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ON CARATHÉODORY OPERATORS

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1. This note represents a direct continuation of section 2 of [1]; definitions and notation of that paper are used here throughout. Our aim is to prove the following assertion:

Theorem A. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I; G)$. *Then there exist classical Carathéodory operators* T_i, $i \in \mathcal{N}$ *such that for each* $\varphi \in \mathbf{C}(I; G)$, $\lim \varrho(\mathsf{T}_i\varphi, \mathsf{T}\varphi) = 0$.

2. First we prove some auxiliary results on polynomials in n variables. Let $\pi_n(x) = x_1 x_2 \dots x_n$. A real polynomial P_n in $x_1, \dots, x_n$ is said to be distinguished iff it is of the form

$$(2.1)\quad P_n(x_1, \dots, x_n) = \alpha\pi_n(x) + \sum_{j=1}^{n} \beta_j \frac{\pi_n(x)}{x_j} + \sum_{\substack{j,k=1 \\ j<k}}^{n} \gamma_{jk} \frac{\pi_n(x)}{x_j x_k} + \dots + \sum_{j=1}^{n} \delta_j x_j + \varepsilon$$

The same definition applies also to other sets of variables.

In what follows we prove that distinguished polynomials in $x_1, \dots, x_n$ have with respect to n-dimensional cubes properties analogical to those of linear functions on segments.

We shall use the following convention: the symbol $\tilde{x}_j$ signifies that x_j does not enter as a variable in our considerations.

3. Lemma. *Let* $P_n(x_1, \dots, x_n)$ *be a distinguished polynomial in* $x_1, \dots, x_n$, *and let* $\vartheta_{j_1}, \dots, \vartheta_{j_r}$, $1 \leqq r \leqq n$, *be real numbers. Then* $P_n(x_1, \dots, \vartheta_{j_1}, \dots, \vartheta_{j_r}, \dots, x_n)$ *is a distinguished polynomial in* $x_1, \dots, \tilde{x}_{j_1}, \dots, \tilde{x}_{j_r}, \dots, x_n$.

Proof. It is sufficient to prove this for $r = 1$, but then is it obvious.

4. Put $K = \langle 0, 1\rangle \times \dots \times \langle 0, 1\rangle$ (n times), $N = 2^n$, and given $j \in \{1, \dots, n\}$, $\vartheta \in \mathscr{R}$, let $\{x_j = \vartheta\} = \{[x_1, \dots, x_n] \in \mathscr{R}^n;\ x_j = \vartheta\}$. Let $\{j_1, \dots, j_r\}$ be a non-empty subset of $\{1, \ldots, n\}$, and let $\vartheta_k = 0$ or 1 for $k = 1, \ldots, r$; then the $(n - r)$-dimensional sides of K, denoted $K\{x_{j_1} = \vartheta_1, \ldots, x_{j_r} = \vartheta_r\}$, are defined by the formula $K \cap \cap \{x_{j_1} = \vartheta_1\} \cap \ldots \cap \{x_{j_r} = \vartheta_r\}$. If $r = n$, we get the vertices of the cube K, N in number; otherwise we have positive-dimensional sides of K, considered in the sequel as lower- dimensional cubes.

5. Lemma. *Let v_j, $j = 1, \ldots, N$, denote the vertices of K, and let $a_j \in \mathscr{R}$ for $j = 1, \ldots, N$. There exists one and only one distinguished polynomial P_n in $x_1, \ldots, x_n$ such that*

$$P_n(v_j) = a_j\,, \quad j = 1, \ldots, N \tag{5.1}$$

Proof. If we put $\varepsilon = P_n(0, 0, \ldots, 0)$, $\delta_1 = P_n(1, 0, \ldots, 0) - \varepsilon$, etc., then it is from (2.1) clear that all coefficients may be successively determined for (5.1) to be satisfied.

The numbers a_j corresponding to the vertices v_j of K, $j = 1, \ldots, N$, are considered to be fixed in some further sections; thus, the above polynomial will be denoted simply by $P_n(x_1, \ldots, x_n; K)$.

6. Lemma. *Let $1 \leqq r \leqq n - 1$, and let $Q = K\{x_{j_1} = \vartheta_1, \ldots, x_{j_{n-r}} = \vartheta_{n-r}\}$ be an r-dimensional side of K. Then*

$$P_n(x_1, \ldots, x_n; K) \,|\, Q = P_r(x_1, \ldots, \tilde{x}_{j_1}, \ldots, \tilde{x}_{j_{n-r}}, \ldots, x_n; Q) \tag{6.1}$$

Proof. In virtue of Lemma 3, the left-hand side is a distinguished polynomial in $x_1, \ldots, \tilde{x}_{j_1}, \ldots, \tilde{x}_{j_{n-r}}, \ldots, x_n$ taking on the prescribed values. Now we apply Lemma 5.

7. Lemma. *For each $j = 1, \ldots, n$, we have*

$$\begin{aligned} P_n(x_1, \ldots, x_n; K) = {} & (1 - x_j)\, P_{n-1}(x_1, \ldots, \tilde{x}_j, \ldots, x_n; K\{x_j = 0\}) + \\ & + x_j P_{n-1}(x_1, \ldots, \tilde{x}_j, \ldots, x_n; K\{x_j = 1\}) \end{aligned} \tag{7.1}$$

Proof. In virtue of Lemma 5 it is sufficient to note that the right-hand side is a distinguished polynomial in $x_1, \ldots, x_n$ taking on the prescribed values.

8. Lemma. *For each $\xi = [\xi_1, \ldots, \xi_n] \in K$, we have*

$$\min\{a_1, \ldots, a_N\} \leqq P_n(\xi_1, \ldots, \xi_n; K) \leqq \max\{a_1, \ldots, a_N\} \tag{8.1}$$

Proof. For $n = 1$ is it clear. Suppose the assertion is true for $n - 1$. As a consequence of (7.1), we have

$$\begin{aligned} P_n(\xi_1, \ldots, \xi_n; K) = {} & (1 - \xi_n)\, P_{n-1}(\xi_1, \ldots, \xi_{n-1}; K\{x_n = 0\}) + \\ & + \xi_n P_{n-1}(\xi_1, \ldots, \xi_{n-1}; K\{x_n = 1\}) \end{aligned}$$

Using induction assumption, we get from it e.g.

$$P_n(\xi_1, \ldots, \xi_n; K) \leqq (1 - \xi_n) \max \{a_1, \ldots, a_N\} + \xi_n \max \{a_1, \ldots, a_N\} = \\ = \max \{a_1, \ldots, a_N\}$$

9. Let $i \in \mathcal{N}$. The division $\mathcal{D}_i^1$ of $\mathcal{R}$ of the i-th rank is the set of all points of the form $k\, 2^{-i}$, where $i \in \mathcal{N}$ and k is an integer. The division $\mathcal{D}_i^n$ of $\mathcal{R}^n$ of the i-th rank is the set $\mathcal{D}_i^1 \times \ldots \times \mathcal{D}_i^1$ (n times); from now on, we write merely $\mathcal{D}_i$, instead of $\mathcal{D}_i^n$. It is clear that $\mathcal{D}_i$ induces a decomposition of $\mathcal{R}^n$ into non-overlapping cubes, with the edge 2^{-i} each. Let $\{\mathcal{D}_i\}$ denote the set of all these cubes.

10. After the preliminaries, we are going to prove the theorem stated in section 1 Here, $P(x_1, \ldots, x_n; a_1, \ldots, a_N)$ is a new notation for the polynomial P_n of Lemma 5 Also, if f denotes a point of $\mathcal{R}^n$ or a vector function, then $f^{(j)}$, $j = 1, \ldots, n$, denotes the j-th component of it. The vector-space operations on $\mathcal{R}^n$ are denoted in the usual manner.

Let $i \in \mathcal{N}$. For each $v \in \mathcal{D}_i \cap G$, let f_v be a vector function on I such that $[f_v] = \mathsf{T}\hat{v}$. Let us now define a vector function f_i on $I \times G$, generating a classical Carathéodory operator. The construction will be carried out for $f_i^{(1)}$ only; for other components of f_i is it analogous.

Let $K_0 \in \{\mathcal{D}_i\}$ be such that $K_0 = \{x \in \mathcal{R}^n;\ 0 \leqq x - v \leqq 2^{-i}\}$, for some $v \in \mathcal{D}_i$. Suppose that

$$K_0 \cap G \neq \emptyset \tag{10.1}$$

and let $v_1 = v, v_2, \ldots, v_N$ denote the vertices of K. Let us define the finite functions $a_j \mid I, j = 1, \ldots, N$, as follows: if $v_j \in G$, put $a_j(t) = f_{i,v_j}^{(1)}(t)$; otherwise put $a_j(t) = v_j^{(1)}$. Now, let for $t \in I$, $x \in K_0$

$$f_i^{(1)}(t, x) = P(2^i(x^{(1)} - v_1^{(1)}), \ldots, 2^i(x^{(n)} - v_1^{(n)});\ a_1(t), \ldots, a_N(t)) \tag{10.2}$$

and similarly for $f_i^{(j)}$, $j = 2, \ldots, n$, and all cubes of $\{\mathcal{D}_i\}$, satisfying (10.1). It follows from Lemma 6 that using (10.2), f_i may be well-defined on $I \times G$; we show that it generates a Carathéodory operator, i.e., it satisfies the conditions of Theorem (2,4) in [1].

Let $x \in G$. Then $f_i(., x)$ is measurable on I, as a "polynomial" of measurable functions. Let $t \in I$. Then $f_i(t, .)$ is continuous on G, as a simple consequence of (10.2) and Lemma 6.

We are going to prove that, for each $\varphi \in \mathbf{C}(I; G)$, $\lim \varrho([f_i \circ \varphi], \mathsf{T}\varphi) = 0$; thus, to prove the theorem, it suffices to put $\mathsf{T}_i\varphi = [f_i \circ \varphi]$.

Let $\varphi \in \mathbf{C}(I; G)$. Given $i \in \mathcal{N}$, there exists $\delta_i > 0$ such that

$$|t - t'| < \delta_i\,, \quad t, t' \in I \Rightarrow |\varphi(t) - \varphi(t')| < 2^{-i} \tag{10.3}$$

Let $\tau_0 = \tau < \tau_1 < \ldots < \tau_k = \tau + \alpha$, $\max\{\tau_j - \tau_{j-1}; j = 1, \ldots, k\} < \delta_i$. In virtue of (10.3), for each $j = 1, \ldots, k$, there exists $s_i(j) \in \mathscr{D}_i$ such that

$$t \in \langle \tau_{j-1}, \tau_j \rangle \Rightarrow |\varphi(t) - s_i(j)| < 2^{-i} \tag{10.4}$$

Let us define $s_i \mid I = \hat{s}_i(1) \mid \langle \tau_0, \tau_1) \oplus \ldots \oplus \hat{s}_i(k) \mid \langle \tau_{k-1}, \tau_k \rangle$. From (10.4) we infer that s_i converge to φ uniformly on I. For each $i \in \mathscr{N}$, it holds

$$\varrho(\mathsf{T}\varphi, \mathsf{T}_i\varphi) \leqq \varrho(\mathsf{T}\varphi, \mathsf{T}s_i) + \varrho(\mathsf{T}s_i, \mathsf{T}_i s_i) + \varrho(\mathsf{T}_i s_i, \mathsf{T}_i \varphi)$$

As a consequence of the definition of T_i, $\varrho(\mathsf{T}s_i, \mathsf{T}_i s_i) = 0$. In virtue of Lemma (2,6) of [1], $\lim \varrho(\mathsf{T}\varphi, \mathsf{T}s_i) = 0$. Thus, it is sufficient to prove that

$$\lim \varrho(\mathsf{T}_i s_i, \mathsf{T}_i \varphi) = 0 \tag{10.5}$$

Let $\nu = 3^n$, and let $\varepsilon_1, \ldots, \varepsilon_\nu$ be the points of $\mathscr{R}^n$ such that $\varepsilon_k^{(j)}$, $k = 1, \ldots, \nu$, $j = 1, \ldots, n$, equals to -1, 0 or 1, independently.

First we prove an estimation, which could easily be made more precise but which suffices for our purpose. In virtue of Lemma 8, we have for $i \in \mathscr{N}$, $t \in I$ and each $j = 1, \ldots, n$

$$|f_i^{(j)}(t, \varphi(t)) - f_i^{(j)}(t, s_i(t))| \leqq \sum_{k,l=1}^{\nu} |f_i^{(j)}(t, s_i(t) + 2^{-i}\varepsilon_k) - f_i^{(j)}(t, s_i(t) + 2^{-i}\varepsilon_l)| \tag{10.6}$$

Now, put $s_i(t) + 2^{-i}\varepsilon_k = s_i(t; k)$. The step functions $s_i(.\,; k) \mid I$ thus defined evidently converge to φ uniformly on I, for each $k = 1, \ldots, \nu$.

We have from (10.6)

$$\begin{aligned} |f_i(t, \varphi(t)) - f_i(t, s_i(t))| &= \max\{|f_i^{(1)}(\ldots) - f_i^{(1)}(\ldots), \ldots\} \leqq \\ &\leqq \max\{\sum_{k,l=1}^{\nu} |f_i^{(1)}(t, s_i(t; k)) - f_i^{(1)}(t, s_i(t; l))|, \ldots\} \leqq \\ &\leqq \sum_{\substack{k,l=1 \\ k<l}}^{\nu} \max\{\ldots\} = \sum_{\substack{k,l=1 \\ k<l}}^{\nu} |f_i(t, s_i(t; k)) - f_i(t, s_i(t; l))| \end{aligned}$$

Hence we get

$$\min(1, |\mathsf{T}_i\varphi - \mathsf{T}_i s_i|) \leqq \sum_{\substack{k,l=1 \\ k<l}}^{\nu} \min(1, |\mathsf{T}_i s_i(.\,; k) - \mathsf{T}_i s_i(.\,; l)|)$$

so that

$$\begin{aligned} \varrho(\mathsf{T}_i\varphi, \mathsf{T}_i s_i) &\leqq \sum_{\substack{k,l=1 \\ k<l}}^{\nu} \varrho(\mathsf{T}_i s_i(.\,; k), \mathsf{T}_i s_i(.\,; l)) = \\ &= \sum_{\substack{k,l=1 \\ k<l}}^{\nu} \varrho(\mathsf{T}s_i(.\,; k), \mathsf{T}s_i(.\,; l)) \leqq \sum_{\substack{k,l=1 \\ k<l}}^{\nu} \varrho(\mathsf{T}s_i(.\,; k), \mathsf{T}\varphi) + \\ &+ \sum_{\substack{k,l=1 \\ k<1}}^{\nu} \varrho(\mathsf{T}\varphi, \mathsf{T}s_i(.\,; l)) \end{aligned}$$

Now, (10.5) follows in virtue of Lemma (2,6) of [1]. This proves Theorem A.

11. Theorem B. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I; G)$ *and let* $\{f_i\}$ *be the sequence of vector functions on* $I \times G$ *defined in the proof of Theorem A. Suppose that there exists* $N \subset I$ *such that*

$$(11.1) \qquad p(N) = 0$$

$$(11.2) \qquad t \in I - N \Rightarrow f_i(t, .) \text{ converge uniformly on compact subsets of } G$$

Then $f \mid I \times G$, *defined for* $t \in I - N$, $x \in G$ *as* $f(t, x) = \lim f_i(t, x)$, *satisfies the conditions* (2.4.1), (2.4.2) *of Theorem* (2,4) *of* [1], *generating thus a classical Carathéodory operator. Further,* $\mathsf{T}\varphi = [f \circ \varphi]$ *for each* $\varphi \in \mathbf{C}(I; G)$.

Proof. From (11.2) we see that $f(t, .)$ is continuous on G for each $t \in I - N$. Further, $f(., x)$ is measurable on I for each $x \in G$, as a limit a.e. of measurable functions. To prove the second assertion of this theorem, it is sufficient to note that $\mathsf{T}\hat{\xi} = [f \circ \hat{\xi}]$ for each $\xi \in \bigcup_{i=1}^{\infty} D_i \cap G$; see Corollary (2,7) in [1].

12. The author believes that the assumptions of Theorem B are fulfilled for each Carathéodory operator; to solve Problem C of [1], it would be enough to prove this hypothesis. In section 14, we prove that this is true for classical Carathéodory operators.

13. Lemma. *Let* f *be a finite continuous function on* $\langle 0, 1 \rangle$. *For each* $i \in \mathcal{N}$, *let* $t_0, t_1 \ldots,, t_{2^i}$, *where* $0 = t_0 < t_1 < \ldots < t_{2^i} = 1$, *denote the points of* $\mathcal{D}_i^1 \cap \langle 0, 1 \rangle$. *Let* $l_i(t_k) = f(t_k)$, $k = 0, 1, \ldots, 2^i$, *and let* l_i *be linear on each* $\langle t_{k-1}, t_k \rangle$. *Then* l_i *converge to* f *uniformly on* I.

Proof. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $t_1', t_2' \in \langle 0, 1 \rangle$, $|t_1' - t_2'| \leqq \delta \Rightarrow$ $\Rightarrow |f(t_1') - f(t_2')| \leqq \varepsilon$. For each $i \in \mathcal{N}$, let $t(i) \in \mathcal{D}_i^1 \cap \langle 0, 1 \rangle$ be such that $|t - t(i)| \leqq$ $\leqq 2^{-i}$. Let $i_0 \in \mathcal{N}$ be such that $2^{-i_0} \leqq \delta$. Now using linearity of l_i, we get for each $i \geqq i_0$ and $t \in I$ that $|f(t) - l_i(t)| \leqq |f(t) - f(t(i))| + f(t(i))| + |f(t(i)) - l_i(t(i))| +$ $+ |l_i(t(i)) - l_i(t)| \leqq \varepsilon + 0 + \varepsilon = 2\varepsilon$ which proves the lemma.

14. Theorem C. *Let* T *be a classical Carathéodory operator on* $\mathbf{C}(I; G)$, *represented by a vector function* $f \mid I \times G$. *Then, the assumptions of Theorem B are fulfilled.*

Proof. This is an easy consequence of the preceding lemma.

15. Corollary. *Let* f *be the vector function of Theorem C. Let* λ^n *denote the Lebesgue measure on* R^n. *Then* f *is measurable on* $(I \times G, p \times \lambda^n)$.

Reference

[1] *K. Karták:* A generalization of the Carathéodory theory of differential equations, Czech. Math. Journal, *17* (*92*) (1967), 482—514.

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