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ON THE DEFINITION OF AN ABSOLUTELY FREE ALGEBRA

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Some mathematicians mean by an absolutely free algebra a freely generated algebra in the sense of [II], Definition 2.1. See, e.g., [I], p. 109. But the author of the present paper wishes to define an absolutely free algebra as a *lawless* algebra, i.e., an algebra which satisfies no non-tautological law in the sense of [III], Definition 3.1. In other words, we wish to

(1) $\begin{cases}\text{call an algebra } \mathfrak{A} \text{ absolutely free if } (\Phi\mathfrak{C})\, \mathfrak{A} \text{ is the equality} \\ \text{relation on } \mathfrak{C} \text{ for every freely generated algebra } \mathfrak{C}.\end{cases}$

(1) is open to the objection that it involves the indefinitely wide class of all freely generated algebras. It is the main purpose of this paper to show how this difficulty can be avoided by speaking of a fixed freely generated algebra of order $\geqq 2$ instead of speaking of "every freely generated algebra".

Some results of this paper were announced in a paper read at the 581st meeting of the American Mathematical Society in Seattle, Washington, U.S.A., June 13–16, 1961. See [VI].

Roman numerals in square brackets refer to the bibliography at the end of this paper. The notation introduced in [II] will be used in this paper. In particular, it is understood that S is a set, and that σ is an S-system of sets. An algebra $\mathfrak{A}$ of species σ over a set A (shortly: an algebra $\mathfrak{A}$ over A) is determined by σ, A, and a system $\{f_s;\, s \in S\}$ in which f_s is a σs-operator on A for every element s of S, and which is denoted $\langle\mathfrak{A}\rangle$. See [II], Definitions 1.1 to 1.4. $\mathfrak{C}$, $\mathfrak{C}_0$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are freely generated algebras on the sets C, C_0, C_1 and C_2, respectively, and D, D_0, D_1 and D_2 are the free bases of $\mathfrak{C}$, $\mathfrak{C}_0$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$, respectively. In the following Lemma 1, $\mathfrak{A}$ and $\mathfrak{B}$ are algebras on the sets A and B, respectively.

Lemma 1. *Let a' and a'' be elements of A with $a'(\boldsymbol{P}_0\mathfrak{A})\, a''$.* (See II, Definition 1.6.) *Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then $(ha')\,(\boldsymbol{P}_0\mathfrak{B})\,(ha'')$.*

Proof. Let s be an element of S, let a be an element of $A^{\sigma s}$, and let k be an element of σs such that $ak = a'$ and $(\langle\mathfrak{A}\rangle\, s)\, a = a''$. Then $(h\,.\,a)\, k = ha'$ and $(\langle\mathfrak{B}\rangle\, s)\,(h\,.\,a) = h((\langle\mathfrak{A}\rangle\, s)\, a) = ha''$. Hence $(ha')\,(\boldsymbol{P}_0\mathfrak{B})\,(ha'')$.

Lemma 2. *Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be two different elements of $\mathfrak{C}_1$ with $\mathfrak{c}_1'(\boldsymbol{P}\mathfrak{C}_1)\,\mathfrak{c}_1''$.* (See II, Definition 1.7.) *Let h be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Then $h\mathfrak{c}_1' \neq h\mathfrak{c}_1''$ and $(h\mathfrak{c}_1')\,(P\mathfrak{C}_2)\,(h\mathfrak{c}_1'')$.*

Proof. See Lemma 1 and [II], Theorem 2.12.

Lemma 3. *Let $\mathfrak{c}_1'$ and $\mathfrak{c}_1''$ be elements of $\mathfrak{C}_1$ such that $(P\mathfrak{C}_1)\,\mathfrak{c}_1' \neq (P\mathfrak{C}_1)\,\mathfrak{c}_1''$.* (See [II], Definition 3.2.) *Let $|C_2| \geqq 2$. Then there exists a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h\mathfrak{c}_1' \neq h\mathfrak{c}_1''$.*

Proof. Let $\mathfrak{d}_0$ be an element of $(P\mathfrak{C}_1)\,\mathfrak{c}_1'$ which does not belong to $(P\mathfrak{C}_1)\,\mathfrak{c}_1''$. Let h_1 be any homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Let h_2 be a homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h_2\mathfrak{d} = h_1\mathfrak{d}$ for $\mathfrak{d} \in (P\mathfrak{C}_1)\,\mathfrak{c}_1''$ and $h_2\mathfrak{d}_0 \neq h_1\mathfrak{d}_0$. h_1 and h_2 exist by [II], Theorem 2.16. By [II], Theorem 3.9, $h_2\mathfrak{c}_1'' = h_1\mathfrak{c}_1''$ and $h_2\mathfrak{c}_1' \neq h_1\mathfrak{c}_1'$. Hence $h_1\mathfrak{c}_1' \neq h_1\mathfrak{c}_1''$ or $h_2\mathfrak{c}_1' \neq h_2\mathfrak{c}_1''$, proving the lemma.

Lemma 4. *Let $\mathfrak{c}_1$ be an element of C_1 and let $\mathfrak{d}_1$ be an element of D_1 different from $\mathfrak{c}_1$. Let $|C_2| \geqq 2$. Then there exists a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h\mathfrak{c}_1 \neq h\mathfrak{d}_1$.*

Proof. Because of Lemma 3, we may assume that $(P\mathfrak{C}_1)\,\mathfrak{c}_1 = (P\mathfrak{C}_1)\,\mathfrak{d}_1$. Then $[\mathfrak{d}_1] = (P\mathfrak{C}_1)\,\mathfrak{c}_1$, $\mathfrak{d}_1 \in (P\mathfrak{C}_1)\,\mathfrak{c}_1$, and $\mathfrak{d}_1(\boldsymbol{P}\mathfrak{C}_1)\,\mathfrak{c}_1$ by [II], Theorem 3.4. Let h be any homomorphism of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Then $h\mathfrak{c}_1 \neq h\mathfrak{d}_1$ by Lemma 2.

Theorem 1. *Let $|C_2| \geqq 2$. Then $(\Phi\mathfrak{C}_1)\,\mathfrak{C}_2$ is the equality relation on C_1.*

Proof. Let E be the set of all elements $\mathfrak{c}$ of C_1 such that, for every element $\mathfrak{c}_1$ of C_1 different from $\mathfrak{c}$, there exists a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ with $h\mathfrak{c}_1 \neq h\mathfrak{c}$. Then $E \supset D_1$ by Lemma 4. Let s be an element of S, and let c be an element of $E^{\sigma s}$. Let $\mathfrak{c}_1$ be an element of C_1 different from $(\langle\mathfrak{C}_1\rangle\, s)\, c$. If $\mathfrak{c}_1 \in D_1$, Lemma 4 implies that there exists a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ with $h\mathfrak{c}_1 \neq h((\langle\mathfrak{C}_1\rangle\, s)\, c)$. Let $\mathfrak{c}_1 \notin D_1$. Let s' be an element of S, and let c' be an element of $C_1^{\sigma s'}$ such that $(\langle\mathfrak{C}_1\rangle\, s')\, c' = \mathfrak{c}_1$. s' and c' exist by [II], Theorem 1.4. If $s' \neq s$ then $(\langle\mathfrak{C}_2\rangle\, s')\,(h\,.\,c') \neq (\langle\mathfrak{C}_2\rangle\, s)\,(h\,.\,c)$, $h((\langle\mathfrak{C}_1\rangle\, s')\, c') \neq h((\langle\mathfrak{C}_1\rangle\, s)\, c)$, and $h\mathfrak{c}_1 \neq h((\langle\mathfrak{C}_1\rangle\, s)\, c)$ for every homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$. Let $s' = s$. Because $(\langle\mathfrak{C}_1\rangle\, s')\, c' \neq (\langle\mathfrak{C}_1\rangle\, s)\, c$, $c' \neq c$. Let k_0 be an element of σs such that $c'k_0 \neq ck_0$. Then $ck_0 \in E$. Hence there exists a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h(c'k_0) \neq h(ck_0)$, whence $h\,.\,c' \neq h\,.\,c$, $(\langle\mathfrak{C}_2\rangle\, s')\,(h\,.\,c') \neq (\langle\mathfrak{C}_2\rangle\, s)\,(h\,.\,c)$, and $h\mathfrak{c}_1 \neq h((\langle\mathfrak{C}_1\rangle\, s)\, c)$. This completes the proof that $(\langle\mathfrak{C}_1\rangle\, s)\, c \in E$. Hence E is closed with respect to $\mathfrak{C}_1$. Hence $E = C_1$. Hence if $\mathfrak{c}_1$ and $\mathfrak{c}$ are any two different elements of C_1, there exists a homomorphism h of $\mathfrak{C}_1$ into $\mathfrak{C}_2$ such that $h\mathfrak{c}_1 \neq h\mathfrak{c}$. This proves the theorem.

Theorem 2. *Let $\mathfrak{A}$ be an algebra or a set of algebras, let $|C_2| \geqq 2$, and let $(\Phi\mathfrak{C}_2)\,\mathfrak{A}$ be the equality relation on C_2. Then $(\Phi\mathfrak{C}_1)\,\mathfrak{A}$ is the equality relation on C_1.*

Proof. $\mathfrak{C}_2/(\Phi\mathfrak{C}_2)\,\mathfrak{A}$ is isomorphic to $\mathfrak{C}_2$. Hence $(\Phi\mathfrak{C}_1)\,\mathfrak{C}_2 = (\Phi\mathfrak{C}_1)\,(\mathfrak{C}_2/(\Phi\mathfrak{C}_2)\,\mathfrak{A})$, $(\Phi\mathfrak{C}_1)\,\mathfrak{C}_2 = (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})\,((\Phi\mathfrak{C}_2)\,\mathfrak{A})$ by [V], Definition 1.1, and $(\Phi\mathfrak{C}_1)\,\mathfrak{C}_2 \supset (\Phi\mathfrak{C}_1)\,\mathfrak{A}$ by [V], Theorem 1.3. By Theorem 1, $(\Phi\mathfrak{C}_1)\,\mathfrak{C}_2$ is the equality relation on C_1. Hence $(\Phi\mathfrak{C}_1)\,\mathfrak{A}$ is the equality relation on C_1.

Theorem 3. *Let $\mathfrak{A}$ be an algebra or a set of algebras. Let $|C_1| \geqq 2$ and $|C_2| \geqq 2$. Then $(\Phi\mathfrak{C}_2)\,\mathfrak{A}$ is the equality relation on C_2 if and only if $(\Phi\mathfrak{C}_1)\,\mathfrak{A}$ is the equality relation on $\mathfrak{C}_1$.*

Theorem 3 is obvious from Theorem 2.

In the following Definition 1, it is supposed that $|C_0| \geqq 2$. (That a $\mathfrak{C}_0$ with $|C_0| \geqq 2$ exists follows from [IV], Theorem 11.)

Definition 1. An algebra $\mathfrak{A}$ is called absolutely free if and only if $(\Phi\mathfrak{C}_0)\,\mathfrak{A}$ is the equality relation on C_0. A set $\mathfrak{M}$ of algebras is called absolutely free if and only if $\mathfrak{M}$ is not empty, and $(\Phi\mathfrak{C}_0)\,\mathfrak{M}$ is the equality relation on C_0.

Remark. The empty set may be regarded as a set of algebras of any species. But this does not matter because, by Definition 1, the empty set is not absolutely free, irrespective of the species considered.

Also, if $\mathfrak{M}$ is the empty set, $(\Phi\mathfrak{C}_0)\,\mathfrak{M}$ is the all relation on C_0, hence not the equality relation on C_0. Hence any set $\mathfrak{M}$ of algebras of species σ is absolutely free if and only if $(\Phi\mathfrak{C}_0)\,\mathfrak{M}$ is the equality relation on C_0.

Throughout the remainder of this paper, we shall use the word "free" in the sense of "absolutely free".

Theorem 4. *Let $\mathfrak{A}$ be an algebra or a set of algebras. Let $|C| \geqq 2$. Then $\mathfrak{A}$ is free if and only if $(\Phi\mathfrak{C})\,\mathfrak{A}$ is the equality relation on C.*

Theorem 4 is obvious from Theorem 3 and Definition 1. It shows that the truth of the statement that a given algebra or set of algebras is free is independent of the choice of $\mathfrak{C}_0$ in Definition 1.

Theorem 5. *If $|C| \geqq 2$ then $\mathfrak{C}$ is free.*

Proof. See Theorem 1 and Definition 1.

Corollary. *If $|D| \geqq l_\sigma$* (see [V], Definition 2.1) *then $\mathfrak{C}$ is free.*

Proof. See [V], Theorem 2.1.

The following five statements are almost obvious:

(i) If $\mathfrak{A}$ is a free algebra then $|\mathfrak{A}| \geqq 2$.

(ii) If $\mathfrak{M}$ is a free set of algebras then there exists an element $\mathfrak{A}$ of $\mathfrak{M}$ with $|\mathfrak{A}| \geqq 2$.

(iii) An algebra $\mathfrak{A}$ is free if and only if $[\mathfrak{A}]$ is free.

(iv) If $\mathfrak{M}$ is a set of algebras, $\mathfrak{A}$ is an element or a subset of $\mathfrak{M}$, and $\mathfrak{A}$ is free, then $\mathfrak{M}$ is free.

(v) If $\mathfrak{A}$ is an algebra, $\mathfrak{B}$ is a subalgebra or a homomorphic image of $\mathfrak{A}$, and $\mathfrak{B}$ is free, then $\mathfrak{A}$ is free. (See [III], Theorems 3.14 and 3.15.)

An algebra can be free without being freely generated. Let $S = [0]$, and $\sigma 0 = [1]$. Let A be the set of all integers $\geqq 4$, and let B be the set of all integers $\geqq 2$. Define algebras, $\mathfrak{A}$ and $\mathfrak{B}$, on A and B, respectively, by requiring that

$$(\langle\mathfrak{A}\rangle\, 0)\,\{x\} = (\langle\mathfrak{B}\rangle\, 0)\,\{x\} = x + 1 \quad \text{for} \quad x = 4, 5, \ldots, \tag{2}$$

$$(\langle\mathfrak{B}\rangle\, 0)\,\{2\} = 3\,,$$

and

$$(\langle\mathfrak{B}\rangle\, 0)\,\{3\} = 2\,.$$

(In these equations, $\{x\}$ is the function on $[1]$ whose only value is x.) Then $[4]$ is obviously a free basis of $\mathfrak{A}$. Since $|A| \geqq 2$, $\mathfrak{A}$ is free by Theorem 5. By (2), $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. Hence $\mathfrak{B}$ is free by (v). Also, 4 is the only element of B which is different from all $(\langle\mathfrak{B}\rangle\, 0)\,\{x\}$, $x \in B$. But neither the empty set nor $[4]$ is a basis of $\mathfrak{B}$. Hence $\mathfrak{B}$ has no free basis. Thus $\mathfrak{B}$ is free but not freely generated.

Because an algebra may be free without being freely generated, the author thinks that the word "free" should not be used in the sense of "freely generated".

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