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A CHARACTERIZATION OF THE MAXIMAL SUBGROUPS OF THE SEMIGROUP OF $n \times n$ COMPLEX MATRICES

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Introduction. This paper gives a characterization of the maximal subgroups of the multiplicitive semigroup, $\mathscr{G}_n$, of all complex $n \times n$ matrices. In what follows A^* and $R(A)$ will, respectively, denote the conjugate transpose of A and the range space of A for each $A \in \mathscr{G}_n$.

Other developments of the structure of semigroups and, in particular, the structure of certain maximal subgroups of $\mathscr{G}_n$ can be found in [1], [2], [5], and [6]. The following theorem, due to PENROSE [4], will play a vital role in characterizing the maximal groups of $\mathscr{G}_n$.

Theorem 1. *For every complex matrix A, the four equations*

(1) $$AXA = A$$

(2) $$XAX = X$$

(3) $$(AX)^* = AX$$

(4) $$(XA)^* = XA$$

have a unique solution X, denoted $X = A^+$ and called the generalized inverse of A. Moreover, AA^+ and A^+A are, respectively, the orthogonal projection operators on $R(A)$ and $R(A^+) = R(A^*)$.

Main Results. It is well known that each idempotent matrix in $\mathscr{G}_n$ is contained in a unique maximal subgroup of $\mathscr{G}_n$. We will first characterize the maximal subrgroup of $\mathscr{G}_n$ containing a hermitian idempotent $E \in \mathscr{G}_n$ from which we will proceed to characterize the maximal subgroups of $\mathscr{G}_n$ containing non-hermitian idempotents.

Theorem 2. *If* $E^2 = E = E^* \in \mathcal{G}_n$ *then*

i) $\mathcal{H}(E) = \{A \in \mathcal{G}_n : AE = EA$ *and* $R(A) = R(E)\}$ *is the maximal subgroup of* $\mathcal{G}_n$ *containing* E.

ii) *Inversion in* $\mathcal{H}(E)$ *is generalized matrix inversion.*

Proof. Let $H(E)$ denote the maximal subgroup of $\mathcal{G}_n$ containing $E^2 = E = E^* \in \mathcal{G}_n$. We will show that $\mathcal{H}(E)$ is a subgroup of $\mathcal{G}_n$ containing $H(E)$ so that $\mathcal{H}(E) = H(E)$.

To this end, let $A \in H(E)$. Clearly, $AE = EA$ and if $\hat{A}$ denotes the inverse of A in $H(E)$ then $R(E) = R(A\hat{A}) \subset R(A) = R(EA) \subset R(E)$ so that $R(A) = R(E)$ and $H(E) \subset \mathcal{H}(E)$. Now if $A, B \in \mathcal{H}(E)$ then $EAB = AEB = ABE$ and, moreover, since $R(A) = R(B) = R(E)$ and hermitian idempotents are the orthogonal projection operators on their range spaces [3], we may conclude from Theorem 1. that $AA^+ = BB^+ = E$. From this and the fact that A commutes with E, it follows that

$$R(E) = R(A) = R(AE) = R(ABB^+) \subset R(AB) \subset R(A)$$

so that $R(E) = R(A) = R(AB)$ and hence that $\mathcal{H}(E)$ is closed under matrix multiplication.

We will now show that each element $A \in \mathcal{H}(E)$ has an inverse in $\mathcal{H}(E)$ and that this inverse is the generalized inverse of A. First a lemma.

Lemma. *If* $E^2 = E = E^* \in \mathcal{G}_n$ *and* $A \in \mathcal{H}(E)$ *then* $(AE)^+ = EA^+$ *and* $(EA)^+ = A^+E$.

Proof of the Lemma. We need only show that EA^+ and A^+E, respectively, satisfy the four equations of Theorem 1. defining $(AE)^+$ and $(EA)^+$. Indeed,

$$\begin{aligned}
(AE)\,EA^+(AE) &= AA^+AE = AE\,,\\
EA^+(AE)\,EA^+ &= EA^+AA^+ = EA^+\,,\\
[EA^+(AE)]^* &= E^*(A^+A)^*E^* = EA^+(AE)\,,\\
[(AE)\,EA^+]^* &= [AA^+]^* = AA^+ = (AE)\,EA^+
\end{aligned}$$

and similarly $(EA)^+ = A^+E$, Q.E.D.

If $A \in \mathcal{H}(E)$ then $AE = EA = (AA^+)A = A$ so that E is an identity for $\mathcal{H}(E)$. Moreover, the lemma implies that $EA^+ = A^+E = A^+$ so that $R(A^*) = R(A^+) = R(EA^+) \subset R(E) = R(A)$. However, since the respective ranks of A and A^* are equal, $R(A^*)$ cannot be a proper subspace of $R(A)$. It follows that $R(A^+) = R(A^*) = R(A) = R(E)$.

The fact that $R(A^+) = R(A^*) = R(A) = R(E)$, together with Theorem 1., implies that $E = AA^+ = A^+A$. This completes the proof that $\mathcal{H}(E)$ is a group containing $H(E)$. The theorem follows using the maximality of $H(E)$.

We note that * defines an involution on $\mathscr{G}_n$ and that in the course of the proof it was shown that $\mathscr{H}(E) = H(E)$ is self involutory i.e., $A \in \mathscr{H}(E)$ implies $A^* \in \mathscr{H}(E)$.

The following corollary characterizes the maximal groups containing non-hermitian idempotents.

Corollary. *G is a maximal subgroup of $\mathscr{G}_n$ if and only if $G = P\,\mathscr{H}(E)\,P^{-1}$ for some $E^2 = E = E^* \in \mathscr{G}_n$ and some nonsingular $P \in \mathscr{G}_n$.*

Proof. Let G be a maximal subgroup of $\mathscr{G}_n$ with identity $F^2 = F$. Let $E^2 = E = E^*$ denote the orthogonal projection on $R(F)$. Since F and E are idempotent and have the same range, F is similar to E and there exists a nonsingular $P \in \mathscr{G}_n$ such that $F = PEP^{-1}$. Moreover, $P\,\mathscr{H}(E)\,P^{-1}$ is the isomorphic image of a group and hence is itself a group containing F. Theorem 1. implies the maximality of $\mathscr{H}(E)$ which, in turn, implies that maximality of $P\,\mathscr{H}(E)\,P^{-1}$. It follows that $G = P\,\mathscr{H}(E)\,P^{-1}$.

The converse is obvious.

We note, in the corollary, that the group inverse of $B = PAP^{-1} \in P\,\mathscr{H}(E)\,P^{-1}$ is PA^+P^{-1} and, moreover, that if P is orthogonal then $P\,\mathscr{H}(E)\,P^{-1} = \mathscr{H}(E)$.

Finally, we note that the theorem and corollary account for all maximal subgroups of $\mathscr{G}_n$. In fact, if we define an equivalence relation $A \sim B$ if and only if $R(A) = R(B)$, we see that each subspace of the n-dimensional complex Euclidean space gives rise to an equivalence class, namely, all of the elements of $\mathscr{G}_n$ equivalent to the orthogonal projection on that subspace. Indeed, these subspaces exhaust the equivalence classes. In [7], a brief discussion concerning the $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$-classes of $\mathscr{G}_n$ is given with reference to the generalized inverse.

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