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$\mathfrak{m}$-IDEAL TOPOLOGIES IN ORDERED SETS

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1. INTRODUCTION

J. MAYER and M. NOVOTNÝ have defined in [5], for every infinite cardinal number $\mathfrak{m}$ in an ordered set P, certain subsets called $\mathfrak{m}$-ideals and a topology $\tau_{\mathfrak{m}}(P)$ called an $\mathfrak{m}$-ideal topology, as sets of all completely (meet-) irreducible ideals and dual ideals by means of a subbasis for open sets. These notions ($\mathfrak{m}$-ideals and $\mathfrak{m}$-ideal topology) coincide, for $\mathfrak{m} = \aleph_0$, with the notions defined by O. FRINK in [3].

In paper [5] the following problems were set:

1.1. Is it possible to construct for every pair of infinite cardinal numbers $\mathfrak{m} < \mathfrak{n}$ an ordered set P such that $\tau_{\mathfrak{m}}(P) \neq \tau_{\mathfrak{n}}(P)$?

1.2. Is it possible to construct for every cardinal number $\mathfrak{m} > \aleph_1$ such an $\mathfrak{m}$-directed set P that for every pair of infinite cardinal number $\mathfrak{p} < \mathfrak{n} < \mathfrak{m}$ the inequality $\tau_{\mathfrak{p}}(P) \neq \tau_{\mathfrak{n}}(P)$ holds?

At the same time an $\mathfrak{m}$-directed set P stands for an ordered set P in which every non-empty subset $M \subset P$ with the property $|M| < \mathfrak{m}$ has an upper a lower bound in P.

L. FUCHSOVÁ has constructed, in [4], for every cardinal number $\mathfrak{m} > \aleph_1$ an ordered continuum E such that $\tau_{\mathfrak{p}}(E) \neq \tau_{\mathfrak{n}}(E)$ for every pair of infinite cardinal numbers $\mathfrak{p} < \mathfrak{n} < \mathfrak{m}$ with the following property: $\mathfrak{p} = \aleph_\mu$, $\mathfrak{n} = \aleph_{\mu+1}$ does not hold, $\aleph_\mu$ being an infinite irregular cardinal number.

In this paper I present a complete solution of problems 1.1 and 1.2.

2. BASIC NOTIONS AND NOTATION

By an ordered set I understand a partially ordered set; I denote incomparable elements a, b by $a \parallel b$; a chain will stand for an ordered set which does not possess incomparable elements. For a subset M of some ordered set P, M^*, M^+ respectively will denote a set of all upper (lower) bounds in P ([1]); M is called a semi-ideal of P if $\{x\}^+ \subset M$ ([6]) holds for every $x \in M$. An ordinal sum of ordered, mutually disjoint sets P_ι, $\iota \in I$, where $\emptyset \neq I$ is a chain, will be denoted by $\sum P_\iota$ $(\iota \in I)$. For a finite number of ordered, mutually disjoin sets P_i $(1 \leqq i \leqq n)$ I denote their ordinal sum by $P_1 \oplus P_2 \oplus \ldots \oplus P_n$ ([1]). $\mathfrak{m}$, $\mathfrak{n}$ will, in the whole paper, stand for infinite cardinal numbers. The cardinal number of a set M is denoted by $|M|$. I will write "iff" instead of "if an only if". If (P, u) is a topological space and $Q \subset P$, then I denote the relative topology on Q by u/Q.

2.1. Definition. ([5], 3.1). Let P be an ordered set. A subset $I \subset P$ is called an $\mathfrak{m}$-ideal of P iff for every subset M, $\emptyset \neq M \subset I$ with $|M| < \mathfrak{m}$ the inclusion $M^{*+} \subset I$ holds.

The following Lemma is evident:

2.2. Lemma. *Every $\mathfrak{m}$-ideal of an ordered set P is a semi-ideal of P. $\{x\}^+$ is an $\mathfrak{m}$-ideal of P for $x \in P$.*

2.3. Definition. ([5], 4.1). Let P be an ordered set, $I \subset P$ an $\mathfrak{m}$-ideal of P. This ideal is called completely irreducible iff for every family $I_\mu (\mu \in M \neq \emptyset)$ of $\mathfrak{m}$-ideals with $I = \bigcap_{\mu \in M} I_\mu$ there exists an index $\mu_0 \in M$ such that $I_{\mu_0} = I$.

2.4. Definition. ([5], 5.1). Let P be an ordered set. Let $(P, \tau_\mathfrak{m}(P))$ be the topological space in which the topology is defined by taking the family consisting of all completely irreducible $\mathfrak{m}$-ideals and of all completely irreducible dual $\mathfrak{m}$-ideals of P as a subbasis for the open sets. Then $\tau_\mathfrak{m}(P)$ is called the $\mathfrak{m}$-ideal topology on P.

2.5. Definition ([2], 6.1.7). Let R be a chain, $x \in R$. If x is the smallest element in R, then $\chi^l(x) = 1$. In case x fails to be the smallest element, then $\chi^l(x) = \min |M|$ $(M \subset R$, M is cofinal with $\{x\}^+ - \{x\})$. Dually it can be defined $\chi^p(x)$.

2.6. Lemma. *Let R be a chain, $x \in R$. Then $\chi^l(x)$ and $\chi^p(x)$ are regular cardinal numbers.*

Proof can be easily deduced e.g. from [2], 3.8.3.

3. $\mathfrak{m}$-IDEAL TOPOLOGIES IN CHAINS

In this section R will denote a chain and for $x \in R$ we put $\{x\}^+ = (-\infty, x]$, $\{x\}^+ - \{x\} = (-\infty, x)$, $\{x\}^* = [x, \infty)$, $\{x\}^* - \{x\} = (x, \infty)$.

3.1. Lemma. *Let $\emptyset \neq I \subsetneqq R$. Then the following statements are equivalent:*

(A) *I is a completely irreducible $\mathfrak{m}$-ideal of R,*

(B) *$I = (-\infty, x)$, where $x \in R$, $\chi^l(x) = 1$ or $\chi^l(x) \geqq \mathfrak{m}$.*

Proof. I. Let (A) hold. For $y \in R - I$ there is $I \subset (-\infty, y]$, because I is a semi-ideal of R according to 2.2. If $I = \bigcap(-\infty, y]\ (y \in R - I)$, then $y_0 \in R - I$ exists such that $I = (-\infty, y_0]$ (because sets $(-\infty, y]$ are $\mathfrak{m}$-ideals by 2.2). But this is a contradiction. As we see there exists $x \in \bigcap(-\infty, y]\ (y \in R - I) - I$. Then, for any $z < x$ there is $z \in I$; thus $I = (-\infty, x)$.

If $\aleph_0 \leqq \chi^l(x) < \mathfrak{m}$, then we have $\emptyset \neq M \subset I$, $|M| < \mathfrak{m}$, M being cofinal with $(-\infty, x)$. From this $M^* = [x, \infty)$ follows, so that $M^{*+} = (-\infty, x]$ non $\subset I$ which is a contradiction.

Consequently (B) holds.

II. Let (B) hold. If $\chi^l(x) = 1$, then because of 2.2, I is an $\mathfrak{m}$-ideal of R. Let $\chi^l(x) \geqq \mathfrak{m}$, $\emptyset \neq M \subset I$, $|M| < \mathfrak{m}$. Then M is not cofinal with I, consequently $y \in I$ exists such that $y \in M^*$, therefore $M^{*+} \subset (-\infty, y] \subset I$. As we see, I is an $\mathfrak{m}$-ideal.

Let $I = \bigcap I_\alpha(\alpha \in A \neq \emptyset)$, where I_α are $\mathfrak{m}$-ideals of R for any $\alpha \in A$. Since $x \notin I$, then $\alpha_0 \in A$ exists such that $x \notin I_{\alpha_0}$. From 2.2 there follows that $I_{\alpha_0} \subset I$. Thus $I_{\alpha_0} = I$ and consequently I is a completely irreducible $\mathfrak{m}$-ideal of R.

3.2. Theorem. *The system of all sets of the type $(-\infty, x)$ and (y, ∞), where $x \in R$, $y \in R$, $\chi^l(x) = 1$ or $\chi^l(x) \geqq \mathfrak{m}$, $\chi^p(y) = 1$ or $\chi^p(y) \geqq \mathfrak{m}$, together with R form a subbasis for open sets of the topology $\tau_{\mathfrak{m}}(R)$.*

Proof follows from 3.1 and from the dual statement to 3.1.

3.3. Theorem. *Let $\aleph_\nu$ be an irregular infinite cardinal number. Then $\tau_{\aleph_\nu}(R) = \tau_{\aleph_{\nu+1}}(R)$ holds.*

Proof follows from 2.6 and 3.2.

3.4. Remark. From Theorem 3.3 there follows that when solving problems 1.1 and 1.2 one must consider partially ordered sets and not only chains.

4. $\mathfrak{m}$-IDEAL TOPOLOGIES IN ORDINAL SUM

In this section P, Q will denote disjoin ordered sets, P' will stand for a set $Q \oplus P$. $\mathscr{I}$, $\mathscr{I}'$ will denote the family of all $\mathfrak{m}$-ideals of P, P' respectively, $\mathscr{J}$, $\mathscr{J}'$ will denote the family of all dual $\mathfrak{m}$-ideals of P, P' respectively different from P (not containing P, respectively); $\mathfrak{A}$, $\mathfrak{A}'$ will denote the family of all completely irreducible $\mathfrak{m}$-ideals of P, P' respectively, $\mathfrak{B}$, $\mathfrak{B}'$ will denote the family of all completely irreducible dual $\mathfrak{m}$-ideals of P, P' respectively different from P (not containing P, respectively).

Operators * and $^+$ taken into consideration with respect to the ordered set P' will be denoted by $_*$ and $_+$ (e.g. M_*, M_+, M_{*+} etc.), whereas, when taken with respect to the ordered set P, the original notation of these operators will be preserved.

The following Lemmas are evident:

4.1. Lemma. *For $M \subset P'$, $M \cap P \neq \emptyset$ we have $(M \cap P)^* = M_*$. For $M \subset P$, $M^+ \cup Q = M_+$ holds.*

4.2. Lemma. *Let $M \subset P$. If $M^+ = \emptyset$, then $M^{+*} = P$ and $M_{+*} \supset P$. If $M^+ \neq \emptyset$, then $M^{+*} = M_{+*}$.*

4.3. Lemma. $\emptyset \neq I \in \mathscr{I} \Rightarrow I \cup Q \in \mathscr{I}'$, $I' \in \mathscr{I}' \Rightarrow I' \cap P \in \mathscr{I}$, $\mathscr{J} = \mathscr{J}'$.

Proof. I. Let $\emptyset \neq I \in \mathscr{I}$, $M \subset I \cup Q$, $|M| < \mathfrak{m}$. If $M \cap P \neq \emptyset$, then according to 4.1 $M_* = (M \cap P)^*$, $M_{*+} = (M \cap P)^{*+} \cup Q$. Since $(M \cap P)^{*+} \subset I$, then $M_{*+} \subset \subset I \cup Q$. If $M \cap P = \emptyset$, then $M_* \supset P$. If P fails to have the smallest element, then $M_{*+} \subset Q \subset I \cup Q$. If P has the smallest element o, then $o \in I$ according to 2.2; consequently $M_{*+} \subset Q \cup \{o\} \subset Q \cup I$. Thus $I \cup Q \in \mathscr{I}'$.

II. Let $I' \in \mathscr{I}'$, $\emptyset \neq M \subset I' \cap P$, $|M| < \mathfrak{m}$. According to 4.1 we have $M^* = M_*$, $M^{*+} \cup Q = M_{*+} \subset I'$, thus $M^{*+} \subset I' \cap P$. Consequently $I' \cap P \in \mathscr{I}$.

III. From 4.2 there easily follows that $\mathscr{J} \subset \mathscr{J}'$. Let $J' \in \mathscr{J}'$. Since J' non $\supset P$, then from the dual statement to 2.2 there follows that $J' \subsetneqq P$. From 4.2 $J' \in \mathscr{J}$ is consequent. Thus $\mathscr{J} = \mathscr{J}'$.

4.4. Lemma. $\emptyset \neq A \in \mathfrak{A} \Rightarrow$ exists $A' \in \mathfrak{A}'$ *such that* $A' \cap P = A$, $\emptyset \neq A' \in \mathfrak{A}'$, $A' \cap P \neq \emptyset \Rightarrow A' \cap P \in \mathfrak{A}$, $\mathfrak{B} = \mathfrak{B}'$.

Proof. I. Let $\emptyset \neq A \in \mathfrak{A}$. Let us put $A' = A \cup Q$. According to 4.3 we have $A' \in \mathscr{I}'$. Let $A' = \bigcap I'_\mu(\mu \in M \neq \emptyset)$, $I'_\mu \in \mathscr{I}'$. By 4.3 we have $I_\mu = I'_\mu \cap P \in \mathscr{I}$ for any $\mu \in M$ and evidently $\bigcap I_\mu\ (\mu \in M) = A$; consequently $\mu_0 \in M$ exists such that $I_{\mu_0} = A$. Then $I_{\mu_0} = A'$ and thus $A' \in \mathfrak{A}'$.

II. Let $\emptyset \neq A' \in \mathfrak{A}'$, $A' \cap P \neq \emptyset$. According to 4.3 we have $A = A' \cap P \in \mathscr{I}$. Let $A = \bigcap I_\mu(\mu \in M \neq \emptyset)$ where $I_\mu \in \mathscr{I}$. By 4.3 it is $I'_\mu = I_\mu \cup Q \in \mathscr{I}'$ for $\mu \in M$ and we have $\bigcap I'_\mu(\mu \in M') = A'(Q \subset A'$ according to 2.2). Thus $\mu_0 \in M$ exists such that $I'_{\mu_0} = A'$, consequently $I_{\mu_0} = A$. Therefore $A \in \mathfrak{A}$.

III. If $B' \in \mathfrak{B}'$, then from the dual statement to 2.2 $B' \subsetneqq P$ follows and from the equation $\mathscr{J} = \mathscr{J}'$ in 4.3 we get $B' \in \mathfrak{B}$. Let $B \in \mathfrak{B}$. According to 4.3 we have $B \in \mathscr{J}'$ and let us assume that $B = \bigcap J'\mu(\mu \in M \neq \emptyset)$, where J'_μ is a dual $\mathfrak{m}$-ideal of P' for every $\mu \in M$. We have $B \subsetneqq P$. For $p \in P - B$ there exists $\mu(p) \in M$ such htat $p \notin J'_{\mu(p)}$. From the dual statement to 2.2, $J'_{\mu(p)} \in \mathscr{J}'$ is consequent. Evidently $B = \bigcap J'_{\mu(p)}(p \in \in P - B)$. Since $\mathscr{J}' = \mathscr{J}$ according to 4.3 then there exists $p_0 \in P - B$ such that $J'_{\mu(p_0)} = B$. Consequently $B \in \mathfrak{B}'$.

4.5. Lemma. $\tau_{\mathfrak{m}}(P')/P = \tau_{\mathfrak{m}}(P)$.

Proof. Denote $\mathfrak{C} = \mathfrak{A} \cup \mathfrak{B}$, $\mathfrak{C}' = \mathfrak{A}' \cup \mathfrak{B}' \cup \mathfrak{C}''$ where $\mathfrak{C}''$ is the family of all completely irreducible dual $\mathfrak{m}$-ideals of P' which contain P. Evidently $\mathfrak{C}, \mathfrak{C}'$ form a subbasis for the open sets in spaces $(P, \tau_{\mathfrak{m}}(P))$ and $(P', \tau_{\mathfrak{m}}(P'))$. From 4.4 there follows that for $\emptyset \neq Y \in \mathfrak{C}'$, $Y \cap P \neq \emptyset$ we have $Y \cap P \in \mathfrak{C}$ and for every $\emptyset \neq X \in \mathfrak{C}$ $Y \in \mathfrak{C}'$ exists such that $Y \cap Q = X$. Hence the statement follows.

4.6. Theorem. *Let P_ι be an ordered set for every $\iota \in I$, where $\emptyset \neq I$ is a chain and P_ι are mutually disjoint sets. Then for every $\iota_0 \in I$*

$$\tau_{\mathfrak{m}}(\textstyle\sum P_\iota(\iota \in I))/P_{\iota_0} = \tau_{\mathfrak{m}}(P_{\iota_0})$$

holds.

Proof. If ι_0 fails to be the smallest (the greatest) in I, then put $Q = \sum P_\iota(\iota \in I, \iota < \iota_0)$ $(R = \sum P_\iota(\iota \in I, \iota > \iota_0)$, respectively). If ι_0 is the smallest (the greatest) in I, then put $Q = \emptyset$ $(R = \emptyset$, respectively). Then $\sum P_\iota(\iota \in I) = Q \oplus P_{\iota_0} \oplus R$. From 4.5 and from the dual statement to 4.5 we get $\tau_{\mathfrak{m}}(\sum P_\iota(\iota \in I))/P_{\iota_0} = \tau_{\mathfrak{m}}(Q \oplus P_{\iota_0} \oplus \oplus R)/P_{\iota_0} = (\tau_{\mathfrak{m}}(Q \oplus (P_{\iota_0} \oplus R))/P_{\iota_0} \oplus R)/P_{\iota_0} = \tau_{\mathfrak{m}}(P_{\iota_0} \oplus R)/P_{\iota_0} = \tau_{\mathfrak{m}}(P_{\iota_0})$.

5. ORDERED SET $P(\mathfrak{m})$

5.1. Definition. Denote by K the set of all finite sequences composed from 0 and 1. Let an empty sequence be denoted by k_0 and let us take it as an element of the set K. Let us order the set K in the following way: the element k_0 is the smallest element of the set K and for $k_1, k_2 \in K$, $k_1 \neq k_0 \neq k_2$, $k_1 = (a_1, \ldots, a_n)$, $k_2 = (b_1, \ldots, b_m)$ let us put $k_1 \leqq k_2$ iff $n \leqq m$ and $a_i = b_i$ for $1 \leqq i \leqq n$.

5.2. Lemma. *Let $k \in K$. Then $k_1, k_2 \in K$, $k_1 \neq k \neq k_2$ exist such that $\{k_1\}^+ \cap \cap \{k_2\}^+ = \{k\}^+$.*

Proof. If $k = k_0$, then put $k_1 = (0)$, $k_2 = (1)$. Evidently $\{k_1\}^+ = \{k_0, k_1\}$ and $\{k_2\}^+ = \{k_0, k_2\}$.

If $k \neq k_0$, then $k = (a_1, \ldots, a_n)$, where $a_i = 0$ or 1 for $1 \leqq i \leqq n$. Let us put $k_1 = (a_1, \ldots, a_n, 0)$, $k_2 = (a_1, \ldots, a_n, 1)$. Then $\{k_1\}^+ = \{k\}^+ \cup \{k_1\}$, $\{k_2\}^+ = = \{k\}^+ \cup \{k_2\}$.

Since in both cases $k_1 \neq k_2$, there is $\{k_1\}^+ \cap \{k_2\}^+ = \{k\}^+$.

5.3. Lemma. *Let $k_1, k_2 \in K$, $k_1 \parallel k_2$. Then $\{k_1\}^* \cap \{k_2\}^* = \emptyset$.*

Proof. Let $k_1, k_2, k \in K$, $k_1 \leqq k$, $k_2 \leqq k$, $k_1 \neq k_0 \neq k_2$. Then $k_1 = (a_1, \ldots, a_n)$, $k_2 = (b_1, \ldots, b_m)$, $k = (c_1, \ldots, c_r)$ while $r \geqq n$, $r \geqq m$, $a_i = c_i$ for $1 \leqq i \leqq n$,

$b_j = c_j$ for $1 \leqq j \leqq m$. Hence there follows easily that k_1, k_2 are comparable elements.

5.4. Lemma. *Let R be an infinite chain in K. Then $R^* = \emptyset$.*

Proof. For $k = (a_1, \ldots, a_n)$, $a_i = 0$ or 1 $(1 \leqq i \leqq n)$ we have $|\{k\}^+| < \aleph_0$. Hence Lemma follows.

5.5. Definition. Let S be an arbitrary set, $|S| \geqq \mathfrak{m}$, $\mathfrak{S} = \{X \subset S \mid 0 < |X| < \mathfrak{m}\}$, σ be some symbol different from all elements of the set $S \cup (\mathfrak{S} \times K)$. Let us put $P(\mathfrak{m}) = S \cup (\mathfrak{S} \times K) \cup \{\sigma\}$ $(=P(\mathfrak{m}, S, \sigma))$ and let us set $p \leqq q$ for $p, q \in P(\mathfrak{m})$ iff $p = q$ or $p \in S$, $q = \sigma$ or $p \in S$, $q = (X, k)$, where $X \in \mathfrak{S}$, $k \in K$, $p \in X$, or $p = (X, k)$, $q = (X, l)$, where $X \in \mathfrak{S}$, $k, l \in K$, $k \leqq l$. This relation is evidently an ordering.

5.6. Lemma. *Let $I \subset S$. Then the following statements are equivalent:*

(A) *I is an $\mathfrak{n}$-ideal of $P(\mathfrak{m})$,*

(B) *$|I| < \mathfrak{m}$ or $\mathfrak{n} \leqq \mathfrak{m} \leqq |I|$.*

Proof. I. If $|I| \geqq \mathfrak{m}$ and $\mathfrak{m} < \mathfrak{n}$, then we have $M \subset I$, $|M| = \mathfrak{m}$. Then $M^* = \{\sigma\}$, thus $M^{*+} \ni \sigma$; consequently I is not an $\mathfrak{n}$-ideal. The statement (A) implies, thus, the statement (B).

II. Let (B) hold and let $\emptyset \neq M \subset I$, $|M| < \mathfrak{n}$. Then $|M| < \mathfrak{m}$ and $(M, k_0) \in M^*$, $\sigma \in M^*$. Since $\{(M, k_0)\}^+ \cap \{\sigma\}^+ = M \subset I$, we have $M^{*+} \subset I$, and consequently I is an $\mathfrak{n}$-ideal of $P(\mathfrak{m})$.

5.7. Lemma. *Let $I \subset S$. Then the following statements S are equivalent:*

(A) *I is a completely irreducible $\mathfrak{n}$-ideal of $P(\mathfrak{m})$,*

(B) *$|S - I| \leqq 1$ and $\mathfrak{n} \leqq \mathfrak{m}$.*

Proof. I. Let $|S - I| > 1$. Then there exist $s_1, s_2 \in S - I$, $s_1 \neq s_2$. Let us put $X_1 = \{s_1\} \cup I$, $X_2 = \{s_2\} \cup I$.

a) If $|I| < \mathfrak{m}$ then $X_1, X_2 \in \mathfrak{S}$ and $\{(X_1, k_0)\}^+ \cap \{(X_2, k_0)\}^+ = I$, $\{(X_1, k_0)\}^+ \neq I \neq \{(X_2, k_0)\}^+$. Then from 2.2 there follows that I is not a completely irreducible $\mathfrak{n}$-ideal of $P(\mathfrak{m})$.

b) If $\mathfrak{n} \leqq \mathfrak{m} \leqq |I|$, then according to 5.6 X_1, X_2 are $\mathfrak{n}$-ideals of $P(\mathfrak{m})$, $X_1 \cap X_2 = I$; hence it follows that I cannot be a completely irreducible $\mathfrak{n}$-ideal of $P(\mathfrak{m})$. In case (A) holds, then from 5.6 there follows that $|S - I| \leqq 1$ and $\mathfrak{n} \leqq \mathfrak{m}$.

II. Let (B) hold. According to 5.6 I is an $\mathfrak{n}$-ideal of $P(\mathfrak{m})$. Let $I = \bigcap I_\alpha (\alpha \in A \neq \emptyset$, where for $\alpha \in A$, I_α is an $\mathfrak{n}$-ideal of $P(\mathfrak{m})$. Then $\alpha_1 \in A$ exists such that $\sigma \notin I_{\alpha_1}$. If $S - I = \{s_0\}$, then we have $\alpha_2 \in A$ such that $s_0 \notin I_{\alpha_2}$. Since $\sigma \geqq s_0$, we have $\sigma \notin I_{\alpha_2}$

according to 2.2. Consequently $\alpha_0 \in A$ exists such that $\sigma \notin I_{\alpha_0}$ and $I_{\alpha_0} \cap S = I$. If $(X, k) \in I_{\alpha_0}$, where $X \in \mathfrak{S}$, $k \in K$, then $s \in I - X$ exists. Let us set $M = \{s, (X, k)\}$. Evidently $M \subset I_{\alpha_0}$, $0 < |M| < \mathfrak{n}$, and $M^* = \emptyset$; consequently $M^{*+} = P(\mathfrak{m})$, which is a contradiction. Thus $I_{\alpha_0} \cap (\mathfrak{S} \times K) = \emptyset$, and hence it follows that $I_{\alpha_0} = I$.

As we can see, the statement (A) is valid.

5.8. Lemma. *Let $X \in \mathfrak{S}$, $k \in K$. Then $\{(X, k)\}^+$ is not a completely irreducible $\mathfrak{m}$-ideal of $P(\mathfrak{m})$.*

Proof. By 5.2 $k_1, k_2 \in K$, $k_1 \neq k \neq k_2$ exist such that $\{k_1\}^+ \cap \{k_2\}^+ = \{k\}^+$. Thus $\{(X, k_1)\}^+ \cap \{(X, k_2)\}^+ = \{(X, k)\}^+$ and from 2.2 then the statement follows.

5.9. Lemma. *Let $z_1, z_2 \in I \cap (\mathfrak{S} \times K)$ where I is an $\mathfrak{n}$-ideal of $P(\mathfrak{m})$ and let $z_1 \parallel z_2$. Then $I = P(\mathfrak{m})$.*

Proof. We have $z_1 = (X_1, k_1)$, $z_2 = (X_2, k_2)$ where $X_1, X_1 \in \mathfrak{S}$, $k_1, k_2 \in K$. Let us put $M = \{z_1, z_2\}$. If $X_1 \neq X_2$ then $M^* = \emptyset$. If $X_1 = X_2$, then $k_1 \parallel k_2$ and from 5.3 there follows that $M^* = \emptyset$. Consequently $M^{*+} = P(\mathfrak{m}) \subset I$.

5.10. Lemma. *Let I be a completely irreducible $\mathfrak{n}$-ideal of $P(\mathfrak{m})$, $\mathfrak{n} > \aleph_0$ and $I \cap (\mathfrak{S} \times K) \neq \emptyset$. Then $I = P(\mathfrak{m})$.*

Proof. Let us put $A = I \cap (\mathfrak{S} \times K)$. If A is not a chain, then $I = P(\mathfrak{m})$ by 5.9.

If A contains the greatest element (X, k), where $X \in \mathfrak{S}$ and $k \in K$, then according to 2.2 $\{(X, k)\}^+ \subset I$. By 5.8 we have $z \in I - \{(X, k)\}^+$. Evidently $z \notin \mathfrak{S} \times K$ and consequently $\{z, (X, k)\}^{*+} = \emptyset^+ = P(\mathfrak{m}) \subset I$. Thus $I = P(\mathfrak{m})$.

If A is a chain without the greatest element, we have $|A| = \aleph_0$ and according to 5.4 it is $A^* = \emptyset$ and consequently $A^{*+} = P(\mathfrak{m}) \subset I$. Thus $I = P(\mathfrak{m})$.

5.11. Theorem. *Let $\mathfrak{m} < \mathfrak{n}$. Then $\tau_{\mathfrak{m}}(P(\mathfrak{m})) \neq \tau_{\mathfrak{n}}(P(\mathfrak{m}))$.*

Proof. Choose $s \in S$. By 5.7 the $\tau_{\mathfrak{m}}(P(\mathfrak{m}))$ – neighbourhood U of the point s exists such that $\sigma \notin U$. I being a completely irreducible $\mathfrak{n}$-ideal of $P(\mathfrak{m})$, we have $\sigma \in I$ according to 5.7 and 5.10. J being a dual $\mathfrak{n}$-ideal of $P(\mathfrak{m})$, $s \in J$, we have $\sigma \in J$ according to the dual statement to 2.2. Thus, every $\tau_{\mathfrak{n}}(P(\mathfrak{m}))$ – neighbourhood of the point s contains the point σ. Hence the statement follows.

6. ORDERED SET $S(\mathfrak{m})$

6.1. Definition. Let ordered sets $P(\mathfrak{a})$, $\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{m}$ be chosen in such a way that they are mutually disjoint. Let us choose two different symbols ω_1, ω_2 different from all elements of the set $\bigcup P(\mathfrak{a})$ $(\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{m})$. Let us put $S(\mathfrak{m}) = \{\omega_1\} \oplus \oplus \sum P(\mathfrak{a})$ $(\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{m}) \oplus \{\omega_2\}$.

6.2. Main Theorem. *Let* $\mathfrak{m}$ *be a cardinal number* $\geqq \aleph_1$. *Then for every pair of infinite cardinal numbers* $\mathfrak{p} < \mathfrak{n} \leqq \mathfrak{m}$

$$\tau_{\mathfrak{p}}(S(\mathfrak{m})) \neq \tau_{\mathfrak{n}}(S(\mathfrak{m}))$$

holds.

Proof. From Theorem 4.6 $\tau_{\mathfrak{p}}(S(\mathfrak{m}))/P(\mathfrak{p}) = \tau_{\mathfrak{p}}(P(\mathfrak{p}))$, $\tau_{\mathfrak{n}}(S(\mathfrak{m}))/P(\mathfrak{p}) = \tau_{\mathfrak{n}}(P(\mathfrak{p}))$ follows. According to Theorem 5.11 it is $\tau_{\mathfrak{p}}(P(\mathfrak{p})) \neq \tau_{\mathfrak{n}}(P(\mathfrak{p}))$ from where we get the statement.

6.3. Remark. Since an ordered set $S(\mathfrak{m})$ contains the greatest and the least element it is $\mathfrak{m}$-directed. From Theorem 6.2 we get then an affirmative solution of problems 1.1 and 1.2.

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