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ON THE OPTIMAL STABILIZATION OF NONLINEAR SYSTEMS

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This paper is concerned with a nonlinear extension of a problem posed in its linear form by LETOV [1] and called by him "analytical regulator construction" problem.

Consider a control system

$$\dot{x} = f(x, u), \quad f(0, 0) = 0 \tag{1}$$

(x being n-vector, u being m-vector) and a scalar function $V(x, u)$ (the cost function), which is positive definite i.e. $V(0, 0) = 0$, $V(x, u) > 0$ for $(x, u) \neq (0, 0)$. For a given n-vector $\hat{x}$ denote $\Omega_{\hat{x}}$ the set of all measurable functions $u(t)$, $t \in \langle 0, \infty)$ such that the solution of the system $\dot{x} = f(x, u(t))$ starting at $\hat{x}$ at $t = 0$ (we shall denote it by $x(t, u, \hat{x})$), exists on the whole interval $\langle 0, \infty)$, satisfies $\lim_{t \to \infty} x(t, u, \hat{x}) = 0$ and $I(u, \hat{x}) = \int_0^\infty V(x(t, u, \hat{x}), u(t))\, dt < \infty$. The elements of $\Omega_{\hat{x}}$ will be called controls. The control $u_{\hat{x}} \in \Omega_{\hat{x}}$ is called optimal, if $I(u_{\hat{x}}, \hat{x}) = \min_{u \in \Omega_{\hat{x}}} I(u, \hat{x})$; $x(t, u_{\hat{x}}, \hat{x})$ is called an optimal solution.

A function $v(x)$ is called a synthesis of optimal control in the domain D, if for every $\hat{x} \in D$ the optimal control $u_{\hat{x}}$ may be expressed in the following way

$$u_{\hat{x}}(t) = v(x(t, u_{\hat{x}}, \hat{x})) .$$

If such a synthesis exists, we obtain by substituing of $v(x)$ into (1) an asymptotically stable differential system with a certain optimality property.

For the case of f being linear, V being quadratic in both x and u theorems on the existence, uniquenes and synthesis have been proved by several authors (cf. [1], [2], [3], [4]). In this case one gets a linear synthesis function and the results are of global character in initial conditions.

The case of f linear and V non-quadratic is treated in [5]. In the present paper the general nonlinear case is treated. Theorems of local character in initial conditions about the existence, uniqueness and synthesis of optimal control are proved. The general nonlinear case is investigated also in [4]. However, the methods, used in [4] are different from those used in this paper and also the results are of different kind. Moreover, in [4] both f and V are supposed to be analytic in both x and u.

1. THE EXISTENCE THEOREM

Let $|.|$ be the Euclidean norm in R^k, $k \geqq 1$, $S(x, \delta) = \{x' : |x' - x| < \delta\}$. Let us introduce following hypotheses:

(f1) f is defined and continuous together with its first partial derivatives for $(x, u) \in G \times R^m$, G being an open domain in R^n, containing the origin; $f(0, 0) = 0$. The system (1) is locally controllable in the origin, i.e. if we denote $A = (\partial f/\partial x)(0, 0)$, $B = (\partial f/\partial u)(0, 0)$, then the matrix $(B, AB, \ldots, A^{n-1}B)$ is of rank n.

(V1) $V(x, u)$ is defined, continuous and positive definite in $G \times R^m$. For every $\delta > 0$ it holds $\inf\limits_{|x| \geqq \delta} \liminf\limits_{|u| \to \infty} V(x, u) > 0$.

(fV1) The set $Q(x) = \{(y_0, \ldots, y_n) : y_0 \geqq V(x, u), y_i = f_i(x, u), i = 1, 2, \ldots, n, u \in \in R^m\}$ satisfies for every $x \in G$ the semicontinuity property $Q(x) = \bigcap\limits_{\delta > 0} \overline{\mathrm{co}}\, Q(S(x, \delta))$, $\overline{\mathrm{co}}\, X$ being the convex closure of X (cf. [7]).

(fV2) There is a nonnegative continuous function $\varphi(\xi)$ such that $\lim\limits_{\xi \to \infty} \xi^{-1} \varphi(\xi) = \infty$ and $V(x, u) \geqq \varphi(|f(x, u)|)$ for $(x, u) \in G \times R^m$.

Remark 1. Under Hypotheses (f1), (V1), (fV2) from $I(u, \hat{x}) < \infty$ follows $\lim\limits_{t \to \infty} x(t, u, \hat{x}) = 0$. In order to prove this, suppose the contrary. Then, from $I(u, \hat{x}) < \infty$ and (V1) follows that there is a sequence $\{t_k'\}$, $t_k' \to \infty$ such that $\lim\limits_{k \to \infty} x(t_k', u, \hat{x}) = 0$. Hence if $x(t, u, \hat{x})$ does not converges to zero for $t \to \infty$, there is a $\varkappa > 0$ and sequences $\{t_k\}, \{\tau_k\}$, $t_k \to \infty$, $\tau_k > t_k$ such that $|x(t_k)| = 2\varkappa$, $|x(\tau_k)| = \varkappa$, $|x(t, u, \hat{x})| \in \langle \varkappa, 2\varkappa \rangle$ for $t \in \langle t_k, \tau_k \rangle$. We have

$$\varkappa \leqq |x(\tau_k, u, \hat{x}) - x(t_k, u, \hat{x})| \leqq \int_{E_{\nu,k}} |f(x(t, u, \hat{x}), u(t))| \, \mathrm{d}t + \tag{2}$$
$$+ \int_{F_{\nu,k}} |f(x(t, u, \hat{x}), u(t))| \, \mathrm{d}t \leqq \nu(\tau_k - t_k) + \sigma_\nu I(u. \hat{x}),$$

where $E_{\nu,k} = \{t \in \langle t_k, \tau_k \rangle : |f(x(t, u, \hat{x}), u(t))| \leqq \nu\}$, $F_{\nu,k} = \{t \in \langle t_k, \tau_k \rangle : |f(x(t, u, \hat{x}), u(t))| > \nu\}$ and $\sigma_\nu^{-1} = \min\limits_{|\xi| \geqq \nu} \xi^{-1} \varphi(\xi)$. From (fV2) follows that there is a $\nu > 0$ such that $\sigma_\nu < \frac{1}{2}\varkappa[I(u, x)]^{-1}$. For such ν from (2) follows

$$\tau_k - t_k \geqq \tfrac{1}{2}\nu^{-1}\varkappa . \tag{3}$$

According to (V1) there is a $\mu > 0$ such that $V(x, u) \geqq \mu$ for $|x| \in \langle \varkappa, 2\varkappa \rangle$ which together with (3) contradicts $I(u, \hat{x}) < \infty$.

Lemma 1. *Let* (f1) *be satisfied. Let* μ *be a given positive constant. Then, there is an* $\eta > 0$ *such that for every* $\hat{x}$ *such that* $|\hat{x}| < \eta$ *there is a function* $u_{\hat{x}}(t)$ *and a* $T_{\hat{x}} > 0$ *such that* $|u_{\hat{x}}(t)| \leqq \mu$, $x(t, u_{\hat{x}}, \hat{x}) = 0$ *for* $t \geqq T_{\hat{x}}$. *Moreover,* $u_{\hat{x}}$ *may be chosen in such a way that* $T_{\hat{x}} \to 0$ *for* $\hat{x} \to 0$.

Proof. Denote Σ_T^μ the set of all $\hat{x}$ for which a control u exists such that $|u(t)| \leqq \mu$ for $t \in \langle 0, \infty \rangle$ and $x(t, u, \hat{x}) = 0$ for $t \geqq T$. It suffices to prove that for every $T > 0$, Σ_T^μ contains a neighbourhood of the origin.

The proof of this fact is a slight modification of the proof of theorem 4 in [6], therefore we shall give only a brief outline of it.

Denote $\xi = (\xi_1, \ldots, \xi_{mn})$ and let $T > t_1 \ldots > t_n > 0$. Denote

$$u(t, \xi) = [\xi_1 \varphi(t, t_1) + \ldots + \xi_n \varphi(t, t_n)]\, e_1 + \ldots + \\ + [\xi_{(m-1)n+1} \varphi(t, t_1) + \ldots + \xi_{mn} \varphi(t, t_n)]\, e_m .$$

where e_k is the m-vector with k-th component being 1 and the remaining being zero and

$$\varphi(t, h) = \begin{cases} 1 & \text{for} \quad |t| \leqq h . \\ 0 & \text{for} \quad |t| > h . \end{cases}$$

For ξ sufficiently small we have $|u(t, \xi)| \leqq \mu$. Denote $X(t, \xi)$ the solution of (1) with $u = u(t, \xi)$ and $X(T, \xi) = 0$. Then, $X(0, \xi)$ is differentiable with respect to ξ. For the $n \times mn$ matrix $(\partial X / \partial \xi)(T, 0)$ the expression

$$\frac{\partial X}{\partial \xi}(T, 0) = (b_1, \ldots, A^{n-1} b_1, \ldots, b_m, \ldots, A^{n-1} b_m) \begin{pmatrix} V_n \\ \ldots \\ V_n \end{pmatrix} + O(t_1^{n+1})$$

may be obtained, where

$$V_n = \begin{pmatrix} -t_1, \ldots, -t_n \\ \dfrac{t_1^2}{2!}, \ldots, \dfrac{t_n^2}{2!} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-1)^{n-1} \dfrac{t_1^n}{n!}, \ldots, (-1)^{n-1} \dfrac{t_n^n}{n!} \end{pmatrix}$$

and $b_1, \ldots, b_m$ are the column vectors of B. The rank of V_n being n, it follows from (f1) that for t_1 sufficiently small the rank of $(\partial X / \partial \xi)(0, 0)$ will be n. Since T may be choosen arbitrarily small, this proves the lemma.

Lemma 2. *Under hypotheses* (f1), (V1), $\Omega_{\hat{x}}$ *is non-empty for* $\hat{x}$ *from a sufficiently small neighbourhood of the origin and* $\lim_{\hat{x} \to 0} (\inf_{u \in \Omega_{\hat{x}}} I(u, \hat{x})) = 0$.

Proof. Due to Lemma 1 and (f1) for every $T > 0$ sufficiently small there is an $\varepsilon > 0$ such that if $|\hat{x}| < \varepsilon$, then $\hat{x} \in \Sigma_T^1$ and $x(t, u, \hat{x}) < \eta$ for $t \leqq T$ if $|u(t)| \leqq 1$ for $t \in \langle 0, T \rangle$. Hence, if u is such that $x(t, u, \hat{x}) = 0$ for $t \geqq T$ and $|u(t)| \leqq 1$ for $t \in \in \langle 0, T \rangle$, we have

$$I(u, \hat{x}) \leqq T \max_{\substack{|u| \leqq 1 \\ |x| \leqq \eta}} V(x, u),$$

which proves this lemma.

The following lemma is based on the same idea as a similar one in [7]. For its easier formulation, denote $x_0(t, u, \hat{x}) = \int_0^t V(x(s, u, \hat{x}), u(s)) \, ds$, $\bar{x}(t) = (x_0(t), x(t))$. We have then $I(u, \hat{x}) = x_0(\infty, u, \hat{x})$.

Lemma 3. *Let hypotheses* (f1), (V1), (fV1), (fV2) *be satisfied. Let* $\bar{x}^k(t) = \bar{x}(t, u^k, \hat{x}^k)$, $t \in \langle 0, \infty)$, $u^k \in \Omega_{\hat{x}^k}$ *and let* $x^k(t)$ *be uniformly bounded and equicontinuous. Let there be an absolutely continuous function* $x(t)$ *such that* $x^k(t) \to x(t)$ *for* $k \to \infty$ *for every* t, *the convergence being uniform on every compact interval and let* $x_0^k(t) \to x_0(t)$ *pointwise for* $t \in \langle 0, \infty)$. *Then, there is a control* $u \in \Omega_{x(0)}$ *such that* $x(t) = x(t, u, x(0))$ *and* $\int_0^\infty V(x(t, u, x(0)), u(t)) \, dt \leqq x_0(\infty)$.

Proof. For arbitrary $X \subset R^k$ denote $S(X, \delta) = \{x : \inf_{x' \in X} |x - x'| < \delta\}$. Let $\varepsilon > 0$. Let t be such that $\dot{\bar{x}}(t)$ exists (note that $\dot{\bar{x}}(t)$ exists for a.e. t since $x(t)$ is absolutely continuous and $x_0(t)$ is nondecreasing). Since $x^k(t) \to x(t)$ and $x^k(t)$ are equicontinuous, there is a $\delta > 0$ such that for $|h| < \delta$ we have $Q(x^k(t + h)) \subset Q(S(x^k(t), \frac{1}{2}\varepsilon))$ for every k. To given h and $\eta > 0$ there is a k_0 such that for $k > k_0$

$$(4) \qquad |\bar{x}_k(t) - \bar{x}(t)| < \tfrac{1}{2}\varepsilon$$

$$\left| \frac{\bar{x}(t + h) - \bar{x}(t)}{h} - \frac{\bar{x}^k(t + h) - \bar{x}^k(t)}{h} \right| < \eta$$

is valid. We have

$$(5) \qquad h^{-1}[\bar{x}^k(t + h) - \bar{x}^k(t)] =$$

$$= \left(h^{-1} \int_t^{t+h} V(x^k(s), u^k(s)) \, ds, \; h^{-1} \int_t^{t+h} f(x^k(s), u^k(s)) \, ds \right) \in \overline{\text{co}} \; Q(S(x^k(t), \tfrac{1}{2}\varepsilon)).$$

From (4), (5) follows $h^{-1}[\bar{x}(t + h) - \bar{x}(t)] \in S(\overline{\text{co}} \; Q(S(x(t), \varepsilon), \eta)$. For $k \to \infty$, $h \to 0$ we obtain from this

$$\dot{\bar{x}}(t) \in \bigcap_{\varepsilon > 0} \overline{\text{co}} \; Q(S(x(t), \varepsilon)).$$

From this and (fV1) follows $\dot{\bar{x}}(t) \in Q(x(t))$, i.e. there is a $u(t) \in R^m$ such that $\dot{x}(t) =$

$= f(x, u(t))$ and $\dot{x}_0(t) \geqq V(x(t), u(t))$. In the same way as in [7] it may be proved that $u(t)$ may be choosen measurable. Since $x_0(t)$ is nondecreasing, we have

$$\int_0^\infty V(x(t), u(t))\,\mathrm{d}t \leqq \int_0^\infty \dot{x}_0(t)\,\mathrm{d}t \leqq x_0(\infty)$$

cf. [8, chap. 8, theorem 5] which was to be proved.

Theorem 1. (the existence theorem). *Let hypotheses* (f1), (V1), (fV1), (fV2) *be valid. Then, there is an* $\varepsilon > 0$ *such that for every* $\hat{x}$ *such that* $|\hat{x}| < \varepsilon$ *the optimal control exists.*

Proof. Denote

$$\Omega_{\hat{x}}^\eta = \{u \in \Omega_{\hat{x}} : |x(t, u, \hat{x})| \leqq \eta \quad \text{for} \quad t \in \langle 0, \infty)\}\,,$$

$$E = \{x \in R^n : |x| = \eta,\, \Omega_x^\eta \neq \emptyset\}\,,$$

$$W(\hat{x}) = \inf_{u \in \Omega_{\hat{x}}} I(u, \hat{x})\,, \quad W^\eta(\hat{x}) = \inf_{u \in \Omega^\eta_{\hat{x}}} I(u, \hat{x})\,.$$

Suppose $\inf_{\hat{x} \in E} W^\eta(\hat{x}) = 0$. Then, there is a sequence $\hat{x}^k \to \hat{x}$, $u^k \in \Omega_{\hat{x}^k}^\eta$, $I(u^k, \hat{x}^k) \to 0$. Denote $x^k(t) = x(t, u^k, x^k)$ and use the notation of Lemma 3. The functions $x_0^k(t)$ are increasing. Hence, as $x_0^k(\infty) = I(u^k, \hat{x}^k) \to 0$, we have

$$x_0^k(t) \to 0 \quad \text{uniformly}\,. \tag{7}$$

We have $|x^k(t)| \leqq \eta$ and therefore with regard to hypothesis (fV2) we obtain for an arbitrary finite sequence $\langle t_i, \tau_i \rangle$, $i = 1, \ldots, r$ of disjoint intervals:

$$\sum_{i=1}^r |x^k(\tau_i) - x^k(t_i)| \leqq \sum_{i=1}^r \int_{t_i}^{\tau_i} |f(x^k(t), u^k(t))|\,\mathrm{d}t \leqq \sum_{i=1}^r \int_{E_{v,i}} |f|\,\mathrm{d}t + \tag{8}$$
$$+ \sum_{i=1}^r \int_{F_{v,i}} |f|\,\mathrm{d}t \leqq v \sum_{i=1}^r (\tau_i - t_i) + \sigma_v \max_k I(u^k, \hat{x}^k)$$

where $E_{v,i} = \{t \in \langle t_i, \tau_i \rangle : |f| \leqq v\}$, $F_{v,i} = \{t \in \langle t_i, \tau_i \rangle : |f| > v\}$ and $\sigma_v^{-1} = \min_{|\xi| \geqq v} \xi^{-1} \varphi(\xi)$. To given $\varepsilon > 0$ there is a v_0 such that $\sigma_{v_0} \max_k I(u^k, \hat{x}^k) \leqq \frac{1}{2}\varepsilon$. For $\sum_{i=1}^r (\tau_i - t_i) < \frac{1}{2} v_0^{-1} \varepsilon$ we get from (8) $\sum_{i=1}^r |x^k(\tau_i) - x^k(t_i)| \leqq \varepsilon$ which proves that x^k are equiabsolutely continuous. Therefore, we may choose from them a subsequence, which converges uniformly on every finite interval to a certain absolutely continuous function $x(t)$. Due to this and (7) we may suppose that $\bar{x}^k(t)$ converge to a certain function $\bar{x}(t)$, uniformly on every finite interval, where $x_0(t) \equiv 0$, $|x(t)| \leqq \eta$

and $|x(0)| = \eta$. Applying lemma 3 we conclude that there is a control u such that $x(t) = x(t, u, x(0))$ nad $I(u, x(0)) = 0$, which is impossible. Hence,

$$\inf_{\hat{x} \in E} W^{\eta}(\hat{x}) = d > 0 . \tag{9}$$

According to lemma 2 there is an $\varepsilon > 0$ such that $W(\hat{x}) < d$ for $|\hat{x}| < \varepsilon$. For such $\hat{x}$ there is a sequence $u^k \in \Omega_{\hat{x}}$ such that $I(u^k, \hat{x}) \to W(\hat{x})$. From this and (9) follows that $x(t, u^k, \hat{x})$ are uniformly bounded; similarly as above it may be proved that they are equiabsolutely continuous. $x_0^k(t)$ are uniformly bounded and increasing. Hence, using for $\{x_0^k\}$ Helly's principle of choise (cf. [8]), we may choose from $\{\bar{x}^k(t)\}$ a subsequence, such that x_0^k converge towards an increasing function for every t and $x^k(t)$ converge uniformly to an absolutely continuous function $x(t)$. Thus, there is a subsequence of the sequence $\{\bar{x}^k\}$, satisfying the conditions of lemma 3. In virtue of Remark 1, the application of this lemma completes the proof.

2. TWO THEOREMS ABOUT LINEAR DIFFERENTIAL SYSTEMS

In this paragraph, two theorems on asymptotic behaviour of systems of linear differential equations will be proved. They will be used in the proof of the uniqueness-synthesis theorem.

Lemma 4. *Let A be a constant matrix, $B(t)$ a matrix function,* $\lim_{t \to \infty} \int_0^1 |B(t + \tau)| \, d\tau = 0$. *Denote $X(t, t_0)$ the solution of the matrix equation*

$$\dot{x} = [A + B(t)] \, X$$

such that $X(t_0, t_0) = E$, E being the unity matrix. Then, it holds

$$\lim_{t_0 \to \infty} X(t_0 + t, t_0) = \exp \{At\}$$

uniformly with respect to $t \in \langle 0, 1 \rangle$.

Proof. Denote $N = \sup_{\tau \in \langle 0,1 \rangle} |\exp \{A\tau\}|$, $M = \sup_{t_0} \exp \{\int_0^1 |A + B(t_0 + t)| \, dt\}$. We have

$$\frac{d}{dt} |X(t_0 + t, t_0)| \leqq |A + B(t_0 + t)| \, |X(t_0 + t, t_0)|$$

from which we get

$$|X(t_0 + t, t_0)| \leqq \exp \left\{ \int_0^t |A + B(t_0 + \tau)| \, d\tau \right\} \leqq M < \infty .$$

Further, we have

$$X(t_0 + t, t_0) = \exp\{At\} + \int_0^t \exp\{A(t - \tau)\} B(t_0 + \tau) X(t_0 + \tau, t_0) \mid d\tau$$

from which it follows for $t \in \langle 0, 1 \rangle$

$$|X(t_0 + t, t_0) - \exp\{At\}| \leqq MN \int_0^1 |B(t_0 + t)| \, dt \to 0$$

for $t_0 \to \infty$.

Theorem 2. *Let $x(t)$ be a solution of the differential system*

$$\dot{x} = [A + B(t)] x + \varphi(t),$$

where A, B satisfy the assumptions of Lemma 4, $\int_0^1 |\varphi(t + \tau)| \, d\tau \to 0$ for $t \to \infty$. Let G_0 be a constant $m \times n$ matrix such that the matrix $\Gamma_0 = (G_0^*, A^*G_0^*, \ldots \ldots, A^{n-1*}G_0^*)^*$ is of rank n [1]) and let $G(t)$ be an $m \times n$ – matrix function, integrable over every finite interval of the positive semiaxis. Let there be a nonnegative increasing continuous real function $\alpha(\xi)$ defined for $\xi > 0$ such that $\alpha(0) = 0$, and for every $\delta > 0$ there is a positive constant K_σ such that for every $\xi \in \langle 0, \infty)$. $\alpha(\sigma\xi) \leqq K_\sigma \alpha(\xi)$ is valid. Let $\int_0^1 \alpha(|G(t + \tau) x(t + \tau)|) \, d\tau \to 0$ and $\int_0^1 \alpha(|G(t + \tau) - G_0|) \, d\tau \to 0$ for $t \to \infty$. Then, $x(t) \to 0$ for $t \to \infty$.

Proof. Obviously, we may without loss of generality suppose that A has Jordan's canonical form. Denote λ_i, $i = 1, \ldots, p$ its characteristic values. A may be writen in the form

$$A = \begin{pmatrix} A_{1,1}, 0, \ldots\ldots\ldots\ldots\ldots\ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots, 0, A_{1,r_1}, 0, \ldots\ldots\ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots\ldots\ldots, 0, A_{p,1}, 0, \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots\ldots\ldots\ldots\ldots\ldots, 0, A_{p,r_p} \end{pmatrix},$$

where

$$A_{i,j} = \begin{pmatrix} \lambda_i, 1, \ldots\ldots, 0 \\ 0, \lambda_i, 1, \ldots, 0 \\ \ldots\ldots\ldots\ldots \\ 0, \ldots, 0, \lambda_i, 1 \\ 0, \ldots, \quad 0, \lambda_i \end{pmatrix}$$

[1]) By X^* we denote the transposed matrix of X.

is of order $q_{i,j}$ $(i = 1, \ldots, p; j = 1, \ldots, r_i)$. Denote $x_\nu^{(i,j)}$ the component of x and $g_\nu^{(i,j)}$ the column vector of G corresponding to the ν-th row in $A_{i,j}$. We have

$$GA^\nu x = \sum_{i=1}^{p} \sum_{\varkappa=0}^{\nu} \binom{\nu}{\varkappa} \lambda_i^{\nu-\varkappa} c_{i\varkappa} \quad \nu = 1, \ldots, n-1 , \tag{9}$$

where

$$c_{i\varkappa} = \sum_{j=1}^{r_i} \sum_{\mu=1}^{q_{i,j}-\varkappa} g_\mu^{(i,j)} x_{\mu+\varkappa}^{(i,j)}$$

and $x_\mu^{(i,j)}$ is zero for $\mu > q_{i,j}$. Further, we have

$$Ge^{At}x = \sum_{i=1}^{p} \sum_{\varkappa=0}^{\bar{q}_i} c_{i\varkappa} \frac{t^\varkappa}{\varkappa!} e^{\lambda_i t} , \quad \bar{q}_i = \max_j q_{i,j} . \tag{10}$$

Suppose $x(t)$ does not tend to zero for $t \to \infty$. Then, there is a sequence $\{\tau_k\}$, $\tau_k \to \infty$ such that $|x(\tau_k)| \geqq 2\eta > 0$. We have $x(t) = X(t, \tau_k)\, x(\tau_k) + \int_{\tau_k}^t X(s, \tau_k)\, \varphi(s)\, \mathrm{d}s$ (X as in Lemma 3) $|x(t) - x(\tau_k)| \leqq |x(\tau_k)| \,.\, |X(t, \tau_k) - E| + \max_{k, s \in \langle \tau_k, t\rangle} |X(s, \tau_k)| \int_{\tau_k}^t |\varphi(s)|\, \mathrm{d}s$. In virtue of Lemma 4, $|X(t, \tau_k) - E| \to 0$, $\int_{\tau_k}^t |\varphi(s)|\, \mathrm{d}s \to 0$ for $t \to \tau_k$, uniformly with respect to k. From this it follows that there is a $\tau_0 > 0$ such that $|x(t)| \geqq \eta$ for $t \in \langle \tau_k, \tau_k + \tau_0 \rangle$.

Since $\int_0^1 \alpha(|G(t+s) - G_0|)\, \mathrm{d}s \to 0$ for $t \to \infty$, for every $\varepsilon > 0$ there is a $T > 0$ such that $\operatorname{mes}(\{t : |G(t) - G_0| \geqq \varepsilon,\ t \in \langle T, \infty)\}) < \varepsilon$. Hence, for every $\varepsilon > 0$, there is a $t_k \in \langle \tau_k, \tau_k + \tau_0 \rangle$ such that $|G(t_k) - G_0)| < \varepsilon$ for every k sufficiently large. Since the rank of Γ_0 is n, there is a $n \times mn$ matrix H_0 such that $x = H_0 \Gamma_0 x$. If we choose $\varepsilon > 0$ sufficiently small, the rank of the matrix $\Gamma_k = (G(t_k)^*, A^*\, G(t_k)^*, \ldots$ $\ldots, A^{n-1*}\, G(t_k)^*)^*$ will be also n and there will be a matrix H_k such that $x = H_k \Gamma_k x$, where $|H_k - H_0| < \eta$. From this it follows

$$|x(t_k)| = O(|\Gamma_k\, x(t_k)|) = O(\max_{i,\varkappa} |c_{i\varkappa}(t_k)|) \text{ for } k \to \infty . \tag{11}$$

Due to (11), the sequences $y_k = (\max_{i,\varkappa} |c_{i\varkappa}(t_k)|)^{-1}\, x(t_k)$, $d_{i\varkappa k} = (\max_{i,\varkappa} |c_{i\varkappa}(t_k)|)^{-1}\, c_{i\varkappa}(t_k)$ are bounded and $\max_{i,\varkappa} |c_{i\varkappa}(t_k)| > \delta > 0$. Hence, passing to a subsequence, if necessary, we may suppose that $y_k \to y$ and $d_{i\varkappa k} \to d_{i\varkappa}$ for $k \to \infty$, where $d_{i\varkappa}$ are not all zero. We have for $t \in \langle t_k, t_k + 1 \rangle$

$$G(t)\, x(t) = G(t)\, X(t, t_k)\, x(t_k) + G(t) \int_{t_k}^{t} X(t, \tau)\, \varphi(\tau)\, \mathrm{d}\tau$$

$$|G(t)\, X(t, t_k)\, x(t_k)| \leqq |G(t)\, x(t)| + M |G(t)| \int_{t_k}^{t_k+1} |\varphi(\tau)|\, \mathrm{d}\tau \leqq$$

$$\leqq |G(t)\, x(t)| + \Phi(t_k)\, [|G(t) - G_0| + |G_0|] ,$$

where $\Phi(t_k) \to 0$ for $t_k \to \infty$. Further, we have

$$\int_{t_k}^{t_k+1} \alpha(|G(t)\, X(t, t_k)\, x(t_k)|)\, dt \leqq$$

$$\leqq \int_{t_k}^{t_k+1} \alpha(|G(t)\, x(t)| + \Phi(t_k)\, |G(t) - G_0| + \Phi(t_k)\, |G_0|)\, dt \leqq$$

$$\leqq \int_{t_k}^{t_k+1} \alpha(3|G(t)\, x(t)|)\, dt + \int_{t_k}^{t_k+1} \alpha(3\Phi(t_k)\, |G(t) - G_0|)\, dt + \int_{t_k}^{t_k+1} \alpha(3\Phi(t_k)\, |G_0|)\, dt \leqq$$

$$\leqq K_3 \left[\int_{t_k}^{t_k+1} \alpha(|G(t)\, x(t)|)\, dt + \int_{t_k}^{t_k+1} \alpha(\Phi(t_k)\, |G(t) - G_0|)\, dt + \int_{t_k}^{t_k+1} \alpha(\Phi(t_k)\, |G_0|) \right] dt\,.$$

Hence,

$$\int_{t_k}^{t_k+1} \alpha(|G(t)\, X(t, t_k)\, x(t_k)|)\, dt \to 0 \quad \text{for} \quad k \to \infty\,.$$

Since $(\max_{i,\varkappa} |c_{i\varkappa}|)^{-1} < \delta^{-1}$, we have

(12)

$$\int_{t_k}^{t_k+1} \alpha(|G(t)\, X(t, t_k)\, y_k|)\, dt \leqq K_{\delta^{-1}} \int_{t_k}^{t_k+1} \alpha(|G(t)\, X(t, t_k)\, x(t_k)|)\, dt \to 0 \quad \text{for} \quad k \to \infty$$

$$\int_{t_k}^{t_k+1} \alpha(|G_0 e^{A(t-t_k)} y_k|)\, dt \leqq K_3 \left[\int_{t_k}^{t_k+1} \alpha(|G_0| \,.\, |e^{A(t-t_k)} - X(t, t_k)| \,.\, |y_k|)\, dt + \right.$$

$$\left. + \int_{t_k}^{t_k+1} \alpha(|G(t) - G_0| \,.\, |X(t, t_k)| \,.\, |y_k|)\, dt + \int_{t_k}^{t_k+1} \alpha(|G(t)\, X(t, t_k)\, y_k|)\, dt \right].$$

The first term tends to zero by lemma 4, the second by the assumption of the theorem and the third does by (12). Thus, $\int_0^1 \alpha(|G_0 e^{At} y_k|)\, dt \to 0$ for $k \to \infty$. Since $y_k \to y$ for $k \to \infty$, we get from this $G_0 e^{At} y = 0$ for $t \in \langle 0, 1 \rangle$. From this and (10) follows

$$G_0 e^{At} y = \sum_{i=1}^{p} \sum_{\varkappa=0}^{\bar{q}_i} d_{i\varkappa} \frac{t^\varkappa}{\varkappa!} e^{\lambda_i t} = 0 \quad \text{for} \quad t \in \langle 0, 1 \rangle\,.$$

Since the functions $(\varkappa!)^{-1}\, t^\varkappa e^{\lambda_i t}$ are linearly independent, this is inconsistent with $d_{i\varkappa} \neq 0$. This completes the proof.

Theorem 3. *Let A, $B(t)$, $G(t)$, G_0 satisfy the assumptions of Theorem 2. Let $x(t)$ be a solution of the differential system*

$$\dot{x} = [A + B(t)]\, x$$

such that $G(t)\, x(t) = 0$ for $t \in \langle 0, \infty)$. Then, $x(t) = 0$ for $t \geqq 0$.

Proof. Suppose the contrary. Then, there is an increasing sequence $\{t_k\}$, $t_k \to \infty$ such that $x(t_k) \neq 0$ and $y_k = |x(t_k)|^{-1}\, x(t_k)$ converge to a certain non-zero limit y. We have (cf. (12))

$$\int_0^1 \alpha(|G_0 e^{At} y|)\, dt = \lim_{k\to\infty} \int_0^1 \alpha(|G(t_k + t)\, X(t_k + t, t_k)\, y_k|)\, dt =$$

$$= \lim_{k\to\infty} \int_0^1 \alpha(|x(t_k)|^{-1}\, |G(t_k + t)\, x(t_k + t)|)\, dt = 0\,.$$

Hence, $G_0 e^{At} y = 0$ for $t \in \langle 0, 1\rangle$, what was shown to be impossible in the end of the proof of Theorem 2.

3. THE UNIQUENESS AND SYNTHESIS THEOREM

For the uniqueness and synthesis theorem, let us introduce some new hypotheses:

(f1') f satisfies (f1) and moreover $(\partial^2 f_i/(\partial u_i\, \partial u_j)\,(x, u))$ is continuous for $|x| \leqq \eta$, $|u| \leqq \eta$, $\eta > 0$, $i, j = 1, \ldots, m$.

(V1') (V1) is valid and moreover the first partial derivatives of V are continuous for $|x| \leqq \eta$, u arbitrary and the second partial derivatives of V are continuous for $|x| \leqq \eta$, $|u| \leqq \eta$. If we denote $P = \frac{1}{2}(\partial^2 V/(\partial x_i\, \partial x_j)\,(0, 0))$, $Q = (\partial^2 V/(\partial x_i\, \partial u_j)\,(0, 0))$, $R = \frac{1}{2}(\partial^2 V/(\partial u_i\, \partial u_j)\,(0, 0))$, then the quadratic form $V_0(x, u) = x^* P x + x^* Q u + u^* R u$ is positive definite.

(V2) There are constants L, $p_1 \in (0, 1)$ and $p_2 > 0$ such that $|V_x| \leqq L V^{p_1}$, $|V_u| \leqq L V^{p_2}$ for $|x| \leqq \eta$, $|u| \geqq \eta$.

(fV3) There are constants K, $r_1 \in (0, 1\rangle$, $r_2 > 0$ such that $|f_x| \leqq K V^{r_1}$, $|f_u| \leqq K V^{r_2}$ for $|x| \leqq \eta$, $|u| \geqq \eta$. For every $\delta > 0$, there is a $\mu > 0$ such that $|V_u| \geqq \mu |f_u|$ for $|x| \leqq \delta$ and $|u| \geqq \delta$.

Theorem 4. (*the uniqueness and synthesis theorem*). *Let hypotheses* (f1'), (V1'), (V2), (fV3) *be valid. Let there be an* $\varepsilon_1 > 0$ *such that for* $|\hat{x}| < \varepsilon_1$ *the optimal control exists. Then, there is an* $\varepsilon \in \langle 0, \varepsilon_1)$ *such that for* $|\hat{x}| < \varepsilon$ *it is unique and there is a synthesis of it, which is continuous together with its first partial derivatives.*

Proof. Since V_0 is positive definite, there is a $\gamma > 0$ such that $V(x, u) \geqq \gamma(|x|^2 + |u|^2)$ for $|x| \leqq \eta$, $|u| \leqq \eta$. Hence, there are constants $\gamma_1, \gamma_2, \beta_1, \beta_2$ such that

$$(13)\qquad |V_u|^2 \leqq \gamma_1 V\,, \quad |V_x|^2 \leqq \gamma_2 V\,, \quad |f_x(x, u) - f_x(0, 0)|^2 \leqq \beta_1\, V(x, u)\,,$$
$$|f_u(x, u) - f_u(0, 0)|^2 \leqq \beta_2\, V(x, u) \quad \text{for} \quad |x| \leqq \eta\,, \quad |u| \leqq \eta\,.$$

From this and hypotheses (V2), (fV3) follows that

$$(14) \qquad \alpha(|V_u|) \leqq V, \quad \alpha(|f_u(x, u) - f_u(0, 0)|) \leqq V(x, u) \quad \text{for} \quad |x| \leqq \eta,$$

where

$$\alpha(\xi) = \begin{cases} \mu\xi^2 & \text{for} \quad \xi \leqq \eta, \\ \mu\eta^{2-\varrho}\xi^2 & \text{for} \quad \xi > \eta, \end{cases}$$

$\varrho = \min\{2, p_2^{-1}, r_2^{-1}\}$ and μ is a suitable constant.

Suppose that for $\hat{x}$ there is an optimal control $u_{\hat{x}}(t)$. Further, we shall denote the solution $x(t, u_{\hat{x}}, \hat{x})$ simply by $x_{\hat{x}}(t)$. From the maximum principle ([9], § 24) follows that there is a non-zero $n+1$-vector function $\bar{\psi}(t) = (\psi_0, \psi(t))$, satisfying the system of equations

$$(15) \qquad \dot{\psi} = -\psi_0 V_x(x_{\hat{x}}(t), u_{\hat{x}}(t)) - \left[\frac{\partial f}{\partial x}(x_{\hat{x}}(t), u_{\hat{x}}(t))\right]^* \psi,$$

$\psi_0 = \text{const} \leqq 0$ such that the function $H(x, \bar{\psi}, u) = \psi_0 V(x, u) + \psi^* f(x, u)$ satisfies $\max_{u \in R^u} H(x_{\hat{x}}(t), \bar{\psi}(t), u) = H(x_{\hat{x}}(t), \bar{\psi}(t), u_{\hat{x}}(t))$. For this, it is necessary that the equation

$$(16) \qquad \psi_0 V_u(x_{\hat{x}}(t), u_{\hat{x}}(t)) + f_u^*(x_{\hat{x}}(t), u_{\hat{x}}(t))\,\psi(t) = 0$$

is statisfied for $t \geqq 0$.

Since $x_{\hat{x}}(t) \to 0$, we have $|x_{\hat{x}}(t)| \leqq \eta$ for $t \geqq T > 0$. From (14), (16) follows $\alpha(|f_u^*(x_{\hat{x}}(t), u_{\hat{x}}(t))\,\psi(t)|) \leqq \sigma V(x_{\hat{x}}(t), u_{\hat{x}}(t))$ for $t \geqq T$, σ being a suitable constant. System (15) may be rewritten as follows

$$\dot{\psi} = [-A + (A - f_x(x_{\hat{x}}(t), u_{\hat{x}}(t))]^* \psi - \psi_0 V_x(x_{\hat{x}}(t), u_{\hat{x}}(t)).$$

Now, from hypotheses (V2), (V3) and (13), (14) follows that the assumptions of theorems 2, 3 are satisfied with $B(t) = (A - f_x(x_{\hat{x}}(t), u_{\hat{x}}(t))^*$, $\varphi(t) = -\psi_0 V_x(x_{\hat{x}}(t), u_{\hat{x}}(t))$, $G(t) = f_u(x_{\hat{x}}(t), u_{\hat{x}}(t))$, $G_0 = B^*$ and $\sigma = \varrho$. Hence, from theorem 2 follows

$$(17) \qquad \psi(t) \to 0 \quad \text{for} \quad t \to \infty$$

and from theorem 3 follows $\psi_0 \neq 0$. Therefore, we may set $\psi_0 = -1$.

From (16) and (17) follows $|V_u(x_{\hat{x}}(t), u_{\hat{x}}(t))| = o(|f_u(x_{\hat{x}}(t), u_{\hat{x}}(t))|)$ for $t \to \infty$. Due to (fV3) this is possible only if $u_{\hat{x}}(t) \to 0$ for $t \to \infty$. From (V1) follows, that there is an $\varepsilon_0 \leqq \eta$ such that for $|x| + |u| + |\psi| < \varepsilon_0$, (16) has a unique solution

$$(18) \qquad u = w(x, \psi)$$

such that $w(0, 0) = 0$, which is continuous together with its first partial derivatives.

Hence, there is a $T_1 \geqq T$ such that for $t \geqq T_1$, $u_{\hat{x}}(t)$, $x_{\hat{x}}(t)$, $\psi(t)$ satisfy (18) and, therefore, $x_{\hat{x}}$, ψ satisfy for $t \geqq T_1$ the differential system

$$\text{(19)} \qquad \begin{aligned} \dot{x} &= f(x, w(x, \psi)), \\ \dot{\psi} &= -f_x^*(x, w(x, \psi))\,\psi + V_x(x, w(x, \psi)). \end{aligned}$$

This system may be rewritten as follows

$$\text{(20)} \qquad \begin{aligned} \dot{x} &= (A - \tfrac{1}{2}BR^{-1}Q^*)\,x + \tfrac{1}{2}BR^{-1}B^*\psi + \omega(x, \psi), \\ \dot{\psi} &= (2P - \tfrac{1}{2}QR^{-1}Q^*)\,x + (-A^* + \tfrac{1}{2}QR^{-1}B^*)\,\psi + \vartheta(x, \psi), \end{aligned}$$

where $\omega(x, \psi) = o(|x| + |\psi|)$, $\vartheta(x, \psi) = o(|x| + |\psi|)$.

Under hypotheses (f1′), (V1′), the optimal control for the linearized problem (i.e. for the system $\dot{x} = Ax + Bu$ with the cost function V_0) exists for every x and, therefore, the solutions of the linearized system, corresponding to (20) (i.e. the system without ω and ϑ on the right-hand sides), which tend to zero for $t \to \infty$, form an n-dimensional linear subspace Z^n of R^{2n}, the points fo which satisfy a system of linear algebraic equations $x = S\psi$. (cf. [2]). From this and the theorem of conditional stability ([10], chap. 13, theorems 4.1, 4.2) follows that there is an n-vector function $g(\psi)$ such that any solution $x(t)$, $\psi(t)$ of (19) tends to zero if and only if it in a certain neighbourhood of the origin lies in the manifold $x = g(\psi) = S\psi + o(\psi)$.

Now, using Lemma 2 and Lemma 3, we may in the same way as in [5] prove that there is an $\varepsilon > 0$ such that if $|\hat{x}| < \varepsilon$ and $x(t)$, $\psi(t)$ is a solution of (19) with $x(0) = \hat{x}$, such that $x(t) \to 0$ and $\psi(t) \to 0$ for $t \to \infty$, then $|x(t)| + |\psi(t)| \leqq \varepsilon_1$ and $\psi(t) = g(x_{\hat{x}}(t))$ for $t \in \langle 0, \infty)$. If we denote $v(x) = w(x, g(x))$, then it follows that for $|\hat{x}| < \varepsilon$, $u_{\hat{x}}(t) = v(x_{\hat{x}}(t))$ is valid. This completes the proof.

Corollary. *Under hypotheses of Theorem* 3 *the trivial solution of the differential system*

$$\text{(21)} \qquad \dot{x} = f(x, v(x))$$

is asymptotically stable and the function $W(x) = \int_0^\infty V(x(t), v(x(t))\,\mathrm{d}t$ ($x(t)$ *being the solution of* (21) *with* $x(0) = x$) *is a Lyapunov function of it in the sense of* [11].

This corollary gives a reason for calling our problem an "optimal stabilization" problem.

Joining the results of Theorem 1 and Theorem 4, we obtain the following result:

Theorem 5. *Let hypotheses* (f1′), (V1′), (V2), (fV1), (fV2), (fV3) *be valid. Then, there is an* $\varepsilon > 0$ *such that for* $|\hat{x}| < \varepsilon$ *the optimal control exists, is unique and admits a synthesis, which is continuous together with its first partial derivates.*

A part of the assumptions, posed on V under which Theorem 2 is proved, seem to be rather unnatural (e.g. hypothesis (V2)); one may expect that this theorem may be proved under less restrictive assumptions. This expectation may be strenghtened by the fact that in a special case we shall present in this paragraph, the conditions V has to satisfy, may be considerably reduced.

In this section we shall suppose that u is scalar and f is linear in u, i.e. $m = 1$ and $f(x, u) = g(x) + h(x)\,u$. Let us note that in this case, (fV2) is equivalent with the following hypothesis:

(fV2′) There is a nonnegative continuous function $\varphi(\xi)$ such that $\lim_{\xi \to \infty} \xi^{-1}\, \varphi(\xi) = \infty$ and $V(x, u) \geqq \varphi(|u|)$ for $(x, u) \in G \times R^m$.

Instead of (V1′), let us introduce the following less restrictive hypothesis:

(V1″) $V(x, u)$ is positive definite and continuous in $G \times R^m$, the first and second partial derivatives of it are continuous for $|x| \leqq \eta$, $|u| \leqq \eta$, $\eta > 0$ and $V_0(x, u)$ (cf. (V1′)) is positive definite.

Note that if f is linear in u, (fV3) follows from (V1), (fV2).

Lemma 5. *Let $m = 1$. Let $V(x, u)$ satisfy (V1″), (fV2′). Then, there is a function $\tilde{V}(x, u)$ with following properties:*

1° *$\tilde{V}(x, u)$ is defined, positive definite, convex in u and continuous together with its first partial derivatives for $|x| \leqq \eta_0$, u arbitrary, $0 < \eta_0 \leqq \eta$.*

2° *$\tilde{V}(x, u) = V(x, u)$ for $|x| \leqq \eta_0$, $|u| \leqq \eta_0$,*
$\tilde{V}(x, u) \leqq V(x, u)$ for $|x| \leqq \eta_0$, u arbitrary.

3° *$\tilde{V}(x, u)$ satisfies hypothesis (fV2′) with G replaced by the region $|x| \leqq \eta_0$ and φ replaced by φ_0, where φ_0 is moreover convex and $\varphi_0(0) = 0$.*

4° *There is a constant $\varkappa > 0$ such that*

$$|\tilde{V}(x', u) - \tilde{V}(x, u)| \leqq \varkappa(1 + |u|)\,|x' - x|$$

for $|x| \leqq \eta_0$, $|x'| \leqq \eta_0$.

Proof. There is a sequence of positive numbers $\{\xi_k\}$ such that $\xi_{k+1} - \xi_k \geqq \frac{1}{2}$, $\varphi(\xi) \geqq k\xi$ for $\xi \geqq \xi_k$. From (V1″) follows that there are positive numbers $\eta_0 \leqq \leqq \eta_1 \leqq \xi_1$ such that

(i) $V(x, u)$ is convex in u for $|x| \leqq \eta_0$, $|u| \leqq \eta_1$,

(ii) $0 \leqq V_u(x, \eta_1) \leqq 1$ for $|x| \leqq \eta_0$,
$0 \geqq V_u(x, -\eta_1) \geqq -1$ for $|x| \leqq \eta_0$,

(iii) $V(x, \eta_1) + V_u(x, \eta_1)(u - \eta_1) \leqq V(x, u)$ for $|x| \leqq \eta_0$, $\eta_1 \leqq u \leqq \xi_1$,
$V(x, -\eta_1) + V_u(x, -\eta_1)(u + \eta_1) \leqq V(x, u)$ for $|x| \leqq \eta_0 - \xi_1 \leqq u \leqq -\eta_1$.

Define $\tilde{V}(x, u)$ recurrently as follows:

$$\tilde{V}(x, u) = \begin{cases} V(x, u) & \text{for } |x| \leqq \eta_0,\ |u| \leqq \eta_1 \\ V(x, \quad \eta_1) + V_u(x, \quad \eta_1)(u - \eta_1) & \text{for } |x| \leqq \eta_0, \\ & \eta_1 \leqq u \leqq \xi_1 \\ V(x, -\eta_1) + V_u(x, -\eta_1)(u + \eta_1) & \text{for } |x| \leqq \eta_0, \\ & -\xi_1 \leqq u \leqq -\eta_1 \\ \tilde{V}(x, \quad \xi_k) + \tilde{V}_u(x, \quad \xi_k)(u - \xi_k) + \psi(u - \xi_k) & \text{for } \quad \xi_k \leqq u \leqq \xi_{k+1} \\ \tilde{V}(x, -\xi_k) + \tilde{V}_u(x, -\xi_k)(u + \xi_k) + \psi(u + \xi_k) & \text{for } -\xi_{k+1} \leqq u \leqq -\xi_k \end{cases}$$

$k = 1, 2, \ldots$, where

$$\psi(\xi) = \begin{cases} \xi^2 & \text{for } |\xi| \leqq \frac{1}{2}, \\ \frac{1}{4} + |\xi| & \text{for } |\xi| > \frac{1}{2}. \end{cases}$$

It is easy to verify that $V(x, u)$ has the desired properties, with $\varphi_0(\xi)$ defined recurrently by $\varphi_0(\xi) = 0$, $|\xi| \leqq \xi_1$, $\varphi_0(\xi) = \varphi_0(\xi_k) + (k - 1)(\xi - \xi_k) + \psi(\xi - \xi_k)$ for $\xi \in \langle \xi_k, \xi_{k+1} \rangle$.

Theorem 6. *Let $m = 1, f(x, u) = g(x) + h(x)\, u$. Let (f1′), (V1″), (fV2′) be satisfied. Then, there is an $\varepsilon > 0$ such that for $|\hat{x}| \leqq \varepsilon$ there is a unique optimal control. Further, there is a synthesis of it, which is continuous together with its first partial derivatives.*

Proof. For the given $V(x, u)$ construct the function $\tilde{V}(x, u)$ as described in Lemma 5. Then, the pair f, $\tilde{V}$ satisfies all assumptions of Theorem 5 (G replaced by $|\hat{x}| \leqq \eta_0$). For this, we have to prove that for $|x| \leqq \eta$, (fV1) is valid, since the remaining assumptions follow directly from the assumptions of this theorem and Lemma 5.

Every point of co $Q(S(x, \delta))$ may be written as a convex combination of $n + 2$ points of $Q(S(x, \delta))$. Thus, we have to prove that $\bar{y}^k \to \bar{y}$, $\bar{y}^k = \sum_{i=1}^{n+2} \lambda^{ik} \bar{y}^{ik}$, $\sum_{i=1}^{n+2} \lambda^{ik} = 1$, $\bar{y}^{ik} \in Q(x^{ik})$, $x^{ik} \to x$ implies $\bar{y} \in Q(x)$. ($\bar{y} = (y_0, \ldots, y_n) = (y_0, y)$, $y = (y_1, \ldots, y_n)$).

From $\bar{y}^{ik} \in Q(x^{ik})$ follows that there are u^{ik} such that $y_0^{ik} \geqq V(x^{ik}, u^{ik})$, $y^{ik} = g(x^{ik}) + h(x^{ik})\, u^{ik}$.

Since y_0^k is convergent and

$$y_k^0 \geqq \sum_{i=1}^{n+2} \lambda^{ik} V(x^{ik}, u^{ik}) \geqq \sum_{i=1}^{n+2} \lambda^{ik} \varphi(|u^{ik}|) \geqq \sum_{i=1}^{n+2} \varphi_0(\lambda^{ik} |u^{ik}|),$$

$\{\lambda^{ik} u^{ik}\}_{k=1}^{\infty}$ must be bounded for every i. Hence, passing to a subsequence, if necessary, we may suppose that the sequences $\{\lambda^{ik} u^{ik}\}_{k=1}^{\infty}$ and, consequently, $\{u^k\}$, are convergent.

Let $u = \lim_{k\to\infty} u^k$. We have

$$y = \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik}[g(x^{ik}) + h(x^{ik})\, u^{ik}] = \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik}\, g(x^{ik}) +$$

$$+ h(x) \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik} u^{ik} + \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik}[h(x^{ik}) - h(x)]\, u^{ik} = g(x) + h(x)\, u$$

(the third term tends to zero because of the continuity of g and boundedness of $\{\lambda^{ik} u^{ik}\}_k$. Further, we have

$$y_0 \geqq \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik}\, \tilde{V}(x^{ik}, u^{ik}) \geqq \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik}\, \tilde{V}(x, u^{ik}) +$$

$$+ \lim_{k\to\infty} \sum_{i=1}^{n+2} \lambda^{ik}[\tilde{V}(x, u^{ik}) - \tilde{V}(x^{ik}, u^{ik})] \geqq \lim_{k\to\infty} \tilde{V}(x, u^k) +$$

$$+ \lim_{k\to\infty} \sum_{i=1}^{n+2} \varkappa|\lambda^{ik} + \lambda^{ik}|\, u^{ik}|\,|\, |x - x^{ik}| = V(x, u)\,.$$

This proves $\bar{y} \in Q(x)$.

Due to Theorem 5 there is an $\varepsilon > 0$ such that the optimal control for the pair f, $\tilde{V}$ exists, is unique and admits a once continuously differentiable synthesis v for $|\hat{x}| < \varepsilon$; further, the solutions $x(t, \hat{x})$ of the system $\dot{x} = f(x, v(x))$ with $x(0, \hat{x}) = \hat{x}$, $|\hat{x}| < \varepsilon$ satisfy $|x(t, \hat{x})| \leqq \eta_0$ and $\tilde{W}(\hat{x}) \leqq \inf_{\hat{x}\in\tilde{\Omega}_{\hat{x}}^{\eta_0}} \tilde{W}^{\eta_0}(\hat{x})$ ($\tilde{W}$, $\tilde{W}^{\eta_0}$, $\tilde{\Omega}_{\hat{x}}^{\eta_0}$ defined as in the proof of Theorem 1 with V replaced by $\tilde{V}$).

Let $|\hat{x}| < \varepsilon$ and $u \in \Omega_{\hat{x}}$, $u(t) \not\equiv v(x(t, u, \hat{x}))$. Denote T zero, if $|x(t, u, \hat{x})| \leqq \eta_0$ for all t and the last number for which $|x(t, u, \hat{x})| = \eta_0$ in the opposite case. We have for $|\hat{x}| < \varepsilon$:

$$\int_0^\infty V(x(t, \hat{x}), v(x(t, \hat{x}))\, \mathrm{d}t = \int_0^\infty \tilde{V}(x(t, \hat{x}), v(x(t, \hat{x}))\, \mathrm{d}t <$$

$$< \int_T^\infty \tilde{V}(x(t, u, \hat{x}), u(t))\, \mathrm{d}t \leqq \int_T^\infty V(x(t, u, \hat{x}), u(t))\, \mathrm{d}t \leqq \int_0^\infty V(x(t, u, \hat{x}), u(t))\, \mathrm{d}t\,.$$

This proves that for $|\hat{x}| < \varepsilon$, $v(x)$ is the synthesis of optimal control also for V and that this optimal control is unique for all $|\hat{x}| < \varepsilon$.

References

[1] *Летов А. М.:* Аналитическое конструирование регуляторов. Автоматика и телемеханика *21* (1960), 436—441.

[2] *Курцвейль Я.* (Kurzweil J.): К аналитическому конструированию регуляторов. Автоматика и телемеханика *22* (1961), 688—695.

[3] *Popov V. M.:* Hyperstability and optimality of automatic systems with several control functions. Rev. Roum. Sci. Techn.-Electrotechn. et Energ., *9* (1964), 629—690.

[4] *Зубов В. И.:* Теория оптимального управления, Ленинград 1966.

[5] *Brunovský P.:* Об аналитическом конструировании регуляторов с неквадтатическим минимизируемым функционалом. Čas. pěst. mat. *90* (1965), 290—310.

[6] *Lee E. B., Markus L.:* Optimal control for nonlinear processes. Arch. Rat. Mech. Anal. *8* (1961), No 1.

[7] *Cesari L.:* Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints I, Transactions of the Am. Math. Soc. *124* 1966, pp. 369—412.

[8] *Натансон И. П.:* Теория функций вещественной переменной, Москва 1957.

[9] *Понтрягин Л. С., Болтянский В. Г., Гамкрелидзе Р. В., Мищенко Е. Ф.:* Математическая теория оптимальных процессов, Москва 1961.

[10] *Coddington Earl A., Levinson N.:* Theory of ordinary differential equations, New York 1955.

[11] *La Salle J. P., Lefschetz S.:* Stability by Liapunov's direct method, New York 1961.

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