_{\alpha \in H} \operatorname{wpdim} G_\alpha$ and $\prod_{\alpha \in H} G_\alpha \cong \prod_{\alpha \in H} G'_\alpha \subseteqq \prod_{\alpha \in H} (\prod_{\varkappa \in K_\alpha} L_\varkappa) \cong \prod_{\varkappa \in K} L_\varkappa$. From this it follows $\operatorname{wpdim} \prod_{\alpha \in H} G_\alpha \leqq \operatorname{card} K = \sum_{\alpha \in H} \operatorname{wpdim} G_\alpha$ according to 4.2.

4.11. Note. The relation $\leqq$ also here cannot be substituted by $=$. This follows from 4.6. and 3.18.

4.12. Corollary. *Let G be an ordered set, let H be an antichain. Then* $\operatorname{wpdim} G^H \leqq \leqq \operatorname{card} H \,.\, \operatorname{wpdim} G$.

4.13. Corollary. *Let G, H be ordered sets. Then* $\operatorname{wpdim} G^H \leqq \operatorname{card} H \,.\, \operatorname{wpdim} G$.

Proof. Similarly as in 3.21.

4.14. Theorem. *Let H be a set, let G_α be a well-ordered set for every $\alpha \in H$. Then* $\operatorname{wpdim} \prod_{\alpha \in H} G_\alpha = \operatorname{card} H$.

Proof. According to 4.10. we have $\operatorname{wpdim} \prod_{\alpha \in H} G_\alpha \leqq \operatorname{card} H$. Assume $\operatorname{wpdim} \prod_{\alpha \in H} G_\alpha = m < \operatorname{card} H$ and let $\{L_\varkappa, f_\varkappa \mid \varkappa \in K\}$ be a well pseudorealizer of the set $\prod_{\alpha \in H} G_\alpha$ of cardinality m. Choose for any $\alpha \in H$ two elements $x_\alpha \in G_\alpha$, $y_\alpha \in G_\alpha$ such that $x_\alpha < y_\alpha$ and for every $\alpha_0 \in H$ denote – similarly as in 3.22. – φ_{α_0}, ψ_{α_0} the elements of $\prod_{\alpha \in H} G_\alpha$ defined in the following way:

$$\varphi_{\alpha_0}(\alpha) = \begin{cases} x_\alpha & \text{for } \alpha \neq \alpha_0 \\ y_\alpha & \text{for } \alpha = \alpha_0 \end{cases} \qquad \psi_{\alpha_0}(\alpha) = \begin{cases} y_\alpha & \text{for } \alpha \neq \alpha_0 \\ x_\alpha & \text{for } \alpha = \alpha_0 \end{cases}$$

It is easy to see that $\varphi_{\alpha_1} < \psi_{\alpha_2}$ for $\alpha_1 \neq \alpha_2$ and $\varphi_{\alpha_0} \parallel \psi_{\alpha_0}$ in $\prod_{\alpha \in H} G_\alpha$. This implies that there exists at least one element $\varkappa_0 \in K$ such that $f_{\varkappa 0}(\psi_{\alpha_1}) < f_{\varkappa 0}(\varphi_{\alpha_1})$ and $f_{\varkappa 0}(\psi_{\alpha_2}) < f_{\varkappa 0}(\varphi_{\alpha_2})$ where $\alpha_1 \neq \alpha_2$. As $\varphi_{\alpha_1} < \psi_{\alpha_2}$ and $\varphi_{\alpha_2} < \psi_{\alpha_1}$ we have $f_{\varkappa 0}(\psi_{\alpha_1}) < f_{\varkappa 0}(\varphi_{\alpha_1}) \leqq f_{\varkappa 0}(\psi_{\alpha_2}) < f_{\varkappa 0}(\varphi_{\alpha_2}) \leqq f_{\varkappa 0}(\psi_{\alpha_1})$, i.e. $f_{\varkappa 0}(\psi_{\alpha_1}) < f_{\varkappa 0}(\psi_{\alpha_1})$ which is impossible. Hence $\operatorname{wpdim} \prod_{\alpha \in H} G_\alpha = \operatorname{card} H$.

4.15. Corollary. *Let L be a well-ordered set, let K be an antichain. Then* $\operatorname{wpdim} L^K = \operatorname{card} K$.

5. EXAMPLES

5.1. *Let G be the set of all real numbers with the natural ordering. Then* $\operatorname{wpdim} G = \aleph_0$.

Proof. According to [9] there is $\mathbf{2} - \operatorname{pdim} G = \operatorname{sep} G = \aleph_0$.[3]) From this there follows $\operatorname{wpdim} G \leqq \mathbf{2} - \operatorname{pdim} G = \aleph_0$. Assume that $\operatorname{wpdim} G < \aleph_0$, i.e. $\operatorname{wpdim} G = m$ where m is a finite number. Then according to 4.3. $G \cong G' \subseteqq L^K$ where L is a suitable well-ordered set and K is an antichain with $\operatorname{card} K = m$. According to 3.15. the set L^K satisfies the descending chain condition and this is a contradiction because G contains an infinite descending chain.

5.2. *Let G be the set of all rational numbers with the natural ordering. Then* $\operatorname{wpdim} G = \aleph_0$.

Proof. As $G \subseteqq H$ implies $\operatorname{wpdim} G \leqq \operatorname{wpdim} H$ for any ordered sets G, H, 5.1. implies $\operatorname{wpdim} G \leqq \aleph_0$. The converse inequality can be proved in the same way as in 5.1. because G again contains an infinite descending chain.

5.3. *Let G be a chain of type* ω_α^*. *Then* $\operatorname{wpdim} G = \aleph_\alpha$.

Proof. According to 4.4. we have $\operatorname{wpdim} G \leqq \aleph_\alpha$. Assume $\operatorname{wpdim} G = m < \aleph_\alpha$.

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[3]) Sep G denotes the separability of G i.e. the minimal cardinality of a subset $H \subseteqq G$ which is dense in G.

Then according to 4.2. there exists a set K with card $K = m$ and for every $\varkappa \in K$ a well-ordered set L_α such that $G \cong G' \subseteqq \prod_{\varkappa \in K} L_\varkappa$. Thus $G' = \{\varphi_0, \varphi_1, \ldots, \varphi_\lambda, \ldots \mid \varphi_0 > > \varphi_1 > \ldots > \varphi_\lambda > \ldots, \lambda < \omega_\alpha, \varphi_\lambda \in \prod_{\varkappa \in K} L_\varkappa\}$. This implies $\varphi_0(\varkappa) \geqq \varphi_1(\varkappa) \geqq \ldots \geqq \geqq \varphi_\lambda(\varkappa) \geqq \ldots$ for $\lambda < \omega_\alpha$ and $\varkappa \in K$. Denote $W_\varkappa = \{\lambda \mid \lambda \in W(\omega_\alpha), \varphi_\lambda(\varkappa) > \varphi_{\lambda+1}(\varkappa)\}$ for any $\varkappa \in K$.

Then it holds: every $W_\varkappa$ is a finite set and for every $\lambda \in W(\omega_\alpha)$ there exists a $\varkappa$ such that $\lambda \in W_\varkappa$. This implies $W(\omega_\alpha) = \bigcup_{\varkappa \in K} W_\varkappa$. But card $\bigcup_{\varkappa \in K} W_\varkappa \leqq \sum_{\varkappa \in K}$ card $W_\varkappa$; the last cardinal number is finite if $m < \aleph_0$; if $m \geqq \aleph_0$ then $\sum_{\varkappa \in K}$ card $W_\varkappa \leqq \sum_{\varkappa \in K} \aleph_0 = m \,.\, \aleph_0 = = m$; at the same time card $W(\omega_\alpha) = \aleph_\alpha > m$ and this is a contradiction. Hence wpdim $G = \aleph_\alpha$.

5.4. *Let G be an antichain such that $\aleph_0 \leqq$ card $G \leqq 2^{\aleph_0}$. Then* wdim $G = \aleph_0$.

Proof. In [10] there is proved: If G is an antichain with card $G = \aleph_\alpha$ then **2** − pdim G $= m$ where m is the smallest cardinal number such that $2^m \geqq \aleph_\alpha$. Hence if G is an antichain of cardinality $2^{\aleph_0}$ then **2** − pdim $G = \aleph_0$ so that wdim $G = =$ wpdim $G \leqq \aleph_0$. Thus it is sufficient to prove that if G is an antichain with card $G = = \aleph_0$ then wdim $G \geqq \aleph_0$. Suppose wdim $G = m < \aleph_0$. Then there exists a well realizer $\{L_i, f_i \mid i = 1, \ldots, m\}$ of the set G of cardinality m. Write all elements of the set G in the form of a sequence: $G = \{x_0, x_1, \ldots, x_n, \ldots\}$. Now, f_1 is a one-one mapping of G into L_1 and L_1 is a well-ordered set; thus, the set $f_1(G)$ is well-ordered, so that $f_1(G) = \{l_0^1, l_1^1, \ldots, l_\lambda^1, \ldots \mid \lambda < \alpha(\alpha < \omega_1), l_0^1 < l_1^1 < \ldots < l_\lambda^1 < \ldots\}$. Now for every $\lambda < \omega_0$ there exists a non-negative integer n_λ such that $f_1^{-1}(l_\lambda^1) = x_{n_\lambda}$; simultaneously for $\lambda_1 \neq \lambda_2$ there is $n_{\lambda_1} \neq n_{\lambda_2}$. In the sequence $\{n_\lambda\}_{\lambda < \omega_0}$ there exists an increasing subsequence $\{n_{\lambda_k}\}_{k < \omega_0}$. Write more briefly $n_k^1 = n_{\lambda_k}$ and denote $G^1 = = \{x_{n^1{}_k}\}_{k < \omega_0}$. Then there holds $n_{k_1}^1 < n_{k_2}^1$ and $f_1(x_{n^1{}_{k_1}}) < f_1(x_{n^1{}_{k_2}})$ for $k_1 < k_2$. Now, $f_2(G^1) \subseteqq L_2$ and L_2 is well-ordered so that $f_2(G^1) = \{l_0^2, l_1^2, \ldots, l_\lambda^2, \ldots \mid \lambda < < \beta(\beta < \omega_1), l_0^2 < l_1^2 < \ldots < l_\lambda^2 < \ldots\}$. For every $\lambda < \omega_0$ there exists again a non-negative integer k_λ such that $f_2^{-1}(l_\lambda^2) = x_{n^1{}_{k_\lambda}}$, where $k_{\lambda_1} \neq k_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$.

In the sequence $\{k_\lambda\}_{\lambda < \omega_0}$ there exists an increasing subsequence $\{k_{\lambda_i}\}_{i < \omega_0}$. Write again n_i^2 instead of $n_{k\lambda_i}^1$. If we denote $G^2 = \{x_{n^2{}_k}\}_{k < \omega_0}$, there will hold $n_{k_1}^2 < n_{k_2}^2$ and $f_1(x_{n^2{}_{k_1}}) < f_1(x_{n^2{}_{k_2}}), f_2(x_{n^2{}_{k_1}}) < f_2(x_{n^2{}_{k_2}})$ for $k_1 < k_2$. When repeating this proceeding m-times we get on to a set $G^m \subseteqq G$, $G^m = \{x_{n^m{}_k}\}_{k < \omega_0}$, where for $k_1 < k_2$ there holds $n_{k_1}^m < n_{k_2}^m$ and $f_i(x_{n^m{}_{k_1}}) < f_i(x_{n^m{}_{k_2}})$ for all $i = 1, \ldots, m$ which implies $x_{n^m{}_{k_1}} < x_{n^m{}_{k_2}}$ in G, because $\{L_i, f_i \mid i = 1, \ldots, m\}$ is a well realizer of G and this is a contradiction. Thus, wdim $G \geqq \aleph_0$.

5.5. *Let G be the set of all pairs $[x, y]$ where x, y are real numbers ordered in the following way*: $[x_1, y_1] < [x_2, y_2] \Leftrightarrow x_1 = x_2$ *and* $y_1 < y_2$. *Then* wpdim $G = \aleph_0$.

Proof. It is easy to see that $G \cong \sum_{\alpha \in H} G_\alpha$ where H is an antichain with card $H = 2^{\aleph_0}$

and each G_α is a chain with $\bar{G}_\alpha = \lambda$.[4]) We have therefore wdim $H = \aleph_0$ according to 5.4 and wpdim $G_\alpha = \aleph_0$ for every $\alpha \in H$ according to 5.1. Then wpdim $G =$ $=$ wpdim $\sum_{\alpha \in H} =$ sup $\{$wdim H, wpdim $G_\alpha(\alpha \in H)\} = \aleph_0$ according to 4.8.

5.6. Problem. *Let G be an antichain with* card $G = \aleph_\alpha$. *Determine* wdim G.

References

[1] *G. Birkhoff:* Lattice Theory. New York 1948.

[2] *G. Birkhoff:* Generalized Arithmetic. Duke Math. Journ. *9* (1942), 283–302.

[3] *M. M. Day:* Arithmetic of Ordered Systems. Trans. Amer. Math. Soc. *58* (1945), 1–43.

[4] *B. Dushnik - E. W. Miller:* Partially Ordered Sets. Am. Journ. Math. *63* (1941), 600–610.

[5] *T. Hiraguchi:* On the Dimension of Partially Ordered Sets. Sci. Rep. of the Kanazawa Univ. *1* (1951), 77–94.

[6] *T. Hiraguchi:* A note on a Mr. Komm's Theorems. Sci. Rep. of the Kanazawa Univ. *2* (1953), 1–3.

[7] *H. Komm:* On the Dimension of Partially Ordered Sets. Am. Journ. Math. *70* (1948), 507 to 520.

[8] *V. Novák:* O dimensi lexikografického součtu částečně uspořádaných množin. Čas. pěst. mat. *86* (1961), 385–391.

[9] *V. Novák:* On the Pseudodimension of Ordered Sets. Czech. Math. Journ. *13* (1963), 587 to 598.

[10] *V. Novák:* On the ω_ν-dimension and ω_ν-pseudodimension of ordered sets. Ztschr. f. math. Logik und Grundlagen d. Math. *10* (1964), 43–48.

[11] *E. Szpilrajn:* Sur l'extension de l'ordre partiel. Fund. Math. *16* (1930), 386–389.

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[4]) λ denotes the order type of the set of all real numbers with the natural ordering.