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# HIGHER DEGREES OF DISTRIBUTIVITY IN LATTICES AND LATTICE-ORDERED GROUPS

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Let $\alpha$ and $\beta$ be cardinal numbers. The $(\alpha, \beta)$-distributivity in Boolean algebras was studied by several authors (for references, cf. SIKORSKI [7, p. 61]). For each regular cardinal $\alpha$ there exists a complete Boolean algebra $B_\alpha$ that is $(\beta, \gamma)$-distributive for every $\beta < \alpha$ and that is not $(\alpha, \alpha)$-distributive (SCOTT [6]). PIERCE [3] and WEINBERG [9, 10] examined the $(\alpha, \beta)$-distributivity for lattice-ordered groups ($l$-groups). Pierce proved the following theorem ([3, Theorem 7.1]): an $\alpha$-complete Boolean algebra $B$ is $\alpha$-distributive if and only if the $l$-group $C(X(B))$ of all continuous real functions defined on the Stone space of $B$ is $\alpha$-distributive. Combining this with the theorem of Scott, we obtain that for each regular cardinal $\alpha$ there exists an $l$-group $G$ which is $\beta$-distributive for each $\beta < \alpha$ and which is not $\alpha$-distributive.

The aim of the present paper is the study of convex sublattices $S$ of an infinitely distributive lattice satisfying a certain maximality condition with respect to the $(\alpha, \beta)$-distributivity. The main idea is as follows. Suppose, at first, that $L$ is a complete modular lattice, $a \in L$. Let $\mathscr{L}$ be the system of all distributive intervals $[u, v]$ $(u \leqq v)$ of $L$ containing the element $a$. The system $\mathscr{L}$ (partially ordered by the set-theoretical inclusion) need not have a greatest element. To see this it suffices to take for $L$ the modular lattice with five elements that is not distributive. Let us now replace the condition of modularity of $L$ and that of distributivity of intervals by the condition of infinite distributivity of $L$ and the $(\alpha, \beta)$-distributivity of intervals, respectively. We shall prove that in this case the system $\mathscr{L}$ always contains a greatest element $L(a)$. The system $\{L(a)\}_{a \in L} = R$ is a partition of the set $L$ and the equivalence relation on $L$ that is defined by the partition $R$ is a congruence relation on the lattice $L$. If the lattice $L$ is not complete, we get analogical results by considering convex sublattices instead of intervals. In the case when $L$ is an $l$-group (the group operation being written additively) the set $L(0)$ is an $l$-ideal and $L(a) = a + L(0)$ for any $a \in L$. If, moreover, $L$ is relatively complete, then the $l$-ideal $L(0)$ is a direct factor of the $l$-group $L$.

Let us recall some basic concept and notations. The symbols $\cap$, $\cup$, $\bigcap$, $\bigcup$ and $\boldsymbol{\bigcap}$, $\boldsymbol{\bigcup}$,

$\cap$, $\cup$ denote the lattice-theoretical and set-theoretical operations, respectively. Let $L$ be a lattice, $a, b \in L$, $a \leqq b$. The interval $[a, b]$ is the set of all $x \in L$ fulfilling $a \leqq \leqq x \leqq b$. $[a, b]$ is nontrivial, if $a < b$. A sublattice $A \subset L$ is convex, if $a_1, a_2 \in A$, $a_1 \leqq a_2$ implies $[a_1, a_2] \subset A$. A subset $A$ of $L$ is a $c$-sublattice of $L$, if any least upper bound and any greatest lower bound of a subset of $A$ belongs to $A$. The convex $c$-hull of a set $B \subset L$ is the set $B_0 = \bigcap B_i$ where $\{B_i\}$ is the system of all convex $c$-sublattices $B_i$ of $L$ with $B \subset B_i$. A property $(p)$ of convex sublattices of $L$ is said to be hereditary, if each convex sublattice $B$ that is a subset of a convex sublattice $A$ satisfying $(p)$ posesses the property $(p)$ as well. The cardinality of a set $M$ is denoted by card $M$. If $\alpha_i, \beta_i$ $(i = 1, 2)$ are cardinals, we write $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$, if $\alpha_1 < \alpha_2$, $\beta_1 \leqq \beta_2$, or $\alpha_1 \leqq \alpha_2$, $\beta_1 < \beta_2$. For any cardinal $\alpha$ we denote by $\alpha+$ the first cardinal that is greater than $\alpha$.

Let $\{A_i\}_{i \in I}$ be a system of lattices.[1]) The complete direct product $P = \prod_{i \in I} A_i$ is the system of all functions $f : I \to \bigcup A_i$ with $f(i) \in A_i$ for each $i \in I$. If $f(i) = a_i$, then we write also $f = (\ldots, a_i, \ldots)$; $a_i$ is the component of $a$ in $A_i$. The lattice-operations in $P$ are performed component-wise. A complete direct product is non-trivial if at least two of the factors $A_i$ have the cardinalities greater than one. If $I = \{1, 2, \ldots, n\}$, then we write also $P = A_1 \times \ldots \times A_n$; $P$ is the direct product of the lattices $A_1, \ldots, A_n$.

Returning to the general case, let $f_0 \in P$ and let us denote (for a fixed $i \in I$ and for any $f \in P$)

$$A_i(f_0) = \{f : f \in P, f(j) = f_0(j) \text{ for each } j \in I, j \neq i\} .$$

Let $Q$ be a sublattice of $P$ such that $\bigcup_{i \in I} A_i(f_0) \subset Q$. Then we shall say that $Q$ is a complete subdirect product of lattice $A_i$ with respect to the element $f_0$. (For the concept of the complete subdirect product cf. RANEY [5], ŠIK [8] and WEINBERG [10].) If $\varphi$ is an isomorphism of a lattice $L$ into the lattice $P$, $a \in L$, $\varphi(a) = f, f(i) = a_i$, then $a_i$ is said to be the projection of $a$ into $A_i$.

## 1. FUNDAMENTAL NOTIONS

In this section some definitions concerning $(\alpha, \beta)$-distributivity are given and simple consequences of these definitions are deduced.

**1.1.** A lattice $L$ is said to be infinitely distributive if the following condition $(\mathrm{d}_1)$ and the condition dual to $(\mathrm{d}_1)$ are satisfied:

$(\mathrm{d}_1)$ If $x \in L$, $\{x_i\} \subset L$ and if the element $\cup x_i$ exists in $L$, then

$$x \cap (\cup x_i) = \cup (x \cap x_i) .$$

[1]) We suppose that $I \neq \emptyset$; in the sequel, this assumption remains valid for any set of indices.

Let $\alpha$ and $\beta$ be cardinal numbers and let $T$, $S$ be non-empty sets with card $T \leqq \alpha$, card $S \leqq \beta$. The lattice $L$ is $(\alpha, \beta)$-distributive if the folowing identities hold in $L$

$$(1.1) \qquad \bigcap_{t\in T} \bigcup_{s\in S} x_{t,s} = \bigcup_{\varphi\in S^T} \bigcap_{t\in T} x_{t,\varphi(t)},$$

$$(1.1') \qquad \bigcup_{t\in T} \bigcap_{s\in S} x_{t,s} = \bigcap_{\varphi\in S^T} \bigcup_{t\in T} x_{t,\varphi(t)}$$

under the assumption that all meets and unions standing in (1.1) and (1.1') exist in $L$. (The symbol $S^T$ denotes the system of all mappings of the set $T$ into the set $S$.) $L$ is $\alpha$-distributive if it is $(\alpha, \alpha)$-distributive. $L$ is completely distributive, if it is $\alpha$-distributive for each cardinal $\alpha$.

**1.1.1.** Thorough the present paper it is supposed that $L$ is infinitely distributive and card $L = \alpha_0$. By examining whether a convex sublattice $A$ of $L$ is $\alpha$-distributive or not we shall consider only the condition (1.1); by a dual argument one can verify whether the dual condition (1.1') is true or not. Analoguously, by deducing consequences from the supposition that $A$ is not $(\alpha, \beta)$-distributive we shall suppose that (1.1) is not satisfied; the case when (1.1') does not hold can be treated in a similar manner.

**1.2.** Let us remark the following simple fact. Suppose that $A$ is a convex sublattice of $L$, $\{a_i\} \subset A$. If the least upper bound of $\{a_i\}$ in the lattice $L$ or in $A$ exists, then we denote this element by $\bigcup a_i$ and $\sup_A a_i$, respectively. If $\sup_A a_i = b$ exists, then $\bigcup a_i$ also exists and $b = \bigcup a_i$. An analogical statement holds for the operation $\cap$. From this it follows: if a convex sublattice $A$ of $L$ is not $(\alpha, \beta)$-distributive, then $L$ is not $(\alpha, \beta)$-distributive, (or, conversely, if $L$ is $(\alpha, \beta)$-distributive, then each convex sublattice of $L$ is $(\alpha, \beta)$-distributive).

**1.3.** Suppose that $L$ is not $(\alpha, \beta)$-distributive. Then (cf. 1.1.1) there exists a system $\{x_{t,s}\}_{t\in T, s\in S}$ (card $T \leqq \alpha$, card $S \leqq \beta$) such that

$$(1.2) \qquad x = \bigcap_{t\in T} \bigcup_{s\in S} x_{t,s},$$

$$(1.2') \qquad y = \bigcup_{\varphi\in S^T} \bigcap_{t\in T} x_{t,\varphi(t)}$$

and $y < x$. Let us denote $(x_{t,s} \cap x) \cup y = z_{t,s}$. We have $y \leqq z_{t,s} \leqq x$ and from (1.2), (1.2') using the infinite distributivity we get

$$(1.3) \qquad x = \bigcap_{t\in T} \bigcup_{s\in S} z_{t,s},$$

$$(1.3') \qquad y = \bigcup_{\varphi\in S^T} \bigcap_{t\in T} z_{t,\varphi(t)}.$$

Hence the interval $[y, x]$ is not $(\alpha, \beta)$-distributivite. From (1.3) and from $z_{t,s} \in [y, x]$ it follows

$$(1.4) \qquad \bigcup_{s\in S} z_{t,s} = x \quad \text{for each} \quad t \in T;$$

in the same way from (1.3') we get

$$(1.4') \qquad \bigcap_{t\in T} z_{t,\varphi(t)} = y \quad \text{for each} \quad \varphi \in S^T.$$

**1.3.1. Corollary.** *If each interval of $L$ is $(\alpha, \beta)$-distributive, then $L$ is $(\alpha, \beta)$-distributive as well.*

**1.3.2.** It is easy to see that if (1.4) and (1.4′) hold and if $y < x$, then the interval $[y, x]$ is not $(\alpha, \beta)$-distributive.

If the lattice $L$ is $(\alpha, \beta)$-distributive and if $(\alpha_1, \beta_1) < (\alpha, \beta)$, then clearly $L$ is $(\alpha_1, \beta_1)$-distributive, too. Denote $2^{\alpha_0} = \alpha^*$. From the axiom of choice it follows $\alpha^* = \alpha_0^{\alpha_0}$.

**1.4.** *If $L$ is $(\alpha^*, \alpha_0)$-distributive, then $L$ is completely distributive.*

Proof. Let us assume that $L$ is not completely distributive and that it is $(\alpha^*, \alpha_0)$-distributive. Then there exists $(\alpha, \beta) > (\alpha^*, \alpha_0)$ such that $L$ is not $(\alpha, \beta)$-distributive. We shall be using the same notations as in 1.3. Let $t \in T$ be fixed. Since card $\{z_{t,s}\} \leqq$ $\leqq$ card $L = \alpha_0$, there exists a set $S_1 = S_1(t)$ and a system $\{v_{t,s}\}_{s \in S_1}$ such that card $S_1 =$ $= \alpha_0$, $\{v_{t,s}\}_{s \in S_1} = \{z_{t,s}\}_{s \in S}$. We may suppose that $S_1(t) = S_1(t')$ holds for each $t, t' \in T$ (since only the power of the set $S_1(t)$ is essential for our consideration). Hence we have by (1.4)

$$(1.5) \qquad \bigcup_{s \in S_1} v_{t,s} = x \quad \text{for each} \quad t \in T.$$

For every $\varphi \in S^T$ there exists $\varphi_1 \in S_1^T$ such that

$$(1.6) \qquad z_{t,\varphi(t)} = v_{t,\varphi_1(t)} \quad \text{for each} \quad t \in T$$

(since each $z_{t,s}$ equals to some suitably choosen element $v_{t,s_1}$). Conversely, for each $\varphi_1 \in S_1^T$ there exists $\varphi \in S^T$ such that (1.6) holds. From (1.6) and (1.4′) we get

$$(1.5') \qquad \bigcap_{t \in T} v_{t,\varphi_1(t)} = y \quad \text{for each} \quad \varphi_1 \in S_1^T.$$

From (1.5) and (1.5′) it follows that $L$ is not $(\alpha, \alpha_0)$-distributive. For any $t, t' \in T$ let us put $t \sim t'$ if $v_{t',s} = v_{t,s}$ for every $s \in S_1$. Then $\sim$ is an equivalence relation on the set $T$; we pick out an element from each class of the corresponding partition of $T$ and we denote the set of all these elements by $T_1$. If $t \in T_1$, let $\psi_t : S_1 \to L$ be a function satisfying $\psi_t(s) = v_{t,s}$ for each $s \in S_1$. Then the correspondence $t \to \psi_t$ is a one-to-one mapping of the set $T_1$ into the set $L^{S_1}$; hence card $T_1 \leqq \alpha^*$.

Let us now consider the system $\{v_{t,s}\}_{t \in T_1, s \in S_1}$. For each $\varphi_2 \in S_1^{T_1}$ there exists $\varphi_1 \in S_1^T$ such that

$$\{v_{t,\varphi_2(t)}\}_{t \in T_1} = \{v_{t,\varphi_1(t)}\}_{t \in T}$$

(for $t' \in T$ it suffices to choose $\varphi_1(t') = \varphi_2(t)$, where $t \sim t'$, $t \in T_1$). Therefore by (1.5′) we have

$$(1.5'') \qquad \bigcap_{t \in T_1} v_{t,\varphi_2(t)} = y \quad \text{for each} \quad \varphi_2 \in S_1^{T_1}.$$

Since $T_1 \subset T$, card $S_1 \leqq \alpha_0$ and card $T_1 \leqq \alpha^*$ it follows from (1.5) and (1.5″) that the lattice $L$ is not $(\alpha^*, \alpha_0)$-distributive; this is a contradiction.

**1.4.1. Corollary.** *If $L$ is $\alpha^*$-distributive, then it is completely distributive.*

**1.5.** *Assume that $L$ is not $(\alpha, \beta)$-distributive. Let $x$, $y$ have the same meaning as in 1.3; $y < x$. Then no non-trivial interval of the lattice $[y, x]$ is $(\alpha, \beta)$-distributive.*

Proof. Let $[u, v]$ be a non-trivial interval of the lattice $[y, x]$; put $y_{t,s} = (z_{t,s} \cap \cap v) \cup u$. From the infinite distributivity and from (1.3), (1.3′) it follows

$$v = \bigcap_{t\in T} \bigcup_{s\in S} y_{t,s} > \bigcup_{\varphi\in S^T} \bigcap_{t\in T} y_{t,\varphi(t)} = u\,.$$

**1.6.** Let $A$ be a convex sublattice of $L$. If $A$ is not completely distributive, then we shall denote by $dA$ the least cardinal number $\gamma$ for which $A$ is not $\gamma$-distributive. For any cardinals $\alpha$, $\beta$ we shall examine the following conditions:

($p_1$) $A$ is $(\alpha, \beta)$-distributive.
($p_2$) If $(\alpha_1, \beta_1) < (\alpha, \beta)$, then $A$ is $(\alpha_1, \beta_1)$-distributive.
($p_3$) $A$ is $\alpha$-distributive.
($p_4$) $A$ is $\alpha_1$-distributive for each $\alpha_1 < \alpha$.
($p_5$) $d[a, b] = \alpha +$ for every non-trivial interval $[a, b] \subset A$.
($p_6$) $d[a, b] = \alpha$ for every non-trivial interval $[a, b] \subset A$.

If necessary, we shall be using a more detailed notation, i.e. $(p_j(\alpha, \beta))$ instead of $(p_j)$ for $j = 1, 2$, and $(p_j(\alpha))$ instead of $(p_j)$ for $j = 3, 4, 5, 6$. Let us put $J = \{1, \ldots, 6\}$. The following statements are immediate consequences of the definition 1.6:

**1.7.** Each condition $(p_j)$ $(j \in J)$ is hereditary. If $a \in L$, then the one-element interval $\{a\}$ satisfies $(p_j)$ for each $j \in J$. The implications

$$(p_1(\alpha, \beta)) \Rightarrow (p_2(\alpha, \beta)),\ (p_3(\alpha)) \Rightarrow (p_4(\alpha)),\ (p_5(\alpha)) \Leftrightarrow (p_6(\alpha+))$$

are fulfilled for any cardinals $\alpha$, $\beta$.

**1.8.** For each $j \in J$ we shall consider also the following condition which is in a certain sense complementary to the condition $(p_j)$:

($p'_j$) No non-trivial interval of the lattice $A$ satisfies the condition $(p_j)$.

It is obvious that the condition $(p'_j)$ is hereditary.

## 2. THE CONDITIONS $(p_j)$ FOR INTERVALS

In this section it is assumed that $\alpha$ and $\beta$ are fixed cardinals. We shall prove at first some simple lemmas on transposed intervals (in 2.1, 2.2 and 2.3 it would be sufficient to suppose that $L$ is distributive rather than infinitely distributive). Let us remark that for $a, b \in L$, $a \cap b = u$, $a \cup b = v$ the intervals $[u, a]$, $[b, v]$ are called transposed.

**2.1.** (Cf. BIRKHOFF [1].) *Any transposed intervals are isomorphic.*

**2.2.** *Let $a, b, c \in L$, $a \leqq b \leqq c$ and let $[x_1, x_2]$ be a non-trivial interval of $L$, $[x_1, x_2] \subset [a, c]$. Then there exists a non-trivial interval $[y_1, y_2] \subset [x_1, x_2]$ which is transposed to an interval contained in $[a, b]$ or in $[b, c]$.*

Proof. For each $z \in [a, c]$ let us denote $z' = b \cap z$, $z'' = b \cup z$. The following statement follows from the distributivity of $L$: if $z_1, z_2 \in [a, c]$ and $z_1' = z_2'$, $z_1'' = z_2''$, then $z_1 = z_2$. Therefore we have $x_1' < x_2'$ or $x_1'' < x_2''$. Let the first case be considered. Then it suffices to put $y_1 = x_1$, $y_2 = x_1 \cup x_2'$. The second case is analogical to the first one.

**2.3.** *Let $a, b, c \in L$, $a \leqq b$. The interval $[a \cup c, b \cup c]$ is transposed to an interval contained in $[a, b]$.*

Proof. For this purpose it is sufficient to take the interval $[b \cap (a \cup c), b]$.

Remark. Obviously, the statement dual to 2.3 also holds.

**2.4.** *Let $a \in L$, $\{b_i\} \subset L$, $a \leqq b_i$ for each $b_i$, $\bigcup b_i = b$. Let $[x_1, x_2]$ be a non-trivial interval of $L$, $[x_1, x_2] \subset [a, b]$. Then there exists a non-trivial interval $[y_1, y_2] \subset$ $\subset [x_1, x_2]$ and an element $b_{i_0} \in \{b_i\}$ such that $[y_1, y_2]$ is transposed to an interval contained in $[a, b_{i_0}]$.*

Proof. For each $z \in [a, b]$ we put $z^i = z \cap b_i$. Then we have

$$z = z \cap b = z \cap \left(\bigcup b_i\right) = \bigcup (z \cap b_i) = \bigcup z^i . \tag{2.1}$$

From this and from $x_1 < x_2$ it follows that there exists at least one $i$ satisfying $x_1^i < x_2^i$. It suffices to apply now the statement dual to 2.3.

**2.5.** *Let $a, b, c \in L$, $a \leqq b \leqq c$. If the intervals $[a, b]$, $[b, c]$ satisfy $(\mathrm{p}_1)$, then $[a, c]$ also fulfils $(\mathrm{p}_1)$.*

Proof. For $z \in [a, c]$ let $z'$ and $z''$ have the same meaning as in 2.2. Let $\{x_{t,s}\}_{t\in T, s\in S} \subset [a, c]$, card $T \leqq \alpha$, card $S \leqq \beta$. Let us assume that all meets and unions standing in (1.1) exist in $L$. The element on the left or on the right side of (1.1) will be denoted by $x$ or $y$ respectively. From the infinite distributivity of $L$ it follows

$$x' = \bigcap_{t\in T} \bigcup_{s\in S} (x_{t,s})' , \quad y' = \bigcup_{\varphi\in S^T} \bigcap_{t\in T} (x_{t,\varphi(t)})' .$$

Since we assume that $[a, b]$ satisfies $(\mathrm{p}_1)$, we have $x' = y'$. Similarly $x'' = y''$ holds, and therefore $x = y$.

**2.6.** *Let $a, b, c \in L$, $a \leqq b \leqq c$, $j \in J$. If the intervals $[a, b]$, $[b, c]$ satisfy the condition $(\mathrm{p}_j)$, then $[a, c]$ also fulfils $(\mathrm{p}_j)$.*

Proof. For $j = 1$ the statement is proved in 2.5 and for $j = 2$ it is an easy consequence of 2.5. For $j = 3$ it suffices to put $\alpha = \beta$. The statement for $j = 4$ follows easily from the case $j = 3$.

For $j = 6$ we proceed as follows. Let $\alpha_1 < \alpha$. By our assumption the intervals $[a, b]$, $[b, c]$ are $\alpha_1$-distributive, hence by 2.5 the interval $[a, c]$ is $\alpha_1$-distributive, too. Let us suppose now that there exists a non-trivial $\alpha$-distributive interval $[x_1, x_2] \subset [a, c]$. Then by 2.2 and 2.1 there exists a non-trivial $\alpha$-distributive interval contained in $[a, b]$ or in $[b, c]$. This is a contradiction.

From the statement for $j = 6$ the statement for $j = 5$ follows.

**2.7.** *Let $a \in L$, $\{b_i\} \subset L$, $a \leqq b_i$ for each $b_i$, $\bigcup b_i = b$, $j \in J$. If each interval $[a, b_i]$ satisfies* $(\mathrm{p}_j)$, *then $[a, b]$ also fulfils* $(\mathrm{p}_j)$.

Proof. Let us consider at first the case $j = 1$. For $z \in [a, b]$ let $z^i$ have the same meaning as in 2.4. Let $\{x_{t,s}\}_{t \in T, s \in S} \subset [a, b]$, card $T \leqq \alpha$, card $S \leqq \beta$. Let us assume that (1.2) and (1.2') holds. By the infinite distributivity

$$x^i = \bigcap_{t \in T} \bigcup_{s \in S} (x_{t,s})^i\,, \quad y^i = \bigcup_{\varphi \in S^T} \bigcap_{t \in T} (x_{t,\varphi(t)})^i\,.$$

The elements standing in these equations belong to $[a, b_i]$; since $[a, b_i]$ is $(\alpha, \beta)$-distributive, we have $x^i = y^i$. Therefore by (2.1) $x = y$. The statements for $j = 2, 3$, are easy consequences of the case $j = 1$. Let $j = 6$ and $\alpha_1 < \alpha$. As we have already proved the interval $[a, b]$ is $\alpha_1$-distributive. Assuming the existence of a non-trivial $\alpha$-distributive interval $[x_1, x_2] \subset [a, b]$ we get that by 2.4 and 2.1 there exists a non-trivial $\alpha$-distributive interval which is contained in some interval $[a, b_i]$; this is a contradiction. The proof for $j = 6$ is complete. Hence by 1.7 the statement holds also for the case $j = 5$.

**2.8.** Let $a, b, c \in L$, $a \leqq b$, $j \in J$. If $[a, b]$ satisfies $(\mathrm{p}_j)$, then $[a \cup c,\ b \cup c]$ satisfies $(\mathrm{p}_j)$, as well. The proof follows from 2.3 and 2.1.

Remark. By an analogical argument one can prowe the statements dual to 2.7 and 2.8.

**2.9.** The statements 2.6, 2.7 and 2.8 remain valid if the condition $(\mathrm{p}_j)$ is replaced by $(\mathrm{p}'_j)$ $(j \in J)$.

This follows from 2.1, 2.2, 2.4 and 2.3 and from the fact that $(\mathrm{p}'_j)$ is hereditary.

## 3. THE SUBLATTICES $L_j(a)$

Let $\alpha$ and $\beta$ be fixed cardinals and let $a \in L$, $j \in J$. With the aid of 2.6, 2.7 and 2.8 some further results on the condition $(\mathrm{p}_j)$ will be deduced. Let us denote by $L_j^1(a)$ the set of all elements $z \in L$, $z \geqq a$ such that the interval $[a, z]$ satisfies $(\mathrm{p}_j)$. Analogically let us put $L_j^2(a) = \{z : z \in L,\ z \leqq a,\ [z, a] \text{ satisfies } (\mathrm{p}_j)\}$.

**3.1.** *$L_j^1(a)$ is a convex c-sublattice of $L$ with the least element $a$.*

Proof. Obviously, $a$ is the least element of $L_j^1(a)$. Since $(\mathrm{p}_j)$ is hereditary, it follows from $a \leqq z_1 \leqq z_2$, $z_2 \in L_j^1(a)$ that $z_1 \in L_j^1(a)$ and hence $L_j^1(a)$ is a convex subset of $L$.

Let $z_i \in L_j^1(a)$, $\bigcup z_i = z$. By 2.7 we have $z \in L_j^1(a)$. If $\bigcap z_i = v$, then $a \leqq v \leqq z_i$, and therefore by the convexity of $L_j^1(a)$ the element $v$ belongs to $L_j^1(a)$.

By the dual argument one can show:

**3.1'.** *$L_j^2(a)$ is a convex c-sublattice of $L$ with the greatest element $a$.*

We shall denote by $L_j(a)$[2]) the set-theoretical union of all intervals $[b, c]$ with $b \in L_j^2(a)$, $c \in L_j^1(a)$.

**3.2.** *The set $L_j(a)$ is a convex c-sublattice of $L$ satisfying the condition* $(\mathrm{p}_j)$.

Proof. Let $\{a_i\} \subset L_j(a)$, $\bigcup a_i = a_0$. To each $a_i$ there exist elements $b_i, c_i$ such that $a_i \in [b_i, c_i]$, $b_i \in L_j^2(a)$, $c_i \in L_j^1(a)$. Hence by 3.1 $a \cup a_i \in L_j^1(a)$, $a \cup a_0 = a \cup \cup (\bigcup a_i) = \bigcup (a \cup a_i) \in L_j^1(a)$. For each $b_i$ we have $b_i \leqq a_0 \leqq a \cup a_0$, hence $a_0 \in \in L_j(a)$. Analogously one can prove: if $\bigcap a_i$ exists in $L$, then $\bigcap a_i \in L_j(a)$. This proves that $L_j(a)$ is a $c$-sublattice of $L$. Let $a_1, a_2 \in L_j(a)$, $a_3 \in L$, $a_1 \leqq a_3 \leqq a_2$. If we use the notations analogical to those used above, we have $b_1 \leqq a_3 \leqq c_2$, hence $L_j(a)$ is a convex sublattice of $L$. Since $[a_1, a_2] \subset [b_1, c_2]$, the interval $[a_1, a_2]$ satisfies $(\mathrm{p}_j)$ by 2.6. The proof is complete.

**3.3.** *Let $A$ be a convex sublattice of $L$ satisfying $(\mathrm{p}_j)$ and let $a \in A$. Then $A \subset \subset L_j(a)$.*

Proof. If $z \in A$, then $z \cap a, z \cup a \in A$, therefore the intervals $[z \cap a, a]$, $[a, z \cup a]$ satisfy $(\mathrm{p}_j)$ and $z \cap a \in L_j^2(a)$, $z \cup a \in L_j^1(a)$. This implies $z \in L_j(a)$.

**3.4.** *If $a_1 \in L_j(a)$, then $L_j(a_1) = L_j(a)$.*

Proof. Let $a_1 \in L_j(a)$. By 3.2 and 3.3 we have $L_j(a) \subset L_j(a_1)$. But then $a \in L_j(a_1)$, and hence $L_j(a_1) \subset L_j(a)$.

Since $a \in L_j(a)$, it follows from 3.4 that the system $\{L_j(a)\}_{a \in L}$ is a partition of the set $L$; this partition (and also the corresponding equivalence relation) will be denoted by $R_j$.

**3.5.** *$R_j$ is a congruence relation on $L$.*

Proof. For $z \in L$ we shall denote $\bar{z} = L_j(z)$. Let $a, b, c \in L$, $\bar{a} = \bar{b}$. Put $a \cap b = u$, $a \cup b = v$. Since $\bar{a}$ is a sublattice of $L$, we have $u, v \in \bar{a}$, therefore the interval $[u, v]$ satisfies $(\mathrm{p}_j)$. By 2.8 the interval $[u \cup c, v \cup c]$ also satisfies $(\mathrm{p}_j)$ and hence by 3.3 $\overline{u \cup c} = \overline{v \cup c}$. Since $a \cup c, b \cup c \in [u \cup c, v \cup c]$, we have also $\overline{a \cup c} = \overline{b \cup c}$. Analogically (by using the statement dual to 2.8) we can prove $\overline{a \cap c} = \overline{b \cap c}$. This completes the proof.

In the sections 3.6–3.9 the symbol $\bar{z}$ has the same meaning as in 3.5.

**3.6.** *The congruence $R_j$ has the following property:*

(v) *If $\{a_i\} \subset L$, $\bigcup a_i = a$, then in the factor-lattice $L/R_j$ $\bigcup \bar{a}_i = \bar{a}$ holds.*

---

[2]) If necessary, we write $L_j(a, \alpha)$ instead of $L_j(a)$ for $j = 3, 4, 5, 6$.

Proof. From $\bigcup a_i = a$ it follows $\bar{a}_i \leqq \bar{a}$ for each $\bar{a}_i$. Let $\bar{c} \in L/R_j$ and let $\bar{a}_i \leqq \bar{c}$ for each $\bar{a}_i$. Then we obtain $a_i \cup c = c_i \in \bar{c}$ for each $a_i$. Since $\bar{c}$ is a $c$-sublattice of $L$, the equation $c \cup a = c \cup (\bigcup a_i) = \bigcup(c \cup a_i) = \bigcup c_i$ implies $c \cup a \in \bar{c}$, hence $\bar{c} \cup \bar{a} = \bar{c}$, $\bar{a} \leqq \bar{c}$. This proves that $\bigcup \bar{a}_i = \bar{a}$.

Remark. Analogically one can prove the statement dual to (v). It is well-known that a congruence on a general lattice need not satisfy the condition (v).

**3.7.** *If $L$ is a complete lattice, then each class of the congruence $R_j$ contains a least and a greatest element.*

Proof. If $L$ is complete and $z \in L$, then there exists the least upper bound $z_1$ and the greatest lower bound $z_2$ of the set $\bar{z} = L_j(z)$. By 3.2 $\bar{z}$ is a $c$-sublattice of $L$, therefore $z_1, z_2 \in \bar{z}$.

The previous results 3.1 – 3.7 will be summarized in the following theorem:

**3.8. Theorem.** *Let $L$ be an infinitely distributive lattice. There exists a partition $R_j$ of the set $L$ with the following properties:*

(a) *If $a \in L$, then the class $\bar{a}$ of the partition $R_j$ containing the element $a$ is the greatest element in the system of all convex sublattices of $L$ satisfying $(\mathrm{p}_j)$ and containing $a$.*

(b) *$R_j$ is a congruence relation on $L$ fulfilling the condition* (v) *and the dual one.*

(c) *Each congruence class of $R_j$ is a $c$-sublattice of $L$. If $L$ is complete, then each such class has the least and the greatest element.*

In proving the theorem 3.8 we have been using the propositions 2.6, 2.7 and 2.8 only (without the explicite use of the definition of $(\mathrm{p}_j)$); hence we obtain by 2.9 the following result:

**3.8'.** *The theorem 3.8 remains true if the condition $(\mathrm{p}_j)$ is replaced by $(\mathrm{p}'_j)$.*

**3.9.** *Let $j \in \{1, 2, 3, 4\}$. Let us put $\bar{L} = L/R_j$ and let $\bar{a}, \bar{b} \in \bar{L}$, $\bar{a} < \bar{b}$. The interval $[\bar{a}, \bar{b}]$ does not fulfil the condition $(\mathrm{p}_j)$.*

It is easy to see that it suffices to prove this for $j = 1$. Let $\bar{a}, \bar{b} \in L/R_j$, $\bar{a} < \bar{b}$. Then there exists $a_1 \in \bar{a}$ and $b_1 \in \bar{b}$ such that $a_1 < b_1$. Since $\bar{a}_1 \neq \bar{b}_1$, the interval $[a_1, b_1]$ is not $(\alpha, \beta)$-distributive. Hence there exists a system $\{z_{t,s}\}_{t\in T, s\in S} \subset [a_1, b_1]$ with the properties as in 1.3. By 3.6, (1.3) and (1.3')

$$\bar{x} = \bigcap_{t\in T} \bigcup_{s\in S} \overline{z_{t,s}}, \tag{3.1}$$

$$\bar{y} = \bigcup_{\varphi\in S^T} \overline{\bigcap_{t\in T} z_{t,\varphi(t)}}. \tag{3.1'}$$

Since $[\bar{y}, \bar{x}] \subset [\bar{a}_1, \bar{b}_1] = [\bar{a}, \bar{b}]$ and the interval $[y, x]$ is not $(\alpha, \beta)$-distributive, we have $\bar{y} < \bar{x}$. By (3.1) and (3.1') the interval $[\bar{a}, \bar{b}]$ is not $(\alpha, \beta)$-distributive.

Let the symbol $L'_j(a)$ have the analogical meaning as $L_j(a)$ with the distinction that instead of $(p_j)$ we are dealing with the condition $(p'_j)$. In the section 5 the following simple proposition will be used:

**3.10.** $L_j(a) \cap L'_j(a) = \{a\}$ *holds for each* $a \in L$.

Proof. If $z \in L_j(a) \cap L'_j(a)$, let us denote $u = z \cap a$, $v = z \cup a$. Since $L_j(a)$ and $L'_j(a)$ are convex sublattices of $L$, we have $[u, v] \subset L_j(a) \cap L'_j(a)$. Therefore $[u, v]$ satisfies $(p_j)$ and no non-trivial interval contained in $[u, v]$ satisfies $(p_j)$; hence $u = v$, $z = a$.

## 4. THE RELATIONS AMONG THE SETS $L_j(a)$

In this section the relations among the sets $L_{j_1}(a, \alpha_1)$, $L_{j_2}(a, \alpha_2)$ $(j_1, j_2 \in J)$ will be studied where $\alpha_1$ and $\alpha_2$ are any cardinals. An immediate consequence of the definition 1.6 (cf. also 1.7) is the following proposition:

**4.1.** (a) $\alpha_1 < \alpha_2 \Rightarrow L_3(a, \alpha_2) \subset L_3(a, \alpha_1)$, $L_4(a, \alpha_2) \subset L_4(a, \alpha_1)$.
(b) $L_3(a, \alpha_1) \subset L_4(a, \alpha_1)$, $L_5(a, \alpha_1) \subset L_3(a, \alpha_1)$, $L_6(a, \alpha_1) \subset L_4(a, \alpha_1)$.
(c) $L_5(a, \alpha_1) = L_6(a, \alpha_1 +)$.

**4.2.** $L_3(a, \alpha_1) \cap L_6(a, \alpha_1) = \{a\} = L_5(a, \alpha_1) \cap L_6(a, \alpha_1)$.
The first statement follows from the fact that each interval of the lattice $L_3(a, \alpha_1)$ is $\alpha_1$-distributive and no non-trivial interval of $L_6(a, \alpha_1)$ is $\alpha_1$-distributive. The second statement follows from the first one and from 4.1 (b).

**4.3.** $\alpha_1 \neq \alpha_2 \Rightarrow L_6(a, \alpha_1) \cap L_6(a, \alpha_2) = \{a\}$.

Proof. If a non-trivial interval $[z_1, z_2]$ is a subset of $L_6(a, \alpha_1) \cap L_6(a, \alpha_2)$, then $\alpha_1 = d[z_1, z_2] = \alpha_2$ would be true and this is a contradiction.

**4.4.** $\alpha_1 \leqq \alpha_2 \Rightarrow L_6(a, \alpha_1) \cap L_3(a, \alpha_2) = \{a\}$.

Proof. If $\alpha_1 \leqq \alpha_2$, then no non-trivial interval of $L_6(a, \alpha_1)$ is $\alpha_1$-distributive and each interval of $L_3(a, \alpha_2)$ is $\alpha_1$-distributive.

**4.5.** *Let* $[z_1, z_2]$ *be a non-trivial interval of L. There exists a non-trivial interval* $[y_1, y_2] \subset [z_1, z_2]$ *satisfying* $(p_3(\alpha^*))$ *or* $(p_6(\alpha))$ *for some infinite* $\alpha \leqq \alpha^*$.

Proof. If $[z_1, z_2]$ is completely distributive, then the interval $[y_1, y_2] = [z_1, z_2]$ fulfils the condition $p_3(\alpha^*)$. Let us assume that $[z_1, z_2]$ is not completely distributive and $d[z_1, z_2] = \alpha_1$. Since $L$ is distributive, $\aleph_0 \leq \alpha_1$ and by 1.4.1 $\alpha_1 \leqq \alpha^*$. In the interval $[z_1, z_2]$ there exist elements $x$, $y$ and a system $\{x_{t,s}\}_{t \in T, s \in S}$ satisfying the same conditions as in 1.3 (where we set $\alpha = \beta = \alpha_1$). Let us put $[y_1, y_2] = [y, x]$. Let $[v_1, v_2]$ be a non-trivial interval contained in $[y_1, y_2]$. By 1.5 $[v_1, v_2]$ is not $\alpha_1$-distributive. If $\alpha_2 < \alpha_1$, then $[z_1, z_2]$ is $\alpha_2$-distributive, hence $[v_1, v_2]$ is $\alpha_2$-distributive, too. This implies $d[v_1, v_2] = \alpha_1$, hence $[y_1, y_2]$ fulfils the condition $(p_6(\alpha_1))$.

Remark. Let $[a, z]$ be a non-trivial interval of $L$. There need not exist, in general, a non-trivial interval $[a, y] \subset [a, z]$ satisfying $(p_3(\alpha^*))$ or $(p_6(\alpha))$ for some $\alpha$, $\aleph_0 \leqq \leqq \alpha \leqq \alpha^*$.

Example: Let $\{\alpha_i\}_{i=1,2,\dots}$ be an ascending sequence of regular cardinals. Let $B_i$ $(i = 1, 2, \dots)$ be a Boolean algebra which is $\beta$-distributive for every $\beta < \alpha_i$ and which is not $\alpha_i$-distributive. The least and the greatest element of $B_i$ will be denoted by $u_i$ and $v_i$, respectively. Let $B_0$ be an one-element Boolean algebra $\{0\}$. We define a partial order on the set $L = \bigcup B_i$ $(i = 0, 1, 2, \dots)$ as follows: on each set $B_i$ the partial order has its original meaning; the element 0 will be the least element of $L$; if $b_1 \in B_{i_1}$, $b_2 \in B_{i_2}$, $i_1, i_2 \geqq 1$, $i_1 \neq i_2$, we put $b_1 < b_2$ if $i_2 < i_1$. It is easy to see that $L$ is an infinitely distributive lattice. Let $[0, z]$, $[0, y]$ be non-trivial intervals of $L$, $[0, y] \subset \subset [0, z]$. Then $y \in B_i$ for some $i \geqq 1$. Since $[u_{i+1}, v_{i+1}] \subset [0, y]$ and by our assumption $d[u_{i+1}, v_{i+1}] = \alpha_{i+1}$, the interval $[0, y]$ is not completely distributive and $d[0, y] \leqq \alpha_{i+1}$. Further we have $[u_{i+2}, v_{i+2}] \subset [0, y]$, $d[u_{i+2}, v_{i+2}] = \alpha_{i+2} > > d[0, y]$. From this we obtain that $[0, y]$ does not fulfil any condition $(p_3(\alpha^*))$, $(p_6(\alpha))$ $(\aleph_0 \leqq \alpha \leqq \alpha^*)$.

We shall be using the following notations:

$I = \{0\} \cup \{\alpha : \aleph_0 \leqq \alpha \leqq \alpha^*\}$,
$H_\alpha(a) = L_6(a, \alpha)$ (for $\aleph_0 \leqq \alpha \leqq \alpha^*$),
$H_0(a) = L_3(a, \alpha^*)$,
$H_i^1(a) = \{z : z \in H_i(a), z \geqq a\}$ $(i \in I)$,
$H_i^2(a) = \{z : z \in H_i(a), z \leqq a\}$ $(i \in I)$.

**4.6.** *If $i_1, i_2 \in I$, $i_1 \neq i_2$, $z_1 \in H_{i_1}^1(a)$, $z_2 \in H_{i_2}^1(a)$, then $z_1 \cap z_2 = a$.*

This follows immediately from 4.3 and 4.4.

The convex $c$-envelope of the set-theoretical union of the system $\{H_i^1(a)\}_{i\in I}$ will be denoted by $H^1(a)$.

**4.7.** *Let $L$ be a conditionally complete lattice. Then each element $z \in H^1(a)$ can be written uniquely in the form $z = \bigcup_{i\in I} z_i$, where $z_i \in H_i^1(a)$ for each $i \in I$. Let $z = \bigcup_{i\in I} z_i$, $v = \bigcup_{i\in I} v_i (z, v \in H^1(a)$, $z_i, v_i \in H_i^1(a)$, $i \in I)$; then the inequality $z \leqq v$ holds if and only if $z_i \leqq v_i$ for each $i \in I$.*

Proof. Let $W$ be the set-theoretical union of all intervals $[a, w]$ where $w$ can be expressed in the form $w = \bigcup_{k\in K} w_k$, $\{w_k\}_{k\in K} \subset \bigcup_{i\in I} H_i^1(a)$. Since $H^1(a)$ is a convex $c$-sublattice of $L$ and $\bigcup_{i\in I} H_i^1(a) \subset H^1(a)$, we obtain $W \subset H^1(a)$. On the other hand the least upper bound of any subset of $W$ (if it exists) obviously belongs to $W$; since $W$ has the least element and $W$ is convex in $L$, an analogical assertion holds for the meets of any subset of $W$. Therefore $W$ is a convex $c$-sublattice of $L$, $W \supset \bigcup_{i\in I} H_i^1(a)$, which implies $W = H^1(a)$. Under the same notations for $w \in W$, let us denote $K_i = \{k : k \in \in K, w_k \in H_i^1(a)\}$. If $K_i = \emptyset$, we put $w_i = a$. If $K_i \neq \emptyset$, then the set $\{w_k\}_{k\in K_i}$ is bounded;

since $L$ is conditionally complete and since $H_i^1(a)$ is a $c$-sublattice of $L$, there exists $w_i = \bigcup_{k\in K} w_k$ and $w_i \in H_i^1(a)$ holds. We obtain $w = \bigcup_{i\in I} w_i$.

Let us now suppose that $z \in H^1(a)$. Since $H^1(a) = W$, we can put $w = z$, $w_i = z_i$ and we have $z = \bigcup_{i\in I} z_i$, $z_i \in H_i^1(a)$. If at the same time $z = \bigcup_{i\in I} t_i$, $t_i \in H_i^1(a)$, then by 4.6 $z_{i_1} \cap t_{i_2} = 0$ holds for $i_1, i_2 \in I$, $i_1 \neq i_2$ and therefore $z_{i_1} = z_{i_1} \cap z = z_{i_1} \cap \cap (\bigcup_{i\in I} t_i) = z_{i_1} \cap t_{i_1}$, $z_{i_1} \leqq t_{i_1}$. Analogously we obtain $t_{i_1} \leqq z_{i_1}$. Consequently $z_{i_1} = t_{i_1}$ for each $i_1 \in I$. The proof of the first assertion is complete.

Let $z = \bigcup_{i\in I} z_i$, $v = \bigcup_{i\in I} v_i$, $z_i, v_i \in H_i^1(a)$. If $z_i \leqq v_i$ for each $i \in I$, then, obviously, $z \leqq v$. If $z \leqq v$, then by 4.6 $z_i = z_i \cap v = z_i \cap v_i$, hence $z_i \leqq v_i$.

Under the same notation as in 4.7 let us consider the mapping $\varphi : H^1(a) \to \to \prod_{i\in I} H_i^1(a)$ defined as follows: $\varphi(z) = (\ldots, z_i, \ldots)$ for each $z \in H^1(a)$. Put $\varphi(H^1(a)) = B$. By 4.7 the partially ordered sets $H^1(a)$ and $B$ are isomorphic. Moreover, we have:

**4.8.** *$B$ is a complete subdirect product of the lattices $H_i^1(a)$ $(i \in I)$ with respect to the element $a$.*

Proof. Let $b_1 = (\ldots, z_i, \ldots)_{i\in I} \in B$, $b_2 = (\ldots\ v_i, \ldots)_{i\in I} \in B$. Then $b_1 = \varphi(z)$ and $b_2 = \varphi(v)$, where $z = \bigcup_{i\in I} z_i$, $v = \bigcup_{i\in I} v_i$. From this we obtain $z \cup v \in H^1(a)$, $z \cup v = = \bigcup_{i\in I}(z_i \cup v_i)$, hence $\varphi(z \cup v) = (\ldots, z_i \cup v_i, \ldots)$ and therefore $b_1 \cup b_2 \in B$. By the infinite distributivity and by 4.6 $z \cap v = \bigcup_{i\in I}(z_i \cap v_i)$, hence $b_1 \cap b_2 \in B$, too. This proves that $B$ is a sublattice of $\prod_{i\in I} H_i^1(a)$. If $z = a$, then we have obviously $z_i = a$, for each $i \in I$. To complete the proof we have to show the following (cf. the introduction): if $i_0 \in I$ and $z_i = a$ for each $i \in I$, $i \neq i_0$, $z_{i_0} \in H_{i_0}^1(a)$, then $b = = (\ldots, z_i, \ldots) \in B$. But under our assumptions $z_{i_0} \in H_{i_0}^1(a) \subset H^1(a)$ and hence $\varphi(z_{i_0}) = b$, $b \in B$.

**4.9. Theorem.** *Let $L$ be a conditionally complete infinitely distributive lattice. Then the lattice $H^1(a)$ is isomorphic to a complete subdirect product of lattices $H_i^1(a)$ $(i \in I)$ with respect to the element $a$. If for any system $\{z_i\}_{i\in I}$, $z_i \in H_i^1(a)$ there exists in $L$ the element $\bigcup_{i\in I} z_i$, then $H^1(a)$ is isomorphic to the lattice $\prod_{i\in I} H_i^1(a)$.*

The first assertion follows from 4.7 and 4.8. The second one follows from the fact that under our assumption $\varphi(H^1(a)) = \prod_{i\in I} H_i^1(a)$ holds.

By a dual argument we can prove the analogical proposition for the convex $c$-envelope of the set-theoretical union of the system $\{H_i^2(a)\}_{i\in I}$.

## 5. RELATIVELY COMPLEMENTED DISTRIBUTIVE LATTICES

In this section we shall assume that $L$ is a relatively complemented distributive lattice (because of the following lemma 5.1 we need not explicitely suppose that $L$ is infinitely distributive). If $z \in [u, v] \subset L$, we shall denote by $z'_{[u,v]}$ the relative complement of $z$ with respect to the interval $[u, v]$. It is well-known that any Boolean algebra is infinitely distributive; hence each interval of $L$ is infinitely distributive.

**5.1.** *The lattice $L$ is infinitely distributive.*

Proof. It suffices to prove that the condition $(d_1)$ from 1.1 is satisfied; the proof of the dual condition is a similar one. Let $\{x_k\}_{k \in K} \subset L$, $\bigcup x_k = x$, $y \in L$. Let us choose a fixed index $k_0 \in K$ and denote $x_{k_0} \cup x_k = x_k^0$. Put $x_{k_0} \cap y = u$, $x \cup y = v$. Then the elements $x, y, x_k^0$ belong to the Boolean algebra $[u, v]$, hence $y \cap x = y \cap \cap (\bigcup x_k) = y \cap (\bigcup x_k^0) = \bigcup (y \cap x_k^0) = z$. Since $x_k \leqq x_k^0$, it follows $y \cap x_k \leqq z$ for each $k \in K$. If $z \neq \bigcup (y \cap x_k)$, then there exists an element $z_1 < z$ satisfying $y \cap x_k \leqq \leqq z_1$ for each $k \in K$. Thus $(y \cap x_k) \cup (y \cap x_{k_0}) \leqq z_1$ whence $y \cap x_k^0 \leqq z_1$ and therefore $z \leqq z_1$, which is a contradiction.

Let $a \in L$, $j \in J$; analogously as in the section 3 we set $L_j'^1(a) = \{z : z \in L_j'(a), z \geqq a\}$, $L_j'^2(a) = \{z : z \in L_j'(a), z \leqq a\}$. Further we put $L^1(a) = \{z : z \in L, z \geqq a\}$, $L^2(a) = \{z : z \in L, z \leqq a\}$. We shall prove some propositions on $L_j^1(a)$, $L_j'^1(a)$; analogical propositions hold for $L_j^2(a)$, $L_j'^2(a)$.

**5.2.** *If the interval $[a, c] \subset L$ contains a non-trivial interval satisfying $(p_j)$, then there exists a non-trivial interval $[a, c_1] \subset [a, c]$ satisfying $(p_j)$.*

Proof. If a non-trivial interval $[u, v] \subset [a, c]$ fulfills $(p_j)$, then the non-trivial interval $[a, v \cap u'_{[a,c]}]$ also satisfies $(p_j)$.

Remark. An analogical assertion holds for $(p_j')$.

Let us denote by (1), (2) and (3) the condition that $L$, $L_j^1(a)$ and $L_j'^1(a)$, respectively, contains a greatest element.

**5.3.** *If any two conditions from (1), (2), (3) hold true, then the remaining one also holds. In such a case the greatest element of the set $L_j^1(a)$ is the relative complement of the greatest element of the set $L_j'^1(a)$ with respect to the interval $[a, u]$, where $u$ is the greatest element of $L$.*

This proposition is a consequence of the following lemmas 5.3.1 – 5.3.4.

**5.3.1.** *Let $u$ be an upper bound of the set $L_j^1(a) \cup L_j'^1(a)$. Then $u$ is the greatest element of $L$.*

Proof. Let us suppose that there exists an element $v \in L$, $v > u$. Assume at first that there does not exist a non-trivial interval contained in $[u, v]$ and satisfying $(p_j)$. Then $[u, v]$ fulfils $(p_j')$, hence $[a, u'_{[a,v]}]$ satisfies $(p_j')$, too, and therefore $u'_{[a,v]} \in \in L_j'^1(a)$, $u'_{[a,v]} \leqq u$. This implies $u = v$, which is a contradiction. Let us now suppose, that in the interval $[u, v]$ there exists a non-trivial interval satisfying $(p_j)$. Since the intervals $[u, v]$ and $[a, u'_{[a,v]}]$ are transposed, it follows by 5.2 that there exists a non-trivial interval $[a, c] \subset [a, u'_{[a,v]}]$ satisfying $(p_j)$. Therefore we have $c \in L_j^1(a)$, $c \leqq u$ and at the same time $c \leqq u'_{[a,v]}$, thus $c = a$, which is impossible.

**5.3.2.** *Let $b$ and $c$ be the greatest element of $L_j^1(a)$ and of $L_j'^1(a)$, respectively. Then $u = b \cup c$ is the greatest element of $L$ and $b'_{[a,u]} = c$.*

Proof. The first assertion follows from 5.3.1. The convexity of the sets $L_j^1(a)$, $L_j'^1(a)$ and 3.10 imply the second statement.

**5.3.3.** *Let $u$ and $b$ be the greatest element of $L$ and $L_j^1(a)$, respectively. Then $L_j'^1(a)$ has a greatest elements as well.*

Proof. It is easy to see that $[b, u]$ satisfies $(p_j')$ (in the opposite case there exists an element $b_1 > b$ such that $[b, b_1]$ satisfies $(p_j)$; then $[a, b_1]$ also satisfies $(p_j)$ and therefore $b$ is not the greatest element of $L_j^1(a)$, which is a contradiction). Let us put $c = b'_{[a,u]}$. The interval $[a, c]$ is isomorphic to $[b, u]$, hence it satisfies $(p_j')$. If $z \in L_j'^1(a)$, then $[a, z]$ satisfies $(p_j')$, whence $b \cap z = a$, $z \leqq c$. This proves that $c$ is the greatest element of the set $L_j'^1(a)$.

**5.3.4.** *Let $u$ and $c$ be the greatest element of $L$ and $L_j'^1(a)$, respectively. Then the set $L_j^1(a)$ also has a greatest element.*

Proof. Let us put $c'_{[a,u]} = b$. The interval $[b, u]$ fulfils $(p_j')$ since $[a, c]$ and $[b, u]$ are transposed. From 3.10 it follows that $b$ is an upper bound of the set $L_j^1(a)$. Let us suppose that there exists an element $b_1 < b$ which is an upper bound of the set $L_j^1(a)$. If there exists in $[b_1, b]$ a non-trivial interval satisfying $(p_j)$, then according to 5.2 there exists a non-trivial interval $[b_1, b_2] \subset [b_1, b]$ fulfilling $(p_j)$. In such a case $[a, b'_{1[a,b_2]}]$ satisfies $(p_j)$, thus $b'_{1[a,b_2]} \in L_j^1(a)$ and at the same time $b'_{1[a,b_2]} \nleqq b_1$; this is a contradiction. Hence $[b_1, b]$ does not contain any non-trivial interval satisfying $(p_j)$ and therefore $[b_1, b]$ fulfils $(p_j')$. It follows that $[a, b'_{1[a,b]}]$ also satisfies $(p_j')$, whence $b'_{1[a,b]} \in L_j'^1(a)$, $b'_{1[a,b]} \leqq c$. This implies $a \leqq b'_{1[a,b]} \leqq b \cap c = a$ and hence $b'_{1[a,b]} = a$, $b_1 = b$, a contradiction.

From the distributivity of $L$ and from 5.3 it follows:

**5.4.** *If any two conditions from (1), (2), (3) hold, then the lattice $L^1(a)$ is isomorphic to the direct product of lattices $L_j^1(a)$ and $L_j'^1(a)$.*

**5.5.** *Let the lattice $L^1(a)$ be conditionally complete. Then $L^1(a)$ is isomorphic to the direct product of lattices $L_j^1(a)$ and $L_j'^1(a)$.*

Proof. For $z \in L^1(a)$ we shall denote $z_1 = \sup \{t : t \in L_j^1(a),\ t \leqq z\}$, $z_2 = z'_{1[a,z]}$. By 5.2 and 2.6 the interval $[z_1, z]$ satisfies $(p_j')$, hence $[a, z_2]$ also satisfies $(p_j')$ and $z_2 \in L_j'^1(a)$. With the aid of 3.10 one can easily prove that the correspondence $\varphi(z) = (z_1, z_2)$ is a one-to-one mapping of the set $L^1(a)$ onto the direct product of lattices $L_j^1(a)$, $L_j'^1(a)$ and that for $z, v \in L^1(a)$ we have $z \leqq v$ if and only if $z_1 \leqq v_1$, $z_2 \leqq v_2$.

**5.6.** *The mapping $\psi : z \to (z \cup a, z \cap a)$ is an isomorphism of the lattice $L$ onto the direct product of lattices $L^1(a)$, $L^2(a)$.*

Proof. If $z \in L$, let us put $z_1 = z \cup a$, $z_2 = z \cap a$. From the distributivity of $L$ it follows that $\psi$ is one-to-one. If $u \leqq a \leqq v$, let us denote $z = a'_{[u,v]}$; then $z_1 = v$, $z_2 = u$, hence $\psi(L) = L^1(a) \times L^2(a)$. Let $z, v \in L$. If $z \leqq v$, then, obviously, $z_1 \leqq v_1$, $z_2 \leqq v_2$. Conversely, let $z_1 \leqq v_1$, $z_2 \leqq v_2$. Then we have

$$z = z \cap z_1 \leqq z \cap v_1 = z \cap (v \cup a) = (z \cap v) \cup (z \cap a) \leqq$$
$$\leqq (z \cap v) \cup (v \cap a) = (z \cup a) \cap v \leqq v\,.$$

**5.7. Theorem.** *If any two conditions from* (1), (2), (3) *and any two analogical dual conditions (concerning the lattices $L$, $L_j^2(a)$, $L_j'^2(a)$) hold, then the lattice $L$ is isomorphic to the direct product of lattices $L_j(a)$ and $L'_j(a)$.*

Proof. Let us denote by $\simeq$ the lattice-theoretical isomorphism. By 5.6, 5.4 and by the statement dual to 5.4 we obtain

$$L \simeq L^1(a) \times L^2(a) \simeq (L_j^1(a) \times L_j'^1(a)) \times (L_j^2(a) \times L_j'^2(a))\,.$$

Since the operation of forming direct product is commutative and assosiative, applying 5.6 once more we get

$$L \simeq (L_j^1(a) \times L_j^2(a)) \times (L_j'^1(a) \times L_j'^2(a)) \simeq L_j(a) \times L'_j(a)\,.$$

**5.8. Theorem.** *If $L$ is conditionally complete, then $L$ is isomorphic to the direct product of lattices $L_j(a)$ and $L'_j(a)$.*

The proof is similar to that of 5.7 with the distinction that we use 5.5 instead of 5.4.

An immediate consequence of the definition 1.6 is that if $L = L_4(a)$, then $L'_3(a) = L_6(a)$. From this it follows:

**5.9.** *The propositions* 5.3, 5.4, 5.5, 5.7 *and* 5.8 *remain true if $L$, $L_j(a)$, $L'_j(a)$ are replaced by $L_4(a)$, $L_3(a)$, $L_6(a)$.*

## 6. A SUBDIRECT PRODUCT DECOMPOSITION OF A RELATIVELY COMPLEMENTED LATTICE

In this section it is assumed that the lattice $L$ is relatively complemented and conditionally complete. We shall use the same notations as in the sections 4 and 5.

**6.1.** *Let $[z_1, z_2]$ be a non-trivial interval of $L$. There exists a non-trivial interval $[z_1, z] \subset [z_1, z_2]$ satisfying $(p_3(\alpha^*))$ or one of the conditions $(p_6(\alpha))$ $(\aleph_0 \leqq \alpha \leqq \alpha^*)$.*

This follows from 4.5 and 5.2.

**6.2.** $H^1(a) = L^1(a)$.

Proof. We obviously have $H^1(a) \subset L^1(a)$. Let $c \in L^1(a)$ and let us denote $M = H^1(a) \cap [a, c]$. Suppose that the element $c$ is not the least upper bound of the set $M$. Then there exists an element $b_1 < c$ which is an upper bound of $M$. Let us put $b = b'_{1[a,c]}$. Then $a < b$, hence there exists an element $z \in L$ with the properties as in 6.1 (where $[z_1, z_2] = [a, b]$). Obviously, $z \in H^1(a)$, therefore $z \leqq b_1$, $a \leqq z \leqq \leqq b \cap b_1 = a$, which is a contradiction. From this it follows $c = \sup M$; since $H^1(a)$ is a $c$-sublattice of $L$, we obtain $c \in H^1(a)$.

**6.3. Theorem.** *The lattice $L^1(a)$ is isomorphic to a complete subdirect product of lattices $H_i^1(a)$ $(i \in I)$ (with respect to the element $a$). If for each system $\{z^i\}_{i \in I}$ where $z^i \in H_i^1(a)$ there exists in $L$ the join $\bigcup z^i$, then the lattice $L^1(a)$ is isomorphic to the complete direct product of lattices $H_i(a)$ $(i \in I)$.*

This is a consequence of 4.9 and 6.2.

Remark. The dual statement concerning the lattice $L^2(a)$ can be proved similarly.

**6.4. Theorem.** *The lattice $L$ is isomorphic to a complete subdirect product of lattices $H_i(a)$ $(i \in I)$ (with respect to the element $a$).*

Proof. The situation can be represented by the following scheme:

$$(6.1) \qquad L \overset{(1)}{\simeq} L^1(a) \times L^2(a) \overset{(2)}{\to} \prod_{i \in I} H_i^1(a) \times \prod_{i \in I} H_i^2(a) \simeq$$
$$\simeq \prod_{i \in I} (H_i^1(a) \times H_i^2(a)) \overset{(3)}{\simeq} \prod_{i \in I} H_i(a)\,.$$

Here, the isomorphism (1) is constructed by 5.6; (2) is an isomorphic map of $L^1(a) \times \times L^2(a)$ into $\prod_{i \in I} H_i^1(a) \times \prod_{i \in I} H_i^2(a)$ which is constructed with the aid of the theorem 6.3 and the theorem dual to 6.3; the isomorphism (3) follows from 5.6. Hence the lattice $L$ is isomorphic to a sublattice of the complete direct product of lattices $H_i(a)$ $(i \in I)$. It remains to prove that the condition contained in the definition of the complete subdirect product is satisfied. For $z \in L$ let us denote by $f_z$ the image of $z$ with respect to the mapping $L \to \prod_{i \in I} H_i(a)$ which is defined in (6.1). It is easy to see that $f_a(i) = a$ for each $i \in I$. Choose a fixed $i_0 \in I$ and $c \in H_{i_0}(a)$. Let $f(i) = a$ for $i \neq i_0$ and $f(i_0) = c$. Let us put $z = c$. In the isomorphism (1) the image of $z$ is the pair $(z_1, z_2)$, $z_1 = c \cup a$, $z_2 = c \cap a$. Obviously, $z_1, z_2 \in H_{i_0}(a)$. In the mapping (2) we have $(z_1, z_2) \to (f_1, f_2)$, where $f_1(i) = f_2(i) = a$ for $i \neq i_0$ and $f_1(i_0) = z_1$, $f_2(i_0) = z_2$. The image of $(f_1, f_2)$ is an element $g \in \prod_{i \in I} H_i(a)$ such that for each $i \in I$ $g(i)$ is the relative complement of $a$ with respect to the interval $[f_2(i), f_1(i)]$. From this we obtain $g(i) = a$ for $i \neq i_0$, and $g(i_0) = c$, whence $g = f$. We have proved that $f = f_c$; the proof is complete.

**6.5. Theorem.** *Suppose that $L$ satisfies the following conditions:*

(a) *For each system $\{a^i\}_{i \in I}$ where $a^i \in H_i^1(a)$ the element $\bigcup a^i$ exists in $L$.*

(b) *For each system $\{b^i\}_{i \in I}$ where $b^i \in H_i^2(a)$ the element $\bigcap b^i$ exists in $L$. Then $L \simeq \prod_{i \in I} H_i(a)$.*

Proof. By 4.9, and 6.2 and by the dual statements we have

$$L^1(a) \times L^2(a) \simeq \prod_{i\in I} H_i^1(a) \times \prod_{i\in I} H_i^2(a) ;$$

from this by (6.1) the assertion of the theorem follows.

From 6.5 we get as a corollary:

**6.6. Theorem.** *Let $L$ be a complete Boolean algebra. Then $L \simeq \prod_{i\in I} H_i(0)$.*

**6.7.** For any convex sublattice $A$ of $L$ we put $\pi A = \sup\{\alpha : \alpha \leqq \alpha^*, A$ is $\alpha$-distributive$\}$. The lattice $A$ is said to be $\pi$-homogeneous, if $\pi[z_1, z_2] = \pi[z_3, z_4]$ holds for any two non-trivial intervals $[z_1, z_2] \subset A$, $[z_3, z_4] \subset A$. The following theorem is known (Pierce [4]):

**6.7.1.** *If $L$ is a complete Boolean algebra, then $L$ is isomorphic to a complete direct product of $\pi$-homogeneous Boolean algebras.*

Remark. Our definition of the cardinal $\pi A$ depends (in the case when $A$ is completely distributive) not only on $A$ but on the cardinality of $L$, too; this could be removed e.g., by dealing with the "infinite" cardinal $\infty$; in such a case we should have $\pi A = \infty$ for any completely distributive lattice $A$.

**6.8.** Now we can compare two direct decompositions of a complete Boolean algebra that are constructed by means of the function $d$ (cf. 1.6 and 6.6) and by means of $\pi$ (cf. 6.7.1). Let us consider at first the lattices $H_i(a)$; let $i \in I$ be fixed. From the definition of the set $H_i(a)$ it follows that for each non-trivial interval $[z_1, z_2] \subset H_i(a)$ the following statements hold (cf. 6.7):

$\pi[z_1, z_2] = \alpha^*$, if $i = 0$.

$\pi[z_1, z_2] = \alpha$, if $0 \neq i = \alpha$ and if $\alpha$ is a limit cardinal.

$\pi[z_1, z_2] = \beta$, where $\beta + = \alpha$, if $0 \neq i = \alpha$ and $\alpha$ is a non-limit cardinal.

Hence each lattice $H_i(a)$ is $\pi$-homogeneous. From this it follows that the theorem 6.7.1 is a corollary of 6.6. From 6.4 and 6.5 we get the following result:

**6.8.1. Theorem.** *If $L$ is conditionally complete and relatively complemented, then $L$ is isomorphic to a complete subdirect product of $\pi$-homogeneous lattices. If, moreover, the conditions* (a) *and* (b) *from* 6.5 *hold, then $L$ is isomorphic to a complete direct product of $\pi$-homogeneous lattices.*

**6.9.** Let us suppose that $[z_1, z_2]$ and $[z_3, z_4]$ are intervals of $L$ which are not completely distributive. Suppose that $\alpha$ is a limit cardinal, $d[z_1, z_2] = \alpha$, $d[z_3, z_4] = \alpha +$. Then we have $\pi[z_1, z_2] = \alpha = \pi[z_3, z_4]$. From this we can see that if $[z_1, z_2] \subset A$, $[z_3, z_4] \subset A$ and if $A$ is a $\pi$-homogeneous lattice, then the behaviour of intervals of $A$ with respect to higher orders of distributivitiy need not be equal. The following example shows that such a situation can actually happen. Let $\alpha$ be a regular cardinal. Let us put $\beta = \alpha+$; obviously, $\beta$ is a regular cardinal, too. Let $B_\alpha$ be a complete Boolean algebra which is $\alpha_1$-distributive for every $\alpha_1 < \alpha$ and which is not $\alpha$-distributive; let $B_\beta$ have analogical properties (with $\beta$ instead of $\alpha$). Let us denote $L = B_\alpha \times B_\beta$. Then $L$ is a complete Boolean algebra. Each non-trivial interval $[u, v]$ of $L$ is $\alpha_1$-distributive for every $\alpha_1 < \alpha$ and it is not $\alpha_2$-distributive for $\alpha_2 > \alpha$. Since $\alpha$ is a limit cardinal, $\pi[u, v] = \alpha$ holds, hence $L$ is $\pi$-homogeneous. Let $u_1$ and $v_1$ be the least and the greatest element of $B_\alpha$, respectively, and let $u_2$ and $v_2$ have the analogical meaning with respect to $B_\beta$. Denote $z_0 = (u_1, u_2)$, $z_1 = (v_1, u_2)$, $z_2 = (u_1, v_2)$. Then $d[z_0, z_1] = dB_\alpha = \alpha$, $d[z_0, z_2] = dB_\beta = \beta > \alpha$. This example shows also that for a $\pi$-homogeneous Boolean algebra $L$ the direct decomposition treated in 6.6 can be non-trivial; in our case we obviously have $H_\alpha(z_0) = [z_0, z_1]$, $H_\beta(z_0) = [z_0, z_2]$, $H_i(z_0) = \{z_0\}$ for $i \in I$, $\alpha \neq i \neq \beta$.

## 7. $(\alpha, \beta)$-DISTRIBUTIVITY IN $l$-GROUPS

In this section we shall denote by $L$ a lattice-ordered group ($l$-group). The terminology of [1, chap. XIV] will be used. If $L$ is regarded merely as a lattice, then it will be denoted by $L(\leqq)$. The $l$-group $L$ is said to be complete if the lattice $L(\leqq)$ is conditionally complete. By the same notations as in the previous sections we have $L^+ = L^1(0)$, $L^- = L^2(0)$. For $z \in L$ let us put $z^+ = z \cup 0$, $z^- = z \cap 0$; then $z = z^+ + z^-$. If $z_1, z_2 \in L$, $z_1 \cap z_2 = 0$, then $z_1 \cup z_2 = z_1 + z_2$. The direct product, the complete direct product and the complete subdirect product of $l$-groups are defined analogously as in the case of lattices; the complete subdirect product is always taken with respect to the element 0.

It is well-known that every $l$-group is infinitely distributive (cf. [1]). Let $\alpha, \beta$ be infinite cardinals, $j \in J$. In considering $l$-groups one often uses the fact that for any elements $a, b \in L$ the mappings $\varphi_1(z) = a + z + b$ and $\varphi_2(z) = a - z + b$ is an automorphism of $L(\leqq)$ and a dual automorphism of $L(\leqq)$, respectively.

**7.1.** *Let $z_i \in L_j(0)$, $z_i \geqq 0$, $i = 1, 2$. Then $z_1 + z_2 \in L_j(0)$.*

Proof. By our assumption the intervals $[0, z_i]$ $(i = 1, 2)$ satisfy the condition $(\mathrm{p}_j)$. The interval $[z_1, z_1 + z_2]$ is isomorphic to $[0, z_2]$, hence $[z_1, z_1 + z_2]$ also satisfies $(\mathrm{p}_j)$. Therefore by 2.6 the interval $[0, z_1 + z_2]$ fulfils $(\mathrm{p}_j)$, hence $z_1 + z_2 \in L_j(0)$.

By a similar argument we can prove the analogical assertion for $z_1 \leqq 0$, $z_2 \leqq 0$.

**7.2.** *$L_j(0)$ is a convex $l$-subgroup of $L$.*

Proof. From the previous considerations we know that $L_j(0)$ is a convex sublattice of $L$. Let $z_1, z_2 \in L_j(0)$. Then $z_i^+, z_i^-$ $(i = 1, 2)$ also belong to $L_j(0)$. Since $z_1^- \leqq z_1 \leqq z_1^+$, $-z_2^+ \leqq -z_2 \leqq -z_2^-$, we obtain $z_1^- - z_2^+ \leqq z_1 - z_2 \leqq z_1^+ - z_2^-$ and therefore appling 7.1 and the convexity of the set $L_j(0)$ we get $z_1 - z_2 \in L_j(0)$.

**7.3.** *$L_j(0)$ is an l-ideal of L.*

Proof. Let $a \in L$. Let us consider the mapping $\varphi(z) = a + z - a$. Since $\varphi$ is an automorphism of $L(\leqq)$, the set $\varphi(L_j(0))$ has analogical properties as $L_j(0)$, i.e., for each $z_1 \in \varphi(L_j(0))$ the set $\varphi(L_j(0))$ is the greatest convex sublattice of $L$ containing $z_1$ and satisfying $(p_j)$. Since $0 \in \varphi(L_j(0))$, we obtain $\varphi(L_j(0)) = L_j(0)$.

**7.4.** *$L_j(a) = a + L_j(0)$ for each $a \in L$ (i.e., the classes of the partition $R_j$ are congruence classes with respect to the l-ideal $L_j(0)$).*

This can be proved in a similar way as in 7.3 by considering the mapping $\varphi(z) = a + z$.

**7.5.** *An element $z \in L$ belongs to $L'_j(0)$ if and only if $|z| \cap |x| = 0$ for each $x \in L_j(0)$.*

Proof. The "only if" part holds by 3.10. Let us suppose that $|z| \cap |x| = 0$ for each $x \in L_j(0)$. If there exists a non-trivial interval $[u, v] \subset [0, |z|]$ such that $[u, v]$ satisfies $(p_j)$ then $0 < u_1 \leqq |z|$, $u_1 \in L_j(0)$, where $u_1 = v - u$. If we set $x = u_1$, we have a contradiction. From this it follows that $[0, |z|]$ satisfies $(p'_j)$ and therefore $|z| \in L'_j(0)$. The interval $[-|z|, 0]$ is dually isomorphic to $[0, |z|]$, hence $-|z|$ belongs to $L'_j(0)$, too. From $-|z| \leqq z \leqq |z|$ and from the convexity of $L'_j(0)$ it follows $z \in L'_j(0)$.

**7.6.** *Let $L$ be a complete l-group. For $z \in L$, $z \geqq 0$, $j \in J$ let us denote*

$$z_j = \sup\{t : t \in L_j(0),\ t \leqq z\},$$
$$z'_j = \sup\{t : t \in L'_j(0),\ t \leqq z\}.$$

*Then $z = z_j \cup z'_j = z_j + z'_j$.*

Proof. Let us suppose that $z_j \cup z'_j = u < z$ holds. If the interval $[u, z]$ satisfies $(p'_j)$, then $-u + z \in L'_j(0)$, hence by 2.6 $z'_j < z'_j + (-u + z) \in L'_j(0)$, $z'_j + (-u + z) \leqq z$, which is a contradiction to the definition of $z'_j$. Therefore there exists a nontrivial interval $[b, v] \subset [u, z]$ satisfying $(p_j)$. But then we have $-t + v \in L_j(0)$, thence $z_j < z_j + (-t + v) \in L_j(0)$, $z_j + (-t + v) \leqq z$; according to the definition of $z_j$ his is a contradiction. Thus $z = z_j \cup z'_j$ holds and $z = z_j + z'_j$ by 7.5.

**7.7. Theorem.** *Any complete l-group $L$ is isomorphic to the direct product of l-groups $L_j(0)$, $L'_j(0)$.*

Proof. Let $z \in L$. Since $z^+, -z^- \geqq 0$, we can construct the elements $(z^+)_j$, $(-z^-)_j$, $(z^+)'_j$, $(-z^-)'_j$ by 7.6; let us put $z_j = (z^+)_j - (-z^-)_j$, $z'_j = (z^+)'_j - (-z^-)'_j$; if $z \geqq 0$, then this definition of elements $z_j$, $z'_j$ obviously coincides with the definition 7.6. By 7.5 any two elements $a \in L_j(0)$, $b \in L'_j(0)$ are permutable, hence $z = z_j + z'_j$. This proves that the group $L(+)$ is isomorphic to the direct product of groups $L_j(0)$, $L'_j(0)$, where the isomorphism is given by the mapping $\varphi(z) = (z_j, z'_j)$. It remains to prove that $\varphi$ is an isomorphism also with respect to the partial order. To do this, it suffices to verify that $z \geqq 0$ holds if and only if $z_j \geqq 0$, $z'_j \geqq 0$. From $z = z_j + z'_j$ it follows that our condition is sufficient; moreover it is also necessary by 7.6.

Let the symbols $I$, $H_i(0)$ $(i \in I)$, $H^1(0)$, $H(0)$ have the same meaning as in the section 4. For any $i \in I$ let $z_i$ be the projection of the element $z$ into the direct factor $H_i(0)$ (cf. 7.6 and 7.7).

**7.8.** *Let $L$ be a complete l-group, $z \in L$, $z \geqq 0$. Then $z = \bigcup_{i \in I} z_i$.*

Proof. Obviously, $z_i \leqq z$ for each $i \in I$. Assume that $\bigcup_{i \in I} z_i = u < z$. By 4.5 there exists a non-trivial interval $[t, v] \subset [u, z]$ satisfying $p_3(\alpha^*)$ or $p_6(\alpha)$ for some $\alpha$, $\aleph_0 \leqq \alpha \leqq \alpha^*$. Let us consider the case when $[t, v]$ satisfies $p_3(\alpha^*)$ (in other cases the proof would be analogical). Then we have $-t + v \in H_0(0)$ and $z_0 < z_0 + (-t + v) \in H_0(0)$. By the definition of $z_0$ (cf. 7.6) $z_0 = \sup \{t : t \in H_0(0), t \leqq z\}$, whence $z_0 + (-t + v) \leqq z_0$, which is a contradiction.

Corollary. *$H^1(0) = L^+$ (and dually, $H^2(0) = L^-$). From this we obtain $H(0) = L$.*

**7.9. Theorem.** *Any complete l-group $L$ is isomorphic to the complete subdirect product of l-groups $H_i(0)$ $(i \in I)$.*

Proof. Let us consider the mapping $\varphi(z) = (\ldots, z_i, \ldots)$ of $L$ into the complete direct product of $l$-groups $H_i(0)$ $(i \in I)$. Let us put $\varphi(L) = L_1$. By 7.7 $\varphi$ is a homomorphism with respect to the operations $+$, $\cap$, $\cup$ (since each $l$-group $H_i(0)$ is equal to some $l$-group $L_j(0)$; cf. the section 4). Let us suppose that there exists an element $z \neq 0$ such that $\varphi(z) = 0$, hence $z_i = 0$ for each $i \in I$. Without the loss of generality we can suppose $z > 0$ (in the opposite case we take the element $|z|$ rather than $z$). By 7.8 $z = 0$; a contradiction. Hence the mapping $\varphi$ is an isomorphism of the $l$-group $L$ onto the $l$-group $L_1$.

Let $i_0 \in I$, $z \in H_{i_0}(0)$. Then $z^+ \in H_{i_0}(0)$, hence $z^+ \cap |t| = 0$ for each $t \in H_i(0)$, where $i \in I$, $i \neq i_0$. From this it follows $(z^+)_i = 0$ by 7.6; analogously we get $(-z^-)_i = 0$, hence $z_i = 0$. Obviously, $z_{i_0} = z$. This proves that $L_1$ is a complete subdirect product of $l$-groups $H_i(0)$.

**7.10.** *Let $L$ be a complete l-group, $0 \leqq x^i \in H_i(0)$ for each $i \in I$, $\bigcup_{i \in I} x^i = x$. Then $x_i = x^i$.*

Proof. Obviously, $x^i \leqq x$ holds and hence $x^i \leqq x_i$ by 7.6. Further $x_i = x_i \cap x = x_i \cap (\bigcup_{i \in I} x^i) = x^i$, $x_i \leqq x^i$ holds.

**7.11. Theorem.** *If for each system $\{x^i\}_{i \in I}$ such that $x^i \in H_i(0)$, $x^i \geqq 0$ there exists the element $\bigcup x^i$ in $L$, then the l-group $L$ is isomorphic to the complete direct product of l-groups $H_i(0)$ $(i \in I)$.*

Proof. Put $v = (\ldots, v^i, \ldots) \in \prod_{i \in I} H_i(0)$. According to the assumption the elements $a = \bigcup (v^i)^+$, $b = \bigcup - (v^i)^-$ exist in $L$. From 7.10 it follows that $a_i = (v^i)^+$, $b_i = -(v^i)^-$ for each $i \in I$. Denote $z = a - b$; then $z_i = (v^i)^+ + (v^i)^- = v^i$ for each $i \in I$, hence $\varphi(z) = v$, $v \in L_1$. Since by 7.9 $\varphi : L \to L_1$ is an isomorphism, the proof is complete.

An $l$-group $L$ is said to be ortogonally complete if for each system $\{a_k\}_{k \in K}$, where $a_{k_1} \cap a_{k_2} = 0$ for any $k_1, k_2 \in K$, $k_1 \neq k_2$ there exists the element $\bigcup_{k \in K} a_k$ in $L$ (cf. e.g., [2]). As $H_{i_1}(0) \cap H_{i_2}(0) = \{0\}$ holds for $i_1, i_2 \in I$, $i_1 \neq i_2$, the following proposition follows from 7.11:

**7.12.** *If $L$ is a complete $l$-group which is also ortogonally complete, then $L$ is isomorphic to the complete direct product of $l$-groups $H_i(0)$ $(i \in I)$.*

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