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DECOMPOSITIONS OF THE PLANE INTO SETS, AND COVERINGS OF THE PLANE WITH CURVES*)

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This paper provides complete answers, involving the position of the cardinal number of the continuum in the scale of alephs, to the following two questions concerning the plane.

Let s and t be integers with $s \geqq 2$ and $t \geqq 0$. Given s directions in the plane, can the plane be decomposed into s sets such that every line having the j th of the s given directions intersects the j th set in less than $\aleph_t$ points?

The answer is: if, and only if, $2^{\aleph_0} \leqq \aleph_{s+t-2}$.

The plane is not the union of finitely many curves. It is, however, the union of enumerably many curves, but the "y-axes" of these curves may make up enumerably many different directions. Is the plane the union of at most $\aleph_t$ curves, each of which has its "y-axis" in one of s given directions?

The aswer is: if, and only if, $2^{\aleph_0} \leqq \aleph_{s+t-1}$.

We now proceed to a more precise and formal treatment of these matters.

Denote by P the set of all points in the Euclidean plane. Supposet that $\theta_1, \theta_2, \ldots$ is an ordinary finite or infinite sequence of distinct unsensed directions in the plane, and that $\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots$ are cardinal numbers. We define the relation

$$P = E_1(\theta_1; < \boldsymbol{m}_1) \cup E_2(\theta_2; < \boldsymbol{m}_2) \cup \ldots$$

to mean that P is the union of the sets $E_1, E_2, \ldots$, where, for $j = 1, 2, \ldots$, E_j intersects every straight line with direction θ_j in fewer than $\boldsymbol{m}_j$ points.

Consider the following propositions, where n is a natural number and $k = 0, 1, 2, \ldots, n + 1$:

(H_n) $2^{\aleph_0} \leqq \aleph_n$;

(Q_n^k) $P = E_1(\theta_1; < \aleph_k) \cup E_2(\theta_2; < \aleph_k) \cup \ldots \cup E_{n+2-k}(\theta_{n+2-k}; < \aleph_k)$;

(B_n^k) $P = E_1(\theta_1; < \aleph_k) \cup E_2(\theta_2; < \aleph_{k+1}) \cup \ldots \cup E_{n+2-k}(\theta_{n+2-k}; < \aleph_{n+1})$.

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We are going to prove the following theorems concerning decompositions of the plane:

Theorem 1. *Let n be a natural number, and suppose that $\theta_1, \theta_2, \ldots, \theta_{n+2}$ are $n + 2$ distinct directions in the plane. Then*

$$(H_n) \Rightarrow (Q_n^k) \quad (k = 0, 1, \ldots, n + 1).$$

Theorem 2. *Let n be natural number, k be any one of the numbers* $0, 1, \ldots, n + 1$, *and $\theta_1, \theta_2, \ldots, \theta_{n+2-k}$ be $n + 2 - k$ distinct directions in the plane. Then*

$$(B_n^k) \Rightarrow (H_n).$$

Since it is evident that $(Q_n^k) \Rightarrow (B_n^k)$, we have, as a consequence of these theorems,

Corollary 1. $(H_n) \Leftrightarrow (Q_n^k)$ $(n = 1, 2, \ldots; k = 0, 1, \ldots, n + 1)$.

For $k = 0$, Theorem 1 becomes a theorem proved by Davies [2, p. 278].

For $n = 1$ and $k = 1$, Corollary 1 reduces essentially to a result obtained by Sierpiński [5, pp. 9, 10].

For $n = 2$ and $k = 1$, Theorem 1 is formally analogous to a theorem about Euclidean three-dimensional space proved by Sierpiński [6, p. 6, Theorem 3].

For $k = 0$, Theorem 2 is a special case of a theorem proved by Bagemihl [1, Theorem 1] which in turn generalizes a result due to Davies [2, p. 277].

Call a set C of points in the plane a *curve*, if every line with some fixed direction θ intersects C in exactly one point; we shall then call θ an *axial direction* of C.

Mazurkiewicz proved [4] that P is not the union of finitely many curves.

Proposition (Q_1^1) is equivalent (see [5, pp. 11, 12]) to the assertion that, if θ_1, θ_2 are two distinct directions, then P is the union of enumerably many curves, each of which has either θ_1 or θ_2 as an axial direction; this assertion, in turn, is equivalent [5, p. 12] to (H_1), in view of Corollary 1 for $n = 1$ and $k = 1$.

Davies has shown [3], without the use of any assumption concerning $2^{\aleph_0}$, that P is the union of enumerably many curves.

Now we observe that for $k = 1, 2, \ldots, n + 1$ the proposition (Q_n^k) is equivalent to the following proposition:

(C_n^k) *P is the union of at most $\aleph_{k-1}$ curves, each of which has one of $\theta_1, \theta_2, \ldots, \theta_{n+2-k}$ as an axial direction.*

Hence, in view of Corollary 1, we have

Corollary 2. $(H_n) \Leftrightarrow (C_n^k)$ $(n = 1, 2, \ldots; k = 1, 2, \ldots, n + 1)$.

If we take $k = 1$ in Corollary 2, and take into account the theorem of Mazurkiewicz quoted above, we obtain the following result about covering the plane with enumerably many curves:

Corollary 3. *For $n = 1, 2, 3, \ldots$, P is the union of enumerably many curves, each of which has one of $n + 1$ distinct directions as an axial direction, if, and only if, (H_n) is true.*

For $n = 1$, Corollary 3 reduces to the second result about curves quoted above.

We turn now to the proofs of Theorems 1 and 2.

Proof of Theorem 1. As we remarked earlier, the case $k = 0$ has already been proved. Furthermore, for $k = n + 1$, Theorem 1 is obviously true. Hence we may assume that $1 \leqq k \leqq n$.

As we noted before, the theorem is true for $n = 1$. Suppose now that $n > 1$ and that we have proved the validity of the implication

$$(H_m) \Rightarrow (Q_m^k) \quad (k = 1, \ldots, m)$$

for every natural number $m < n$. We shall show that

$$(H_n) \Rightarrow (Q_n^k) \quad (k = 1, \ldots, n),$$

and this will complete the proof of Theorem 1 by induction.

Instead of assuming (H_n), we may assume that $2^{\aleph_0} = \aleph_n$. For if $2^{\aleph_0} < \aleph_n$, then (H_{n-1}) is true; in view of our induction hypothesis, (Q_{n-1}^{k-1}) is true, for $k = 1, \ldots, n$; and evidently (Q_{n-1}^{k-1}) implies (Q_n^k) $(k = 1, \ldots, n)$.

Assume, then, that $2^{\aleph_0} = \aleph_n$. For $k = n$, (Q_n^k) asserts that

$$P = E_1(\theta_1; < 2^{\aleph_0}) \cup E_2(\theta_2; < 2^{\aleph_0}),$$

and (essentially) according to Sierpiński [5, p. 9, Lemma], this is true. Hence, we may further restrict ourselves to establishing the truth of (Q_n^k) for $k = 1, \ldots, n - 1$.

The remainder of the proof is essentially an appropriate elaboration of an argument given by Davies [2, pp. 278–280].

Fix k in the range $1 \leqq k \leqq n - 1$. A line in the plane is called *special* provided that it has one of the directions $\theta_1, \ldots, \theta_{n+2-k}$. A set N of special lines is called a *network* provided that whenever two of the special lines through a point p belong to N so do all the special lines through p. As Davies shows [2, p. 278, Lemma 1], if M is an infinite set of special lines, then the smallest network N containing M exists and is a set having the same cardinal number as M.

We now prove the following

Lemma. *Let m be an integer satisfying $k \leqq m \leqq n$. If N is a network whose cardinal number is $\aleph_m$, then N can be ordered by a relation $\prec$ with the following property:*

If $l \in N$, then there exist at most $\aleph_{k-1}$ systems of $m - k + 1$ elements $l_1, \ldots, l_{m-k+1}$ of N such that $l, l_1, \ldots, l_{m-k+1}$ are concurrent and

$$l_{m-k+1} \prec \ldots \prec l_1 \prec l.$$

We prove this lemma by induction on m.

If N is a network whose cardinal number is $\aleph_k$, then N can be well-ordered by some relation $\prec$ as a transfinite sequence of type ω_k:

$$k_0 \prec k_1 \prec \ldots \prec k_\xi \prec \ldots \quad (\xi < \omega_k)\,.$$

If $l \in N$, then $l = k_\eta$ for some $\eta < \omega_k$. Hence, there exist at most $\aleph_{k-1}$ systems of one element $l_1 \in N$ for which $l_1 \prec l$, namely the elements k_ξ of N with $\xi < \eta$. This proves the lemma for $m = k$.

Now suppose the lemma is true for some m satisfying $k \leqq m < n$. Let N be a network whose cardinal number is $\aleph_{m+1}$. Then N can be well-ordered as a transfinite sequence of type ω_{m+1}:

$$k_0, k_1, \ldots, k_\xi, \ldots \quad (\xi < \omega_{m+1})\,.$$

For every ordinal number α satisfying $\omega_m \leqq \alpha < \omega_{m+1}$, denote by $N(\alpha)$ the smallest network containing all the lines k_β $(\beta \leqq \alpha)$. Then the cardinal number of $N(\alpha)$ is $\aleph_m$, and because of our current supposition, $N(\alpha)$ can be ordered by a relation $\prec_\alpha$ possessing the property stated in the lemma. Given any line $k \in N$, denote by $k(\alpha)$ the least ordinal number α satisfying $\omega_m \leqq \alpha < \omega_{m+1}$ for which $k \in N(\alpha)$. For any two distinct lines g, h in N, write $g \prec h$ provided that either $\alpha(g) < \alpha(h)$ or $\alpha(g) = = \alpha(h) = \alpha$ and $g \prec_\alpha h$. Then the relation $\prec$ orders N.

To complete the proof of the lemma, let $l \in N$, and let $l_1, \ldots, l_{m-k+2}$ be a system of $m - k + 2$ elements of N such that $l, l_1, \ldots, l_{m-k+2}$ are concurrent and

$$l_{m-k+2} \prec l_{m-k+1} \prec \ldots \prec l_1 \prec l\,.$$

According to the definition of the relation $\prec$, we must have

$$\alpha(l_{m-k+2}) \leqq \alpha(l_{m-k+1}) \leqq \ldots \leqq \alpha(l_1) \leqq \alpha(l)\,.$$

The first inequality implies that $N(\alpha(l_{m-k+2})) \subseteq N(\alpha(l_{m-k+1}))$, so that both l_{m-k+2} and l_{m-k+1} belong to $N(\alpha(l_{m-k+1}))$, and since this set is a network, it contains all the special lines through the point $l_{m-k+2} \cap l_{m-k+1}$. Hence $l \in N(\alpha(l_{m-k+1}))$, which implies that $\alpha(l) \leqq \alpha(l_{m-k+1})$. But then

$$\alpha(l_{m-k+1}) = \ldots = \alpha(l_1) = \alpha(l)\,.$$

If we set $\alpha(l) = \alpha$, then all the concurrent lines $l, l_1, \ldots, l_{m-k+1}$ belong to $N(\alpha)$, and it follows from the definition of $\prec$ that

$$l_{m-k+1} \prec_\alpha \ldots \prec_\alpha l_1 \prec_\alpha l\,.$$

Since the relation $\prec_\alpha$ possesses the property stated in the Lemma, there are at most $\aleph_{k-1}$ such systems $l_1, \ldots, l_{m-k+1}$, and for each such system, there are only finitely many special lines l_{m-k+2} through their point of intersection. This completes the induction.

Now to finish the proof of Theorem 1, we define the sets E_j ($j = 1, \ldots, n + 2 - k$). The set of all special lines in the plane is a network N, and our assumption that $2^{\aleph_0} = \aleph_n$ implies that the cardinal number of this network is $\aleph_n$. According to the lemma with $m = n$, N can be ordered by a relation $\prec$ possessing the property described in the lemma. If $p \in P$, denote by $p(\theta)$ the line through p with direction θ. We assign p to the set E_j provided that

$$p(\theta_i) \prec p(\theta_j) \quad (i = 1, \ldots, n + 2 - k;\ i \neq j)\,.$$

Then

$$P = \bigcup_{j=1}^{n+2-k} E_j\,.$$

Suppose finally that l is any special line. Then l has a direction θ_j, where j is one of the numbers $1, \ldots, n + 2 - k$. If $l \cap E_j \neq \emptyset$, let $p \in l \cap E_j$. Then $l = p(\theta_j)$, and hence by the definition of E_j, if the $n + 1 - k$ lines $p(\theta_i)$ ($i = 1, \ldots, n + 2 - k$; $i \neq j$) are suitably labeled $l_1, \ldots, l_{n-k+1}$, then $l, l_1, \ldots, l_{n-k+1}$ are concurrent and

$$l_{n-k+1} \prec \ldots \prec l_1 \prec l\,.$$

By the lemma, there are at most $\aleph_{k-1}$ such systems $l_1, \ldots, l_{n-k+1}$, and hence there are at most $\aleph_{k-1}$ points $p \in l \cap E_j$. But this means that (Q_n^k) is true, and Theorem 1 is proved.

Proof of Theorem 2. As we have already remarked, Theorem 2 is already known to be true for $k = 0$, so that we have

$$(B_n^0) \Rightarrow (H_n)\,.$$

Assume that k is one of the numbers $1, 2, \ldots, n + 1$, and that (B_n^k) is true. This means that

$$P = E_1(\theta_1;\ < \aleph_k) \cup E_2(\theta_2;\ < \aleph_{k+1}) \cup \ldots \cup E_{n+2-k}(\theta_{n+2-k};\ < \aleph_{n+1})\,.$$

Let $\theta_{n+3-k}, \theta_{n+4-k}, \ldots, \theta_{n+1}, \theta_{n+2}$ be k distinct directions in the plane, each of which is different from every one of the directions $\theta_1, \theta_2, \ldots, \theta_{n+2-k}$, and let the k sets

$$F_1 = F_2 = \ldots = F_k = \emptyset\,.$$

Then

$$P = F_1(\theta_{n+3-k};\ < 1) \cup F_2(\theta_{n+4-k};\ < 1) \cup \ldots \cup F_k(\theta_{n+2};\ < 1) \cup E_1(\theta_1;\ < \aleph_k) \cup \cup E_2(\theta_2;\ < \aleph_{k+1}) \cup \ldots \cup E_{n+2-k}(\theta_{n+2-k};\ < \aleph_{n+1})\,,$$

which implies that

$$P = F_1(\theta_{n+3-k};\ < \aleph_0) \cup F_2(\theta_{n+4-k};\ < \aleph_1) \cup \ldots \cup F_k(\theta_{n+2};\ < \aleph_{k-1}) \cup \cup E_1(\theta_1;\ < \aleph_k) \cup E_2(\theta_2;\ < \aleph_{k+1}) \cup \ldots \cup E_{n+2-k}(\theta_{n+2-k};\ \aleph_{n+1})\,,$$

and since this asserts that (Q_n^0) is true, it follows that (H_n) is true.

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