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ON SEQUENTIAL ENVELOPES DEFINED BY MEANS OF CERTAIN CLASSES OF CONTINUOUS FUNCTIONS

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In this paper the $\mathscr{F}_0$ sequentially regular convergence spaces L as well as the $\mathscr{F}_0$ sequential envelopes of such spaces are defined and the existence of such envelopes proved, $\mathscr{F}_0$ being a subclass of the class of all functions continuous on L. The theory of $\mathscr{F}_0$ sequential envelope is applied to algebras of sets $\mathbf{A}$ in the case when $\mathscr{F}_0$ is the class of all probability measures on $\mathbf{A}$.

According to Čech-Stone compactification theorem each continuous function f defined on a completely regular space P such that $0 \leqq f(x) \leqq 1$, $x \in P$, can be continuously extended on a compactification $\beta(P)$ of P. It is well known that each probability measure defined on an algebra of sets $\mathbf{A}$ can be extended onto the σ-algebra $\mathbf{S}(\mathbf{A})$ generated by $\mathbf{A}$. The problem [1] arises as follows: To define analogous notions as complete regularity and Čech-Stone compactification for systems of sets in order to get $\mathbf{S}(\mathbf{A})$ as an envelope of $\mathbf{A}$. The solution of this problem leads to the notion of $\mathscr{F}_0$ sequentially regular convergence spaces and $\mathscr{F}_0$ sequential envelopes of such spaces.

In the section I the $\mathscr{F}_0$ sequential regularity of a convergence space L is defined, $\mathscr{F}_0$ being a subclass of the class $\mathscr{F}$ of all continuous functions on L. Further the definition of an $\mathscr{F}_0$ sequential envelope is given and it is proved that each $\mathscr{F}_0$ sequentially regular space has an $\mathscr{F}_0$ sequential envelope. In the section II it is shown that each algebra of sets $\mathbf{A}$ is a $\mathscr{P}$ sequentially regular convergence space, $\mathscr{P}$ denoting the class of all probability measures, and it is proved that the σ-algebra $\mathbf{S}(\mathbf{A})$ generated by $\mathbf{A}$ is a $\mathscr{P}$ sequential envelope of $\mathbf{A}$. An example of a set algebra $\mathbf{F}$ is given showing that the $\mathscr{F}$ sequential envelope can substantially differ from the $\mathscr{P}$ sequential envelope of $\mathbf{F}$.

I.

A convergence space $(L, \mathfrak{L}, \lambda)$ is a point set L on which a closure operation λ is defined by means of a convergence $\mathfrak{L}$ on L. The convergence $\mathfrak{L}$ is the set of elements

$(\{x_n\}, x)$ where $\{x_n\}$ is a sequence of points $x_n \in L$ and $x \in L$, fulfilling the properties:

$(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$ implies $x = y$,

$(\{x\}, x) \in \mathfrak{L}$ for each $x \in L$,

$(\{x_n\}, x) \in \mathfrak{L}$ implies $(\{x_{n_i}\}, x) \in \mathfrak{L}$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Instead of $(\{x_n\}, x) \in \mathfrak{L}$ we write $\lim x_n = x$. The closure λA of a set $A \subset L$ is the set of all points $\lim x_n \in L$ such that $\bigcup x_n \subset A$. It is easy to see that $\lambda x = x$ for each $x \in L$, $\lambda(A \cup B) = \lambda A \cup \lambda B$ and $A \subset B$ implies $A \subset \lambda A \subset \lambda B$. In convergence spaces the axiom of the closed closure $\lambda\lambda A = \lambda A$ need not be satisfied. It is possible to construct non decreasing sequences of successive closures $\lambda^\xi A$ where $\lambda^\xi A = \bigcup_{\eta < \xi} \lambda\lambda^\eta A$ and $\lambda^0 A = A$. The set $\lambda^{\omega_1} A$ is the smallest closed set containing A as a subset, ω_1 being the least uncountable ordinal. We say that A is sequentially dense in B if $\lambda^{\omega_1} A = B$.

Examples. The sets R_0 of rational and R of real numbers with the usual convergence. The convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ (of finite or infinite dimension) with the coordinatewise convergence on E defined by means of the usual convergence on R. The system of sets $(\mathbf{X}, \mathfrak{L}, \lambda)$ with the convergence $\mathfrak{L}$ consisting of all elements $(\{A_n\}, A)$ such that $A = \bigcup_{k \geq 1} \bigcap_{n \geq k} A_n = \bigcap_{k \geq 1} \bigcup_{n \geq k} A_n$.

A map g on a convergence space $(L, \mathfrak{L}, \lambda)$ into a convergence space $(M, \mathfrak{M}, \mu)$ is continuous at a point $x_0 \in L$ if $\lim x_n = x_0$ in L implies $\lim g(x_{n_i}) = g(x)$ in M, $\{x_{n_i}\}$ being a suitable subsequence of $\{x_n\}$. It is easy to see that a real valued function f on L into R is continuous on L if and only if $\lim x_n = x$ implies $\lim f(x_n) = f(x)$ for each point $x \in L$. The class of all continuous real valued functions on L will be denoted $\mathscr{F}(L)$ or simply $\mathscr{F}$.

In [1] I defined the notion of a sequentially regular space. Now we are going to generalize this notion as follows[1]). Let L be a convergence space and $\mathscr{F}_0$ a subclass of $\mathscr{F}(L)$. The space L is $\mathscr{F}_0$ sequentially regular if for any point $x_0 \in L$ and any sequence of points $x_n \in L$ no subsequence of which converges to x_0 there is a continuous function $f \in \mathscr{F}_0$ such that $\{f(x_n)\}$ does not converge to $f(x_0)$. Now we shall prove

Theorem 1. *Each $\mathscr{F}_0$ sequentially regular space is homeomorphic to a subspace of the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of the dimension*[2]) card $\mathscr{F}_0$.

Proof. Let $\mathscr{F}_0$ consist of all f_α, $\alpha \in I$, I being an index set of the same power as $\mathscr{F}_0$. Consider the map $\varphi_0(x) = (f_\alpha(x))_{\alpha \in I}$ on L into $(E, \mathfrak{E}, \varepsilon)$. Then φ_0 is a homeomorphism on L onto the subspace $\varphi_0(L)$ of $(E, \mathfrak{E}, \varepsilon)$. The proof of this assertion is analogous as in [1] and may be omitted.

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[1]) Instead of $\mathscr{F}_0$ the Greek letter α is used in [1]. If $\mathscr{F}_0 = \mathscr{F}$, then the symbol $\mathscr{F}_0$ will be omitted and we shall speak simply of a sequentially regular space instead of an $\mathscr{F}$ sequentially regular space; the same concerns the $\mathscr{F}$ sequential envelope.

[2]) The power of a set A will be denoted by card A.

The homeomorphism φ_0 will be called an $\mathscr{F}_0$ homeomorphism[3]) and always denoted by a thick Greek letter.

In [1] I defined the sequential envelope of a sequentially regular space $(L, \mathfrak{L}, \lambda)$ to be a largest sequentially regular overspace S of L such that L is sequentially dense in S and each continuous function on L can be continuously extended onto S. Now we shall generalize this definition.

Definition. Let $(L, \mathfrak{L}, \lambda)$ be an $\mathscr{F}_0$ sequentially regular space. Let $(S, \mathfrak{S}, \sigma)$ be a convergence space. We say that S is an $\mathscr{F}_0$ *sequential envelope of the space* L if

1° L is a sequentially dense subspace of S.

2° Each continuous function $f \in \mathscr{F}_0(L)$ can be extended to a continuous function $\bar{f} \in \mathscr{F}(S)$ and the space S is $\overline{\mathscr{F}}_0(S)$ sequentially regular, $\overline{\mathscr{F}}_0(S)$ being the class of all $\bar{f} \in \mathscr{F}(S)$ such that $\bar{f} \mid L \in \mathscr{F}_0(L)$.

3° There is no convergence space $(T, \mathfrak{T}, \tau)$ containing S as a proper subspace and fulfilling 1° and 2° with regard to L and T.

Now we shall proceed analogously as in [1] to show that each $\mathscr{F}_0$ sequentially regular space has an $\mathscr{F}_0$ sequential envelope. The proofs will be shortened accordingly.

Theorem 2. *Let $(L, \mathfrak{L}, \lambda)$ be an $\mathscr{F}_0$ sequentially regular space. Let φ_0 be an $\mathscr{F}_0$ homeomorphism on L into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of the dimension* card $\mathscr{F}_0$. *Let the space L be sequentially dense in a convergence overspace $(S, \mathfrak{S}, \sigma)$. Then* 2° *holds true if and only if there is a homeomorphism h on S into $\varepsilon^{\omega_1}\varphi_0(L)$ such that $h(x) = \varphi_0(x)$, $x \in L$.*

Proof. Let 2° hold. Since $\varphi_0(L) = \{(f_\alpha(x)) \in E : f_\alpha \in \mathscr{F}_0(L),\ x \in L,\ \alpha \in I\}$ and because there is a one-to-one correspondence on $\mathscr{F}_0(L)$ onto $\overline{\mathscr{F}}_0(S)$ (a function $g \in \overline{\mathscr{F}}_0(S)$ corresponds to $f \in \mathscr{F}_0(L)$ if $g \mid L = f$) there is an $\overline{\mathscr{F}}_0$ homeomorphism ψ_0 on S onto $\psi_0(S) = \{(g_\alpha(x)) \in E : g_\alpha \in \overline{\mathscr{F}}_0(S),\ x \in S,\ \alpha \in I\}$ such that $\psi_0(x) = \varphi_0(x)$, $x \in L$, g_α being the corresponding continuous extension of f_α, $\alpha \in I$. Using the method of transfinite induction it is easy to prove that $\psi_0(S) \subset \varepsilon^{\omega_1} \varphi_0(L)$. Consequently it suffices to put $h = \psi_0$.

Now, let h be a homeomorphism on S into $\varepsilon^{\omega_1} \varphi_0(L)$ such that $h(x) = \varphi_0(x)$, $x \in L$. If $f_{\alpha'} \in \mathscr{F}_0(L)$, then the function ph on S is a continuous extension of the function $f_{\alpha'}$, p being a projection function: $p((z_\alpha)) = z_{\alpha'}$, for each $(z_\alpha) \in \varepsilon^{\omega_1} \varphi_0(L)$. The $\overline{\mathscr{F}}_0$ sequential regularity of the space S remains to be proved. Evidently, it suffices to show that $h(x) = (\bar{f}_\alpha(x))$, $x \in S$, $\alpha \in I$, where $\bar{f}_\alpha \in \overline{\mathscr{F}}_0$ and $\bar{f}_\alpha$ corresponds to $f_\alpha \in \mathscr{F}_0(L)$.

Suppose (transfinite induction) that $h(x) = (\bar{f}_\alpha(x))$, $x \in \sigma^\eta L$, for all $\eta < \xi$, where $0 < \xi \leqq \omega_1$. Let y be any point belonging to the set $\sigma^\xi L - \bigcup_{\eta<\xi} \sigma^\eta L$. Then ξ is isol-

[3]) If $\mathscr{F}_0 = \mathscr{F}$, then in [1] an $\mathscr{F}_0$ homeomorphism is called a special homeomorphism.

ated and there is a sequence of points $y_n \in \sigma^{\xi-1}L$ such that $\lim y_n = y$; therefore $\lim h(y_n) = h(y) \in E$. Denote $h(y) = (t_\alpha)$. Since $h(y_n) = (\bar{f}_\alpha(y_n))$, it follows $\lim \bar{f}_\alpha(y_n) = t_\alpha$, $\alpha \in I$. On the other hand, $\lim \bar{f}_\alpha(y_n) = \bar{f}_\alpha(y)$ and so $h(y) = (\bar{f}_\alpha(y))$.

Theorem 3. *Let $(L, \mathfrak{L}, \lambda)$ be an $\mathscr{F}_0$ sequentially regular space. Let $\varphi_0(x)$, $x \in L$, be an $\mathscr{F}_0$ homeomorphism into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of the dimension* card $\mathscr{F}_0$. *Let L be a sequentially dense subspace of a convergence space $(S, \mathfrak{S}, \sigma)$. Then S is an $\mathscr{F}_0$ sequential envelope of L if and only if there is a homeomorphic map h on S onto $\varepsilon^{\omega_1} \varphi_0(L)$ such that $h(x) = \varphi_0(x)$, $x \in L$.*

Proof. The necessity. From Theorem 2 it follows that there is a homeomorphism h on S into $\varepsilon^{\omega_1} \varphi_0(L)$ such that $h(x) = \varphi_0(x)$, $x \in L$. We are to prove that $h(S) = \varepsilon^{\omega_1} \varphi_0(L)$. Suppose that, on the contrary, $\varepsilon^{\omega_1} \varphi_0(L) - h(S) \neq \emptyset$. Let γ be the least ordinal such that there is a point $b \in \varepsilon^\gamma \varphi_0(L) - h(S)$. Add a new element a to the set S, denote $S' = S \cup a$, put $h'(x) = h(x)$, $x \in S$ and $h'(a) = b$ and define the convergence $\mathfrak{S}'$ on $S' : (\{x_n\}, x) \in \mathfrak{S}'$ if $\lim h'(x_n) = h'(x)$ in E. Then h' is a homeomorphism on S' into $\varepsilon^{\omega_1} \varphi_0(L)$ such that $h'(x) = \varphi_0(x)$, $x \in L$. It is easy to see that L is a sequentially dense subspace in $(S', \mathfrak{S}', \sigma')$. Consequently 1° and also 2° (by Theorem 2) hold with regard to L and S'. This contradicts 3°.

The sufficiency. 1° holds by the supposition and 2° by Theorem 2. Suppose (indirect proof) that 3° is not fulfilled. Then there is a convergence overspace $(\bar{S}, \bar{\mathfrak{S}}, \bar{\sigma})$ of $(S, \mathfrak{S}, \sigma)$, $\bar{S} \neq S$, fulfilling 1° and 2° with regard to L and $\bar{S}$. By Theorem 2, there is a homeomorphism $\bar{h}(x)$ on $\bar{S}$ into $\varepsilon^{\omega_1} \varphi_0(L)$ such that $\bar{h}(x) = \varphi_0(x)$, $x \in L$. Then $\bar{h}(x) = h(x)$, $x \in S$. As a matter of fact, assume that $\bar{h}(x) = h(x)$, $x \in \sigma^\xi L$, for each $\xi < \zeta$ where $0 < \zeta \leqq \omega_1$. If $x_0 \in \sigma^\zeta L - \sigma^{\zeta-1}L$, then there is a sequence of points $x_n \in \sigma^{\zeta-1}L$ such that $(\{x_n\}, x_0) \in \mathfrak{S}$; consequently $h(x_0) = \lim h(x_n) = \lim \bar{h}(x_n) = \bar{h}(x_0)$. Since $S \subset \bar{S} = \bar{\sigma}^{\omega_1}L \neq S$ and $L \subset S \cap \bar{S}$, there is a point $\bar{a} \in \bar{S} - S$ and a sequence of points $t_n \in \bar{S} \cap S$ such that $(\{t_n\}, \bar{a}) \in \bar{\mathfrak{S}}$. Denote $b = \bar{h}(\bar{a})$. Then $b \in \varepsilon^{\omega_1} \varphi_0(L)$ and $b = \lim \bar{h}(t_n) = \lim h(t_n)$. Since h^{-1} is continuous on $\varepsilon^{\omega_1} \varphi_0(L)$, it follows that $h^{-1}(b) \in \sigma \bigcup_{n=1}^{\infty} t_n = S \cap \sigma \bigcup_{n=1}^{\infty} t_n$ so that $(\{t_{n_i}\}, h^{-1}(b)) \in \bar{\mathfrak{S}}$ for a suitable subsequence $\{t_{n_i}\}$ of $\{t_n\}$. However, $(\{t_{n_i}\}, \bar{a}) \in \bar{\mathfrak{S}}$ and so $\bar{a} = h^{-1}(b)$. This is a contradiction.

Theorem 4. *Let $(L, \mathfrak{L}, \lambda)$ be an $\mathscr{F}_0$ sequentially regular space. Then there exists an $\mathscr{F}_0$ sequential envelope of L.*

Proof. Let S be a point set containing L as a subset and such that card $(S - L) =$ card $(\varepsilon^{\omega_1} \varphi_0(L) - \varphi_0(L))$, φ_0 being an $\mathscr{F}_0$ homeomorphism. Let g be a one-to-one map of S onto $\varepsilon^{\omega_1} \varphi_0(L)$ such that $g(x) = \varphi_0(x)$, $x \in L$. Define the convergence $\mathfrak{S}$ on $S : (\{x_n\}, x) \in \mathfrak{S}$ if $\lim g(x_n) = g(x)$ in $\varepsilon^{\omega_1} \varphi_0 L$. Then g is a homeomorphism on $(S, \mathfrak{S}, \sigma)$ onto $\varepsilon^{\omega_1} \varphi_0 L$ and $(S, \mathfrak{S}, \sigma)$ is an $\mathscr{F}_0$ sequential envelope of $(L, \mathfrak{L}, \lambda)$, by Theorem 3.

Theorem 5. *Let $(S_1, \mathfrak{S}_1, \sigma_1)$ and $(S_2, \mathfrak{S}_2, \sigma_2)$ be $\mathscr{F}_0$ sequential envelopes of an $\mathscr{F}_0$ sequentially regular space $(L, \mathfrak{L}, \lambda)$. Then there is a homeomorphism h on S_1 onto S_2 such that $h(x) = x$, $x \in L$.*

Proof. According to Theorem 3 there are homeomorphisms h_i on S_i onto $\varepsilon^{\omega_1} \varphi_0(L)$ such that $h_i(x) = \varphi_0(x)$, $x \in L$, $i = 1, 2$, where φ_0 denotes an $\mathscr{F}_0$ homeomorphism on L onto $\varphi_0(L)$. Consequently it suffices to put $h = h_2^{-1}h_1$.

Theorem 6. *Let $(L, \mathfrak{L}, \lambda)$ be an $\mathscr{F}_0$ sequentially regular space. Let $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \subset \mathscr{F}(L)$. Let $(S_0, \mathfrak{S}_0, \sigma_0)$ and $(S_1, \mathfrak{S}_1, \sigma_1)$ be $\mathscr{F}_0$ and $\mathscr{F}_1$ sequential envelopes of L. Then there is a continuous map m on S_1 into S_0 such that $m(x) = x$, $x \in L$.*

Proof. Let $\mathscr{F}_i = \{f_\alpha; \alpha \in I_i\}$, $i = 0, 1$, where $I_0 \subset I_1$. Let φ_i be an $\mathscr{F}_i$ homeomorphism on L onto $\varphi_i(L) \subset E_i$, $(E_i, \mathfrak{E}_i, \varepsilon_i)$ being the convergence Euclidean space of the dimension card I_i. By Theorem 3, there is a homeomorphism h_i on S_i onto $\varepsilon_i^{\omega_1} \varphi_i(L)$ such that $h_i(x) = \varphi_i(x)$, $x \in L$, $i = 0, 1$. It suffices to put $m(x) = h_0^{-1}\pi\, h_1(x)$, $x \in S_1$, where π denotes the projection map on $\varepsilon_1^{\omega_1} \varphi_1(L)$ onto $\varepsilon_0^{\omega_1} \varphi_0(L)$.

II.

Let X be a point set and $\mathbf{X}$ the system of all subsets of X. Denote $(\mathbf{X}, \mathfrak{L}, \lambda)$ the convergence space, $\mathfrak{L}$ being the usual convergence of sets. Let $\mathbf{A}$ be an algebra of sets on X (i.e. $X \in \mathbf{A}$) and $\mathbf{S}(\mathbf{A})$ the σ-algebra generated by $\mathbf{A}$. Since both $\mathbf{S}(\mathbf{A})$ and $\lambda^{\omega_1}\mathbf{A}$ are convergence subspaces of $\mathbf{X}$ and both are the smallest closed sets in $(\mathbf{X}, \mathfrak{L}, \lambda)$ containing $\mathbf{A}$ as a subset, evidently [2] $\mathbf{S}(\mathbf{A}) = \lambda^{\omega_1}\mathbf{A}$. Consequently, the algebra $\mathbf{A}$ is a sequentially dense subspace of the σ-algebra $\mathbf{S}(\mathbf{A})$. Denote $\mathscr{P}$, or more precisely $\mathscr{P}(\mathbf{A})$, the class of all probability measures defined on the algebra $\mathbf{A}$. It is known [2] that $\mathscr{P} \subset \mathscr{F}(\mathbf{A})$.

Lemma 1. *Each algebra of sets is a $\mathscr{P}$ sequentially regular space.*

Proof. Let A_0 be an element and $\{A_n\}$ a sequence of elements of an algebra of sets $\mathbf{A}$ not converging to A_0. Choose a point $x_0 \in (A \div \text{Lim sup}\, A_n) \cup (A \div \div \text{Lim inf}\, A_n)$. Then the characteristic function $c_A(x_0)$, $A \in \mathbf{A}$, is a probability measure on $\mathbf{A}$ such that $\{c_{A_n}(x_0)\}_{n=1}^{\infty}$ does not converge to $c_{A_0}(x_0)$.

Lemma 2. *Let $\mathbf{B}$ be a σ-algebra of sets on X and $\{B_n\}$ a sequence of elements $B_n \in \mathbf{B}$. Then $\{B_n\}$ converges in $\mathbf{B}$ if and only if there exists* $\lim P(B_n)$ *for each probability measure P on $\mathbf{B}$.*

Proof. If $\text{Lim}\, B_n = B \in \mathbf{B}$ and $P \in \mathscr{P}(\mathbf{B})$, then $\mathscr{P}(\mathbf{B}) \subset \mathscr{F}(\mathbf{B})$ implies that $\lim P(B_n) = P(B)$. Now suppose that $\{B_n\}$ does not converge[4]) in $\mathbf{B}$. Since $\mathbf{B}$ is

[4]) i.e. either Lim B_n does not belong to $\mathbf{B}$ or $\{B_n\}$ does not converge at all.

closed in $\mathbf{X}$, then $\{B_n\}$ does not converge in $\mathbf{X}$. Consequently, there is a point $x_0 \in \in \operatorname{Lim\,sup} B_n - \operatorname{Lim\,inf} B_n$ and $c_B(x_0) \in \mathscr{P}(\mathbf{B})$. It follows that $\{c_{B_n}(x_0)\}$ does not converge.

Theorem 7. *Let* $\mathbf{A}$ *be an algebra of sets on* X. *Then the* σ-*algebra* $\mathbf{S}(\mathbf{A})$ *generated by* $\mathbf{A}$ *is a* $\mathscr{P}$ *sequential envelope of* $\mathbf{A}$.

Proof. It is well known that each probability measure $P \in \mathscr{P}(\mathbf{A})$ can be extended in a unique way to a probability measure $\bar{P}$ on the σ-algebra $\mathbf{S}(\mathbf{A})$; consequently $\bar{\mathscr{P}} = \mathscr{P}(\mathbf{S}(A))$, $\bar{\mathscr{P}}$ denoting the class of all extended probability measures on $\mathbf{S}(\mathbf{A})$. By Lemma 1, the convergence space $\mathbf{S}(\mathbf{A})$ is $\bar{\mathscr{P}}$ sequentially regular so that 1° and 2° hold with respect to $\mathbf{A}$ and $\mathbf{S}(\mathbf{A})$. Let $\varphi_0(A)$, $A \in \mathbf{A}$, be a $\mathscr{P}$ homeomorphism on $\mathbf{A}$ into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of the dimension card $\mathscr{P}$. According to Theorem 2 there is a homeomorphism h on $\mathbf{S}(\mathbf{A})$ into $\varepsilon^{\omega_1} \varphi_0(A)$ such that $h(A) = \varphi_0(A)$, $A \in \mathbf{A}$. We are going to prove that h maps $\mathbf{S}(\mathbf{A})$ onto $\varepsilon^{\omega_1} \varphi_0(A)$.

Suppose, on the contrary, that there is the least ordinal ϑ and a point $(z_\alpha) \in \varepsilon^{\vartheta} \varphi_0(A)$ such that $(z_\alpha) \neq h(A)$ for each $A \in \mathbf{S}(\mathbf{A})$. Then evidently $\vartheta - 1$ exists and there is a sequence of points $(z_\alpha^n) \in \varepsilon^{\vartheta-1} \varphi_0(A)$ such that $\lim z_\alpha^n = z_\alpha$ for each index α. Denote $B_n = h^{-1}((z_\alpha^n))$. Then $z_\alpha^n = \bar{P}_\alpha(B_n)$ and $\lim \bar{P}_\alpha(B_n) = z_\alpha$ for each probability measure $\bar{P}_\alpha \in \bar{\mathscr{P}}$. By Lemma 2 there is an element $B \in \mathbf{S}(\mathbf{A})$ such that $\operatorname{Lim} B_n = B$. Consequently $h(B) = (z_\alpha)$. This is a contradiction.

Hence, in view of Theorem 3, the proof is finished.

Now, consider the relation between the sequential envelope and the $\mathscr{P}$ sequential envelope of the same algebra of sets $\mathbf{A}$. The example shows that both envelopes can substantially differ from each other.

Example. Let X be an infinite point set. The system $\mathbf{F}$ of all subsets $F \subset X$ such that F or $X - F$ is finite is an algebra of sets on X. The algebra $\mathbf{F}$ is a sequential envelope of $\mathbf{F}$ itself.

As a matter of fact, let $\{F_n\}$ be a sequence of sets not converging in $\mathbf{F}$. Two cases are possible: either there is a point $x_0 \in \operatorname{Lim\,sup} F_n - \operatorname{Lim\,inf} F_n$ or there is a set $S = \operatorname{Lim} F_n$ in X such that both S and $X - S$ are infinite sets. In the first case define a set function g on $\mathbf{F} : g(F) = c_F(x_0)$, c_F being the characteristic function of the set F. In the second case suppose that there is a subsequence $\{G_n\}$ of $\{F_n\}$ such that G_n are finite (if nearly all F_n are infinite, then the procedure is analogous). Use the method of mathematical induction: Choose a point $y_k \in S - \bigcup_{i=1}^{k-1} G_{n_i}$ and an element G_{n_k} containing points $y_1, \ldots, y_k$ such that $n_k > n_{k-1} > \ldots > n_1$. Now define a set function g on $\mathbf{F} : g(F) = 1$ if there is an even natural m such that $F \supset \bigcup_{i=1}^{m} y_i$ and $y_{m+1} \notin F$; otherwise put $g(F) = 0$.

It can easily be proved that in both cases g is continuous on $\mathbf{F}$ and the sequence $\{g(F_n)\}$ does not converge at all. From this it follows that if φ is any $\mathscr{F}$ homeomorphism on **F** onto $\varphi(\mathbf{F})$, then $\varepsilon\, \varphi(\mathbf{F}) = \varphi(\mathbf{F})$ so that $\varepsilon^{\omega_1}\, \varphi(\mathbf{F}) = \varphi(\mathbf{F})$. According to Theorem 3 the algebra of sets **F** is an $\mathscr{F}$ sequential envelope of itself; it differs from the σ-algebra $\mathbf{S}(\mathbf{F})$ generated by **F** because $\mathbf{S}(\mathbf{F})$ consists of all elements A such that A or $X - A$ are finite or countably infinite subsets of X.

Now, from Theorem 5 it follows that the $\mathscr{F}$ sequential envelope of **F** is not homeomorphic to the $\mathscr{P}$ sequential envelope of **F**.

In this example the $\mathscr{F}$ sequential envelope of the algebra **A** is **A** itself. On the other hand, V. Koutník has shown in [3] that there are convergence rings of sets **R** such that the $\mathscr{F}$ sequential envelope of **R** is different from **R**.

The following problem arises: Let **A** be an algebra of sets on X. We define a real valued function $f(A)$, $A \in \mathbf{A}$, to be uniformly continuous on **A** if $\operatorname{Lim}(A_n \div B_n) = \emptyset$ implies that $\lim(f(A_n) - f(B_n)) = 0$. Denote $\mathscr{U}$ the class of all bounded uniformly continuous functions on **A**. It is easy to show that $\mathscr{P} \subset \mathscr{U} \subset \mathscr{F}(\mathbf{A})$. Consequently **A** is a $\mathscr{U}$ sequentially regular space and according to Theorem 4 there is a $\mathscr{U}$ sequential envelope of **A**. Is the σ-algebra $\mathbf{S}(\mathbf{A})$ generated by the algebra of sets **A** a $\mathscr{U}$ sequential envelope of **A**?

References

[1] *J. Novák:* On convergence spaces and their sequential envelopes. Czech. Math. Journ. *15* (*90*) 1965, p. 74–100.

[2] *J. Novák:* Über die eindeutigen stetigen Erweiterungen stetiger Funktionen. Czech. Math. Journ. *8* (*83*) 1958, p. 344–355.

[3] *V. Koutník:* On sequentially regular convergence spaces. Czech. Math. Journ. *17* (*92*) 1967, 232–247.

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