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A GENERALIZATION OF THE CARATHÉODORY THEORY OF DIFFERENTIAL EQUATIONS

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0. INTRODUCTION AND NOTATION

The classical theory of differential equations considers the system

$$
\begin{aligned}
&x_1' = f_1(t, x_1, \ldots, x_n), \quad x_1(\tau) = \xi_1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&x_n' = f_n(t, x_1, \ldots, x_n), \quad x_n(\tau) = \xi_n
\end{aligned}
\tag{0.1}
$$

where $f_1, \ldots, f_n$ are continuous in a domain D of the space of $t, x_1, \ldots, x_n$, and $[\tau, \xi_1, \ldots, \xi_n] \in D$. Proofs of fundamental theorems are carried out by investigation of an equivalent system of integral equations

$$
\begin{aligned}
&x_1(t) = \xi_1 + \int_\tau^t f_1(s, x_1(s), \ldots, x_n(s))\, \mathrm{d}s \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&x_n(t) = \xi_n + \int_\tau^t f_n(s, x_1(s), \ldots, x_n(s))\, \mathrm{d}s
\end{aligned}
\tag{0.2}
$$

by means of elementary theorems of integral calculus and Ascoli's theorem on relative compactness in the space of continuous functions. Hence, the solutions are continuously differentiable functions.

In order to widen the scope of the theory, CARATHÉODORY in his book [3] investigated the system (0.2) under following more general conditions; here we take $n = 1$ for simplicity. Let the function $f(t, x)$ be Lebesgue measurable in t and continuous in x, and suppose that there exists a Lebesgue integrable function m, on an interval containing τ, such that $|f(t, x)| \leq m(t)$. It is shown that under these assumptions there exists a solution of (0.2) on an interval containing τ, which is absolutely continuous there so that equality in corresponding (0.1) may hold only almost everywhere. Basic theorems of this theory are also given in the book [4].

It is of interest to inquire into the usefulness of more general integrals in these questions. First, it is natural to insert the condition $m(t) \leqq f(t, x) \leqq M(t)$ with e.g. Perron integrable or even Denjoy-Chinčin integrable m and M, instead of the above condition $|f(t, x)| \leqq m(t)$; in this last case, of course, differentiation is considered in the asymptotical sense. In order to carry out these generalizations simultaneously, it is useful, and may be useful for other questions of analysis, to give an axiomatic definition of the integral. We show that axiomatics given here permits to prove necessary limit passages. Further, the basis of Carathéodory proofs consists in the fact that the mapping $\varphi \to f \circ \varphi$, where $(f \circ \varphi)(t) = f(t, \varphi(t))$, is from continuous functions to measurable functions and that this mapping is continuous in the sense that $\lim \varphi_i = \varphi_0$ pointwise implies $\lim f \circ \varphi_i = f \circ \varphi_0$ pointwise. Instead of this mapping, it will be convenient to introduce here the notion of the Carathéodory operator; in our sense, this is a continuous map from the space $\mathbf{C}$ to the space $\mathbf{S}$ with the additional property of possibility "to take part of it" on arbitrary intervals. Thus, both factors of the compound mapping $\varphi \to \int_\tau^t f \circ \varphi$ are here axiomatized.

Now, existence and uniqueness results, and some theorems on continuous dependence on a parameter forming the central part of this paper, are established in this more general case. In a sense, the present paper shows the natural frame of the Carathéodory theory, and may serve as a certain axiomatization of it. At the end, some complementary remarks and unsolved problems are given.

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In what follows, the symbol $\Rightarrow$ stands for implication. Instead of "if and only if", we write "iff". The letter n denotes a fixed integer $\geqq 1$, $\mathscr{N} = \{1, 2, \ldots\}$. $\mathscr{R}^n$ is the set of all real n-tuples $[x_1, \ldots, x_n]$; instead of $\mathscr{R}^1$, we write merely $\mathscr{R}$. For $x \in \mathscr{R}^n$, $|x| = |[x_1, \ldots, x_n]| = \max \{|x_1|, \ldots, |x_n|\}$; $x \leqq y$ in $\mathscr{R}^n$ means that $x_i \leqq y_i$ for each $i = 1, \ldots, n$. For $\varepsilon \in \mathscr{R}$, let $\vec{\varepsilon}$ denote the point $[\varepsilon, \ldots, \varepsilon] \in \mathscr{R}^n$. It is clear that $|x| \leqq \varepsilon$ iff $-\vec{\varepsilon} \leqq x \leqq \vec{\varepsilon}$. A cube K in $\mathscr{R}^n$ is the set of the form $\langle a_1, a_1 + \delta\rangle \times \ldots \times \langle a_n, a_n + \delta\rangle$, with $[a_1, \ldots, a_n] \in \mathscr{R}^n$, $\delta > 0$. The interior of a set A in $\mathscr{R}^n$ will be denoted by A^0. We put $\overline{\mathscr{R}} = \mathscr{R} \cup \{-\infty\} \cup \{\infty\}$, with usual algebraic and order properties. A mapping f defined on A will be sometimes denoted by $f \mid A$ or $x \to f(x)$, $x \in A$; however, $\to$ denotes also convergence in various spaces, and $\not\to$ stands for non-convergence. For $\emptyset \neq B \subset A$, $f \mid B$ denotes the reduction of f on B. Given $f \mid A, f_1 \mid A_1, \ldots, f_r \mid A_r$, $r \in \mathscr{N}$, we write $f \mid A = f_1 \mid A_1 \oplus \ldots \oplus f_r \mid A_r$, iff $A_1, \ldots, A_r$ are non-empty mutually disjoint sets such that $\bigcup_{i=1}^{r} A_i = A$ and $f \mid A_i = f_i \mid A_i$, $i = 1, \ldots, r$. If V is a proposition, $\{x \in A;\ V(x)\}$ denotes the set of all $x \in A$ such that $V(x)$ is true. Further, if f is a mapping defined on A, $\{f(x);\ V(x)\}$ denotes the image by f of $\{x \in A;\ V(x)\}$. A mapping with values in $\overline{\mathscr{R}}$ resp. $\mathscr{R}$ is called a scalarvalued resp. finite function. If f is a mapping from A to $\mathscr{R}^n$, we also speak of the vector function f, and denote $f = [f_1, \ldots, f_n]$; the finite functions f_i, $i = 1, \ldots, n$, are called the components of f. The function having a constant value ξ on a domain considered will be denoted by $\hat{\xi}$. We say that a vector function s defined on $\langle a, b \rangle$ is a step function, iff there exist intervals (= non-empty connected subsets of $\mathscr{R}$) $I_1, \ldots, I_r$ and $\xi_i \in \mathscr{R}^n$, $i = 1, \ldots, r$ such that $s \mid \langle a, b \rangle = \hat{\xi}_1 \mid I_1 \oplus \ldots \oplus \hat{\xi}_r \mid I_r$. If f is defined on $A \times B$ and $y \in B$, then $f(., y)$ denotes the function on A with value $f(x, y)$ at $x \in A$.

1. INTEGRATION

In this section we introduce the concept of the integral in an axiomatic way. Our interest is focused here on the notion more general than that one used later in differential equations.

Let p be a continuous nondecreasing function on a closed interval $\langle a, b \rangle \subset \mathscr{R}$; it is then possible to introduce fundaments of the theory of the Lebesgue-Stieltjes integral, e.g. measurability, convergence a.e. (= almost everywhere), etc. The symbol $\mathscr{L}(a, b; p)$ stands for the set of all functions f on $\langle a, b \rangle$ with finite Lebesgue-Stieltjes integral $\int_a^b f \,\mathrm{d}p$; here and in the sequel, p denotes also the measure corresponding to the above function p. The Lebesgue measure on $\mathscr{R}$ will be denoted by λ. To include some usual generalizations of the Lebesgue-Stieltjes integral, we introduce integration as follows.

(1,1) Definition. Let p be a continuous nondecreasing function on a closed interval $\langle a, b \rangle$. An $*$-integration on $\langle a, b \rangle$ with respect to p is a correspondence assigning to each closed interval $\langle c, d \rangle \subset \langle a, b \rangle$ a class $\mathscr{F} = \mathscr{F}(c, d; p)$ of scalarvalued p-measurable functions defined on $\langle c, d \rangle$ so that to each $f \in \mathscr{F}(c, d; p)$ there corresponds a finite real number, denoted by $\int f$, $\int_c^d f \,\mathrm{d}p$, or $(\mathscr{F}) \int_c^d f \,\mathrm{d}p$ and called the $\mathscr{F}$-integral of f over $\langle c, d \rangle$ with respect to p, where the following is fulfilled:

(A) $f_1 \in \mathscr{F}(c, d; p,)\ f_2 \in \mathscr{F}(c, d; p) \Rightarrow f_1 + f_2 \in \mathscr{F}(c, d; p)$,

where $f_1(t) + f_2(t)$ of the form e.g. $\infty - \infty$ may be defined in an arbitrary way, and $\int f_1 + \int f_2 = \int(f_1 + f_2)$

(B) $f \in \mathscr{F}(c, d; p)$, $k \in \mathscr{R} \Rightarrow kf \in \mathscr{F}(c, d; p)$, and $\int kf = k \int f$,

(C) $f \in \mathscr{F}(a, b; p)$, $\langle c, d \rangle \subset \langle a, b \rangle \Rightarrow f \mid \langle c, d \rangle \in \mathscr{F}(c, d; p)$,

(D) for each $f \in \mathscr{F}(a, b; p)$, $\int_c^d f \,\mathrm{d}p$ is an additive function of interval $\langle c, d \rangle \subset \subset \langle a, b \rangle$,

(E) $a \leqq c_1 < c_2 < c_3 \leqq b$, $f \mid \langle c_1, c_2 \rangle \in \mathscr{F}(c_1, c_2; p)$, $f \mid \langle c_2, c_3 \rangle \in \mathscr{F}(c_2, c_3; p) \Rightarrow \Rightarrow f \mid \langle c_1, c_3 \rangle \in \mathscr{F}(c_1, c_3; p)$,

(F) for each nonnegative f on $\langle a, b \rangle$, $f \in \mathscr{F}(a, b; p)$ iff $f \in \mathscr{L}(a, b; p)$, and $(\mathscr{F}) \int_a^b f \,\mathrm{d}p$ equals to the corresponding Lebesgue-Stieltjes integral.

(1,2) Examples. It is well known that the integrals $\mathscr{L}$ of Lebesgue-Stieltjes, $\mathscr{P}$ of Perron, $\mathscr{D}_*$ of Denjoy, $\mathscr{D}$ of Denjoy-Chinčin behave according to this definition [11]. On the other hand, e.g. the extension of $\mathscr{L}$-integral by Cauchy's principal value, and the $\mathscr{A}$-integral of Titchmarsh [1] are not $*$-integrals. Further comments will be given in section 10.

We pass to some simple consequences of the above definition. In what follows, $\mathfrak{F}^*$ denotes the class of all $*$-integrations and $\mathscr{F}$ denotes an element of $\mathfrak{F}^*$; we speak also of an $\mathscr{F}$-integration in this case.

(1,3) Theorem. *$f \in \mathscr{F}(a, b; p) \Rightarrow |f| < \infty$ a.e. on $\langle a, b\rangle$.*

Proof. $f \in \mathscr{F} \Rightarrow (-f) \in \mathscr{F}$, hence $f + (-f) \in \mathscr{F}$; then $0 = \int f + \int(-f) = \int[f + (-f)]$. When the sum is of the form e.g. $\infty - \infty$, put $f(t) - f(t) = 1$. Then $f + (-f) \geqq 0$, lies in $\mathscr{F}$, hence in $\mathscr{L}$; thus, $f + (-f) = 0$ a.e. on $\langle a, b\rangle$.

(1,4) Theorem. *$f \in \mathscr{L} \Rightarrow f \in \mathscr{F}$ for each $\mathscr{F} \in \mathfrak{F}^*$, and $(\mathscr{L}) \int f = (\mathscr{F}) \int f$. Further, $f \in \mathscr{F}$, $|f| \in \mathscr{F} \Rightarrow f \in \mathscr{L}$.*

Proof. This is an easy consequence of (A), (B), (F).

(1,5) Theorem. *$f \in \mathscr{F}(a, b; p)$, $f = g$ a.e. on $\langle a, b\rangle \Rightarrow g \in \mathscr{F}(a, b; p)$, and $\int_a^b f \,\mathrm{d}p = \int_a^b g \,\mathrm{d}p$.*

Proof. This follows easily from Theorem (1,4).

(1,6) Remark. We see that a function $f \in \mathscr{F}$ may be defined only a.e. on $\langle a, b\rangle$.

(1,7) Theorem. *$f, g \in \mathscr{F}(a, b; p), f \leqq g$ a.e. on $\langle a, b\rangle \Rightarrow \int f \leqq \int g$.*

Proof. $\int g - \int f = \int(g - f) \geqq 0$.

From now on, instead of "f_i converge to f asymptotically" we shall write $\operatorname{limas} f_i = f$.

(1,8) Theorem. *Let the following hold:*

$$g, h, f_i \in \mathscr{F}(a, b; p)\,, \quad i \in \mathscr{N} \tag{1.8.1}$$

$$g \leqq f_i \leqq h \quad \text{a.e. on} \quad \langle a, b\rangle\,, \quad i \in \mathscr{N} \tag{1.8.2}$$

$$\operatorname{limas} f_i = f \quad \text{on} \quad \langle a, b\rangle \tag{1.8.3}$$

Then $f \in \mathscr{F}(a, b; p)$, and $\int_a^b f \,\mathrm{d}p = \lim \int_a^b f_i \,\mathrm{d}p$.

Proof. We may suppose that all functions considered are finite. Then $0 \leqq f_i - g \leqq h - g$ a.e. on $\langle a, b\rangle$, $f_i - g \in \mathscr{L}$, $\operatorname{limas}(f_i - g) = f - g$; hence from the Lebesgue theorem $\lim \int(f_i - g) = \int(f - g)$, whence the assertion.

(1,9) Theorem. Let

(1.9.1) $$g, h \in \mathscr{F}(a, b; p)$$

(1.9.2) $$g \leqq f \leqq h \quad \textit{a.e. on} \quad \langle a, b \rangle$$

(1.9.3) $$f \quad \textit{be measurable on} \quad \langle a, b \rangle$$

Then $f \in \mathscr{F}(a, b; p)$, *and* $\int g \leqq \int f \leqq \int h$.

Proof. We have $0 \leqq f - g \leqq h - g$ a.e. on $\langle a, b \rangle$, $h - g \in \mathscr{L}$, $f - g$ is measurable. Hence $f - g \in \mathscr{L}$, and $\int(f - g) + \int g = \int f$. Thus, $f \in \mathscr{F}$; the last assertion is a consequence of Theorem (1,7).

The following generalization of Theorem (1,8) plays an important rôle in the investigation of continuous dependence on a parameter.

(1,10) Theorem. *Let the following hold*:

(1.10.1) $$g_i, h_i, g, h \in \mathscr{F}(a, b; p), \quad i \in \mathscr{N}$$

(1.10.2) $$g_i \leqq f_i \leqq h_i \quad \textit{a.e. on} \quad \langle a, b \rangle, \quad i \in \mathscr{N}$$

(1.10.3) $$\operatorname{limas} g_i = g, \quad \operatorname{limas} f_i = f, \quad \operatorname{limas} h_i = h$$

(1.10.4) $$\lim \int_a^b g_i \, \mathrm{d}p = \int_a^b g \, \mathrm{d}p, \quad \lim \int_a^b h_i \, \mathrm{d}p = \int_a^b h \, \mathrm{d}p$$

(1.10.5) $$f_i, \; i \in \mathscr{N}, \quad \textit{are measurable on} \quad \langle a, b \rangle.$$

Then $f_i, f \in \mathscr{F}(a, b; p)$, $i \in \mathscr{N}$, *and* $\lim \int_a^b f_i \, \mathrm{d}p = \int_a^b f \, \mathrm{d}p$.

Proof. According to Theorem (1,9), $f_i \in \mathscr{F}(a, b; p)$ for each $i \in \mathscr{N}$. Further, it is elementary that $g \leqq f \leqq h$ a.e. on $\langle a, b \rangle$; hence $f \in \mathscr{F}$. We prove that $\liminf \int f_i \geqq \int f$. Suppose on the contrary that $\liminf \int f_i < \int f$. Then there exist $i_1, i_2, \ldots$ such that $f_{i_k} \to f$, $g_{i_k} \to g$ a.e. on $\langle a, b \rangle$ and $\lim \int f_{i_k} < \int f$. Using Fatou's lemma we get $\int(f - g) = \int \lim (f_{i_k} - g_{i_k}) \leqq \liminf (\int f_{i_k} - \int g_{i_k}) = \liminf \int f_{i_k} - \int g$; hence $\int f \leqq \liminf \int f_{i_k}$. This is a contradiction. Passing to opposite functions, we obtain $\int f \geqq \limsup \int f_i$.

We join some usual definitions. First, for $f \in \mathscr{F}(a, b; p)$ we put $\int_b^a f \, \mathrm{d}p = -\int_a^b f \, \mathrm{d}p$, and $\int_a^a f \, \mathrm{d}p = 0$. For a $c \in \langle a, b \rangle$, the function $F(t) = \int_c^t f \, \mathrm{d}p$, defined on $\langle a, b \rangle$, will be called an $\mathscr{F}$-antiderivative of f.

Let $f = [f_1, \ldots, f_n]$ be a vector function defined a.e. on $\langle a, b \rangle$. We say that $f \in \mathscr{F}(a, b; p)$ iff $f_j \in \mathscr{F}(a, b; p)$ for each $j = 1, \ldots, n$, and put then $\int_a^b f \, \mathrm{d}p = [\int_a^b f_1 \, \mathrm{d}p, \ldots, \int_a^b f_n \, \mathrm{d}p]$. It is easy to see that all preceding theorems are also valid for vector functions. Further, $f \in \mathscr{L} \Rightarrow |\int f| \leqq \int |f|$.

We say that $\mathscr{F}_1 \in \mathfrak{F}^*$ is not weaker than $\mathscr{F} \in \mathfrak{F}^*$, and write $\mathscr{F} \subset \mathscr{F}_1$, iff $f \in$

$\in \mathscr{F}(a, b; p) \Rightarrow f \in \mathscr{F}_1(a, b; p)$ and $(\mathscr{F}) \int_a^b f \, \mathrm{d}p = (\mathscr{F}_1) \int_a^b f \, \mathrm{d}p$. Clearly, $\subset$ is an order relation in $\mathfrak{F}^*$; to stress it, we shall also denote this set by $(\mathfrak{F}^*, \subset)$. It is known that $\mathscr{L} \subset \mathscr{P} \subset \mathscr{D}_* \subset \mathscr{P} \subset \mathscr{D}$ [11]. Also, by Theorem (1,4), $\mathscr{L} \subset \mathscr{F}$ for each $\mathscr{F} \in \mathfrak{F}^*$.

(1,11) Theorem. *Given $\mathscr{F} \in (\mathfrak{F}^*, \subset)$, there exists a maximal element $\mathscr{F}_m \in (\mathfrak{F}^*, \subset)$ such that $\mathscr{F} \subset \mathscr{F}_m$.*

Proof. For a linearly ordered system $\{\mathscr{F}_\iota\}$ of $*$-integrations, the $*$-integration $\cup \mathscr{F}_\iota$ may be defined in an obvious way. The assertion now follows from Zorn's lemma.

There exist $*$-integrations such that the corresponding antiderivatives needn't be continuous; see (10,1,1) for a simple example. Having in mind the applications of general integration to differential equations, we introduce the following notion.

(1,12) Definition. We say that $\mathscr{F} \in \mathfrak{F}^*$ is an integration, iff

(G) for each $f \in \mathscr{F}(a, b; p)$, $\int_c^d f \, \mathrm{d}p$ is a continuous function of interval on $\langle a, b \rangle$, i.e. $d - c \to 0 \Rightarrow \int_c^d f \, \mathrm{d}p \to 0$.

In what follows, $\mathfrak{F}$ denotes the set of all integrations. It is easy to see that all the preceding assertions concerning $*$-integrations are true when we replace $\mathfrak{F}^*$ by $\mathfrak{F}$.

(1,13) Theorem. *For each $\mathscr{F} \in \mathfrak{F}$, $\mathscr{F}$-antiderivatives are continuous.*

Proof. This follows immediately from (G).

In later sections, we shall need the following lemma on $\mathscr{D}_*$- and $\mathscr{D}$-antiderivatives; its proof is easy and may be found in [7]. For the notion ACG_* see [11]; for the notion ACG, see also section 7.

(1,14) Lemma. *Let $F \in \operatorname{Lip} 1$ on $\langle a, b \rangle$ and let φ be ACG_* resp. ACG on $\langle \alpha, \beta \rangle$. Suppose that $\varphi(t) \in \langle a, b \rangle$ for each $t \in \langle \alpha, \beta \rangle$. Then $F(\varphi)$ is ACG_* resp. ACG on $\langle \alpha, \beta \rangle$.*

2. CARATHÉODORY OPERATORS AND FORMULATION OF THE PROBLEM

In what follows, I denotes a fixed compact interval $\langle \tau, \tau + \alpha \rangle$, $\alpha > 0$, and $G \neq \emptyset$ is a region (= open connected set) in $\mathscr{R}^n$. Measurability notions refer to a fixed continuous nondecreasing function p on I.

The symbol $\mathbf{C} = \mathbf{C}(I; G)$ denotes the set of all continuous mappings from I to G; especially, for $G = \mathscr{R}^n$ we write simply $\mathbf{C}(I)$ instead of $\mathbf{C}(I; \mathscr{R}^n)$. This set is given the usual metric $\|\varphi - \psi\| = \sup \{|\varphi(t) - \psi(t)|; t \in I\}$ so that the corresponding convergence is uniform on I.

The symbol $\mathbf{S} = \mathbf{S}(I; p)$ stands for the set of all measurable mappings from I with values in $\mathscr{R}^n$. This set is given the pseudometric $\varrho(g, h) = \int_I \min(1, |g(t) - h(t)|) \, \mathrm{d}p$ so that the induced convergence is asymptotical. Further, we put $[\mathbf{S}] = \mathbf{S} \bmod \mathbf{Z}$, where $\boldsymbol{Z} = \{z \in \boldsymbol{S};\ z = 0$ a.e. on $I\}$ so that $[\boldsymbol{S}]$ is a linear space with metric ϱ; let $[f]$ denote the image of f in the natural map of $\boldsymbol{S}$ onto $[\boldsymbol{S}]$. Note that if $g \in [\boldsymbol{S}]$ and A is a measurable subset of I, an obvious meaning may be given to $g \mid A$. For $f, g \in [\boldsymbol{S}]$, we write $f \leqq g$ iff there exist $f_1, g_1 \in \boldsymbol{S}$ such that $[f_1] = f$, $[g_1] = g$ and $f_1 \leqq g_1$ a.e. on I; it is clear that $\leqq$ is an order relation on $[\boldsymbol{S}]$. Similarly, for $f \in [\boldsymbol{S}]$ and $g \in \boldsymbol{S}$, we write $f \leqq g$ iff $f \leqq [g]$. For $f \in [\boldsymbol{S}]$, $|f|$ denotes the element of $[\boldsymbol{S}]$ defined by $[|f_1|]$, where $f_1 \in \boldsymbol{S}$ is such that $[f_1] = f$. Note also that Theorem (1,7) enables to introduce $\mathscr{F}$-integrals for some $f \in [\boldsymbol{S}]$.

(2,1) Lemma. *Let* $f_i = [f_i^{(1)}, \ldots, f_i^{(n)}] \in \boldsymbol{S}(I;\, p)$, $i \in \mathscr{N}$, $f = [f^{(1)}, \ldots, f^{(n)}] \in \boldsymbol{S}(I;\, p)$. *Then* $\lim \varrho(f_i, f) = 0$ *iff* $\lim \varrho(f_i^{(j)}, f^{(j)}) = 0$ *for each* $j = 1, \ldots, n$.

Proof. This is a consequence of the inequality

$$\varrho(f_i^{(j)}, f^{(j)}) \leqq \varrho(f_i, f) \leqq \sum_{j=1}^{n} \varrho(f_i^{(j)}, f^{(j)})$$

The symbol $\mathbf{C}^{\oplus}(I;\, G)$ denotes the set of all functions φ on I of the form $\varphi_1 \mid I_1 \oplus \oplus \ldots \oplus \varphi_r \mid I_r$, where $\varphi_i \in \mathbf{C}(I;\, G)$ for $i = 1, \ldots, r$ and $I_1, \ldots, I_r$ are intervals; hence, each step function on I with values in G lies in $\mathbf{C}^{\oplus}(I;\, G)$.

(2,2) Definition. A continuous mapping T from $\mathbf{C}(I;\, G)$ to $[\boldsymbol{S}]$ is a Carathéodory operator on $\mathbf{C}(I;\, G)$ iff the following is satisfied: $\varphi, \psi \in \mathbf{C}(I;\, G)$, J closed interval in I, $\varphi \mid J = \psi \mid J \Rightarrow \mathsf{T}\varphi \mid J = \mathsf{T}\psi \mid J$.

(2,3) Remark. It is clear that, in this definition, a closed interval J may be replaced by an arbitrary interval in I.

It follows that the domain of definition of each Carathéodory operator may now be extended in a natural way to all functions ζ of the form $\zeta = \varphi \mid J$, where $\varphi \in \mathbf{C}(I;\, G)$ and J is an interval such that $J \subset I$; indeed, we put $\mathsf{T}\zeta = \mathsf{T}\varphi \mid J$ in this case. Now for $\varphi \in \mathbf{C}^{\oplus}(I;\, G)$, $\varphi(I) = \varphi_1 \mid I_1 \oplus \ldots \oplus \varphi_r \mid I_r$ put $\mathsf{T}\varphi = \mathsf{T}\varphi_1 \mid I_1 \oplus \oplus \ldots \oplus \mathsf{T}\varphi_r \mid I_r$, the meaning of the right side being obvious. This definition is unambiguous.

If I_1 is a closed subinterval of I and $G_1 \neq \emptyset$ is a region $\subset G$, then evidently T induces a Carathéodory operator on $\mathbf{C}(I_1, G_1)$.

The following theorem shows a fundamental example of a Carathéodory operator.

(2,4) Theorem. *Let* $[t, x] \to f(t, x)$ *be a function from* $I \times G$ *to* $\mathscr{R}^n$ *with the following properties*:

(2.4.1) *for each* $x \in G$, $f(.\,, x)$ *is measurable on* I

(2.4.2) *there exists a set* $N \subset I$ *of zero measure such that for each* $t \in I - N$, $f(t,\,.)$ *is continuous on* G.

Then T *defined by* $\mathsf{T}\varphi = [f \circ \varphi]$, *where* $(f \circ \varphi)(t) = f(t, \varphi(t))$, *is a Carathéodory operator on* $\mathbf{C}(I;\, G)$.

Proof. Let s be a step function, $s \in \mathbf{C}^{\oplus}(I; G)$; $f \circ s$ is measurable, as a consequence of (2.4.1). Let $\varphi \in \mathbf{C}(I; G)$ and let $\{s_i\}$ be a sequence of step functions, $s_i \in \mathbf{C}^{\oplus}(I; G)$, such that $\lim s_i = \varphi$ pointwise on I. Then $(f \circ s_i)(t)$ converge to $(f \circ \varphi)(t)$ for each $t \in I - N$, as a consequence of (2.4.2); hence $f \circ \varphi \in \mathbf{S}$. Other properties to prove are now clear.

(2,5) Remark. The Carathéodory operator described in (2,4) will be called classical. It is not clear whether conversely each Carathéodory operator T is of the form $[f \circ \varphi]$, for a suitable f satisfying (2.4.1) and (2.4.2), or some more general conditions. In section 8, it will be shown that this is true provided T is linear; in [9] we prove an approximation theorem of this kind. Let us still show a simple example of a function f not generating a Carathéodory operator. Put $I = \langle 0, 1 \rangle$, $G = \mathscr{R}$, and let $f(t, x) = 0$ resp. 1 according to $t > x$ resp. $t \leqq x$. Put $\varphi_i(t) = t - i^{-1}$, $i \in \mathscr{N}$, $\varphi(t) = t$. Then $\|\varphi_i - \varphi\| = i^{-1} \to 0$, but $f \circ \varphi_i = \hat{0}$, $f \circ \varphi = \hat{1}$.

(2,6) Lemma. *Let $\varphi_i \in \mathbf{C}^{\oplus}(I; G)$, $i \in \mathscr{N}$, $\varphi \in \mathbf{C}(I; G)$, $\varphi_i \to \varphi$ uniformly on I. Let T be a Carathéodory operator on $\mathbf{C}(I; G)$. Then* $\lim \varrho(\mathsf{T}\varphi_i, \mathsf{T}\varphi) = 0$.

Proof. Let $\Delta > 0$ be such that 2Δ-neighbourhood of the set $\varphi(I)$ lies in G. We may suppose that $|\varphi_i(t) - \varphi(t)| < \Delta$ for each $t \in I$ and each $i \in \mathscr{N}$. Let $t_1^{(i)}, \ldots, t_{\nu_i}^{(i)}$ be the discontinuities of φ_i. For each $i \in \mathscr{N}$ and $j = 1, \ldots, \nu_i$, let $I_j^{(i)}$ denote an open interval $(l_j^{(i)}, r_j^{(i)})$ with centre $t_j^{(i)}$ such that

$$\sum_{j=1}^{\nu_i} p(I_j^{(i)}) < i^{-1} \tag{2.6.1}$$

$$\sup \{|\varphi(t) - \varphi(t')|;\ t, t' \in I_j^{(i)}\} < i^{-1} \tag{2.6.2}$$

Define φ_i^* on I, $i \in \mathscr{N}$, by putting $\varphi_i^*(t) = \varphi_i(t)$ for $t \in I - \bigcup_{j=1}^{\nu_i} I_j^{(i)}$, $\varphi_i^*(t) = (r_j^{(i)} - l_j^{(i)})^{-1} [(r_j^{(i)} - t)\, \varphi_i(l_j^{(i)}) + (t - l_j^{(i)})\, \varphi_i(r_j^{(i)})]$ for $t \in I_j^{(i)}, j = 1, \ldots, \nu_i$. Thus, φ_i^*, $i \in \mathscr{N}$, is continuous on I.

Let $\varepsilon > 0$ be given. We prove that there exists i_0 such that $i \geqq i_0 \Rightarrow \varrho(\mathsf{T}\varphi_i, \mathsf{T}\varphi) \leqq 2\varepsilon$. There exists $\delta > 0$ such that $\delta \leqq \Delta$ and

$$x \in \mathbf{C}(I; G)\,, \quad \|x - \varphi\| \leqq 2\delta \Rightarrow \varrho(\mathsf{T}x, \mathsf{T}\varphi) \leqq \varepsilon \tag{2.6.3}$$

Let q be such that $i > q, t \in I \Rightarrow |\varphi_i(t) - \varphi(t)| \leqq \delta$. Then $i > q$, $t \in I_j^{(i)}, j = 1, \ldots, \nu_i$ implies $|\varphi_i^*(t) - \varphi(t)| \leqq \delta + i^{-1}$. Indeed, in view of (2.6.2) we get (we omit subscripts)

$$\begin{aligned} &|\varphi_i^*(t) - \varphi(t)| = \\ &= |(r - l)^{-1} [(r - t)\, \varphi_i(l) + (t - l)\, \varphi_i(r) - (r - t)\, \varphi(t) - (t - l)\, \varphi(t)]| \leqq \\ &\leqq (r - l)^{-1} \{(r - t)\, |\varphi_i(l) - \varphi(t)| + (t - l)\, |\varphi_i(r) - \varphi(t)|\} \leqq \\ &\leqq (r - l)^{-1} \{(r - t)\, [|\varphi_i(l) - \varphi(l)| + |\varphi(l) - \varphi(t)|] + (t - l)\, [\ldots]\} \leqq \\ &\leqq (r - l)^{-1} \{(r - t)(\delta + i^{-1}) + (t - l)(\delta + i^{-1})\} = \delta + i^{-1} \end{aligned}$$

Thus, $i \geqq \max(\delta^{-1}, q) \Rightarrow \|\varphi_i^* - \varphi\| \leqq 2\delta$ and as $\delta \leqq \Delta$, $\varphi_i^* \in \mathbf{C}(I; G)$. Put $i_0 = \max(\delta^{-1}, q, \varepsilon^{-1})$. Then, as a consequence of (2.6.3) and (2.6.1), $i \geqq i_0 \Rightarrow$ $\Rightarrow \varrho(\mathsf{T}\varphi, \mathsf{T}\varphi_i) \leqq \varrho(\mathsf{T}\varphi, \mathsf{T}\varphi_i^*) + \varrho(\mathsf{T}\varphi_i^*, \mathsf{T}\varphi_i) \leqq \varepsilon + \int_I \min(1, |\mathsf{T}\varphi_i^* - \mathsf{T}\varphi_i|) \, \mathrm{d}p \leqq$ $\leqq \varepsilon + \sum_{j=1}^{\nu_i} p(I_j^{(i)}) \leqq 2\varepsilon$.

(2,7) Corollary. *Let* $\mathsf{T}_1, \mathsf{T}_2$ *be Carathéodory operators on* $\mathbf{C}(I; G)$. *Let* G_1 *be a dense subset of* G. *Then* $\mathsf{T}_1 = \mathsf{T}_2$ *iff* $\mathsf{T}_1\hat{x} = \mathsf{T}_2\hat{x}$ *for each* $x \in G_1$.

We pass to the formulation of our problem. The following sections are devoted to investigation of the equation

$$(\mathscr{E}) \qquad x(t) = \xi + \int_\tau^t \mathsf{T}x \, \mathrm{d}p$$

where T is a Carathéodory operator on $\mathbf{C}(I; G)$ and $\xi \in G$.

We say that $\varphi \in \mathbf{C}(\langle \tau, \tau + \beta \rangle; G)$ is a solution of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$, $0 < \beta \leqq \alpha$, iff there exists $\mathscr{F} \in \mathfrak{F}$ such that $\varphi(t) = \xi + (\mathscr{F}) \int_\tau^t \varphi \, \mathrm{d}p$ for each $t \in \langle \tau, \tau + \beta \rangle$. To speak more precisely, we call φ a right $\mathscr{F}$-solution in this case, the left-hand situation being analogous. We say, then, that $(\mathscr{E})$ has a solution on $\langle \tau, \tau + \beta \rangle$. It is clear that, for each $\gamma \in (0, \beta \rangle$, $\varphi \mid \langle \tau, \tau + \gamma \rangle$ is then an $\mathscr{F}$-solution of $(\mathscr{E})$ on $\langle \tau, \tau + \gamma \rangle$. It follows that each $\mathscr{F}$-solution of $(\mathscr{E})$ is continuous and $\varphi(\tau) = \xi$. Evidently, each $\mathscr{F}$-solution is also an $\mathscr{F}_1$-solution for each $\mathscr{F}_1 \supset \mathscr{F}$, while the contrary needn't be, of course, true.

A right $\mathscr{F}$-solution φ of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$, $\beta \in (0, \alpha \rangle$, is said to be the $\mathscr{F}$-unique solution on $\langle \tau, \tau + \beta \rangle$ iff from "*ψ is an $\mathscr{F}$-solution of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$*" it follows that "*$\varphi = \psi$ on $\langle \tau, \tau + \beta \rangle$*"; φ is said to be strongly $\mathscr{F}$-unique on $\langle \tau, \tau + \beta \rangle$ iff the $\mathscr{F}$-solution $\varphi \mid \langle \tau, \tau + \gamma \rangle$ is $\mathscr{F}$-unique on $\langle \tau, \tau + \gamma \rangle$ for each $\gamma \in (0, \beta \rangle$.

A right $\mathscr{F}$-solution φ of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$, $\beta \in (0, \alpha \rangle$, is said to be the $\{\mathscr{F}\}$-unique solution of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$ iff from "*ψ is an $\mathscr{F}_1$-solution of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$ for some $\mathscr{F}_1 \supset \mathscr{F}$, $\mathscr{F}_1 \in \mathfrak{F}$*", it follows that "*$\varphi = \psi$ on $\langle \tau, \tau + \beta \rangle$*"; φ is said to be strongly $\{\mathscr{F}\}$-unique $\langle \tau, \tau + \beta \rangle$ iff the $\mathscr{F}$-solution $\varphi \mid \langle \tau, \tau + \gamma \rangle$ is $\{\mathscr{F}\}$-unique on $\langle \tau, \tau + \gamma \rangle$ for each $\gamma \in (0, \beta \rangle$.

Similar definitions may, of course, be given for half-open intervals $\langle \tau, \tau + \beta)$.

(2,8) Remark. Theorem (2,4) shows that the usual Carathéodory theory of differential equations, using $\mathscr{L}$-integration with $p = \lambda$, is included here.

3. EXISTENCE OF SOLUTIONS

It is clear that there exist equations $(\mathscr{E})$ without a solution; it suffices to take $\mathsf{T}x = g \in [\mathbf{S}]$ identically, with $g \notin \mathscr{F}$ for arbitrary $\mathscr{F} \in \mathfrak{F}$. In this section we prove two existence theorems, first a global one.

(3,1) Theorem. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I; G)$, $\xi \in G$, $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(3.1.1) *there exist* $m, M \in \mathscr{F}(I; p)$ *such that*

$$\varphi \in \mathbf{C}(I; G) \Rightarrow m \leqq \mathsf{T}\varphi \leqq M$$

(3.1.2) *denoting* $L(t) = \int_\tau^t m \,\mathrm{d}p$, $U(t) = \int_\tau^t M \,\mathrm{d}p$, *it holds* $t \in I \Rightarrow \xi + L(t) \in G$, $\xi + U(t) \in G$

(3.1.3) $$x \in G\,, \quad z \in G\,, \quad x \leqq y \leqq z \Rightarrow y \in G$$

Then $(\mathscr{E})$ *has an* $\mathscr{F}$*-solution on* I.

Proof. Put $L(t) = U(t) = 0$ for $t < \tau$ so that L, U are now defined on $(-\infty, \tau + \alpha\rangle$. Further, put $\alpha_i = \alpha i^{-1}$, $i \in \mathscr{N}$. Let us define the Carathéodory approximations φ_i, $i \in \mathscr{N}$, in the following way:

(3.1.4) $$\varphi_i(t) = \xi \quad \text{for} \quad t \in \langle \tau, \tau + \alpha_i \rangle$$

$$\varphi_i(t) = \xi + \int_\tau^{t-\alpha_i} \mathsf{T}\varphi_i \,\mathrm{d}p \quad \text{for} \quad t \in \langle \tau + \alpha_i, \tau + \alpha \rangle$$

By induction, we prove that each φ_i, $i \in \mathscr{N}$, maps I into G continuously. This is clear, for each $i \in \mathscr{N}$, on $\langle \tau, \tau + \alpha_i \rangle$. Suppose that the assertion is true for $\tau \leqq t \leqq \tau + k\alpha_i$, $1 \leqq k < i$. Then the second of the formulae (3.1.4) defines φ_i on $\langle \tau + k\alpha_i, \tau + (k + 1)\,\alpha_i \rangle$; in view of (3.1.1) and Theorem (1,9), the $\mathscr{F}$-integral exists. According to (3.1.1), it holds

(3.1.5) $$L(t - \alpha_i) = \int_\tau^{t-\alpha_i} m \,\mathrm{d}p \leqq \varphi_i(t) - \xi \leqq \int_\tau^{t-\alpha_i} M \,\mathrm{d}p = U(t - \alpha_i)$$

so that, from (3.1.2) and (3.1.3), the assertion is valid for $1 \leqq k + 1 \leqq j$. Further, we get from (3.1.5) that $\{\varphi_i\}$ are uniformly bounded on I. Let $\langle t_1, t_2 \rangle \subset \langle \tau + \alpha_i, \tau + \alpha \rangle$; then $\varphi_i(t_2) - \varphi_i(t_1) = \int_{t_1-\alpha_i}^{t_2-\alpha_i} \mathsf{T}\varphi_i \,\mathrm{d}p$. Hence we have for each $i \in \mathscr{N}$

(3.1.6) $$L(t_2 - \alpha_i) - L(t_1 - \alpha_i) \leqq \varphi_i(t_2) - \varphi_i(t_1) \leqq U(t_2 - \alpha_i) - U(t_1 - \alpha_i)$$

and equicontinuity of $\{\varphi_i\}$ follows now immediately. From Ascoli's theorem we have existence of a subsequence $\{\varphi_{i_k}\}$ of $\{\varphi_i\}$ converging to a function $\varphi \in \mathbf{C}(I; G)$ uniformly on I. From (3.1.1) and Theorem (1,8) we get

(3.1.7) $$\lim \int_\tau^t \mathsf{T}\varphi_{i_k} \,\mathrm{d}p = \int_\tau^t \mathsf{T}\varphi \,\mathrm{d}p \quad \text{for each} \quad t \in I$$

Also, it holds there $\varphi_i(t) = \xi + \int_\tau^t \mathsf{T}\varphi_i \,\mathrm{d}p - \int_{t-\alpha_i}^t \mathsf{T}\varphi_i \,\mathrm{d}p$. From (3.1.1) and Theorem (1,9) we see that the last integral converges to zero, as $i \to \infty$; (3.1.7) then gives $\varphi(t) = \lim \varphi_{i_k}(t) = \xi + \int_\tau^t \mathsf{T}\varphi \,\mathrm{d}p$ for each $t \in I$. Thus, φ is an $\mathscr{F}$-solution of $(\mathscr{E})$ on I.

(3,2) Remark. Let us prove that $\lim \varphi_i = \varphi$ uniformly on I provided φ is unique on I. Suppose on the contrary that there exists $t_0 \in I$ such that $\varphi_i(t_0) \nrightarrow \varphi(t_0)$. Then there exist $i_1, i_2, \ldots$ such that $\lim \varphi_{i_k}(t_0) \neq \varphi(t_0)$ and $\{\varphi_{i_k}\}$ converge to the solution of $(\mathscr{E})$ on I, hence to φ; a contradiction. Uniform convergence of $\{\varphi_i\}$ results now from pointwise convergence and equicontinuity of $\{\varphi_i\}$.

(3,3) Remark. The preceding theorem is not suitable for the study of linear equations. Another global existence theorem will be proved in section 6.

We pass to a local existence theorem.

(3,4) Theorem. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I; G)$, $\xi \in G$, $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(3.4.1) *for each compact* $K \subset G$, *there exist* $m(., K)$, $M(., K) \in \mathscr{F}(I; p)$ *such that*

$$\varphi \in \mathbf{C}(I; G)\,, \quad \varphi(I) \subset K \Rightarrow m(., K) \leqq \mathsf{T}\varphi \leqq M(., K)$$

Then there exists $\beta > 0$ *such that* $(\mathscr{E})$ *has an* $\mathscr{F}$*-solution on* $\langle \tau, \tau + \beta \rangle$.

Proof. Choose a cube K with centre ξ such that $K \subset G$. Let m, M correspond to this cube according to (3.4.1), and denote $L(t) = \int_\tau^t m \, \mathrm{d}p$ and similarly for U. Let K^0 denote the interior of K. Then there exists $\beta > 0$ such that $\xi + L(t) \in K^0$, $\xi + U(t) \in \in K^0$ for each $t \in \langle \tau, \tau + \beta \rangle$. The assertion now follows from Theorem (3,1) as (3.1.1) and (3.1.3) are satisfied on taking $K^0 = G$.

4. EXTENSION OF SOLUTIONS

Let φ be an $\mathscr{F}$-solution of $(\mathscr{E})$ on $\langle \tau, \tau + \beta \rangle$, and let ψ be an $\mathscr{F}$-solution of $(\mathscr{E})$ on $\langle \tau, \tau + \gamma \rangle$, where $0 < \gamma < \beta \leqq \alpha$. If $\varphi \mid \langle \tau, \tau + \gamma \rangle = \psi$, then φ is called an $\mathscr{F}$-extension of ψ, and ψ is called $\mathscr{F}$-extendable. If there exists no $\mathscr{F}$-extension of φ then φ is called non-$\mathscr{F}$-extendable. Similar definitions may be given for half-open intervals.

First we state the following elementary assertion.

(4,1) Lemma. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I; G)$ *and let* $\xi \in G$, $\mathscr{F} \in \mathfrak{F}$. *For some* $t_0 \in (\tau, \tau + \alpha)$, *let* φ *be an* $\mathscr{F}$*-solution of* $(\mathscr{E})$ *on* $\langle \tau, t_0 \rangle$ *and let* ψ *be an* $\mathscr{F}$*-solution of the equation* $x(t) = \varphi(t_0) + \int_{t_0}^t \mathsf{T}x \, \mathrm{d}p$ *on* $\langle t_0, t_0 + \Delta \rangle$, $\Delta > 0$, $t_0 + \Delta \leqq \tau + \alpha$. *Then* $\chi \mid \langle \tau, t_0 + \Delta \rangle = \varphi \mid \langle \tau, t_0) \oplus \psi \langle t_0, t_0 + \Delta \rangle$ *is an* $\mathscr{F}$*-solution of* $(\mathscr{E})$.

Proof. For $t \in \langle t_0, t_0 + \Delta \rangle$ we have $\chi(t) = \xi + \int_\tau^t \mathsf{T}\varphi \, \mathrm{d}p + \int_{t_0}^t \mathsf{T}\psi \, \mathrm{d}p$.

(4,2) Theorem. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I; G)$, $\xi \in G$, $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(4.2.1) *there exist* $m, M \in \mathscr{F}(I; p)$ *such that*

$$\varphi \in \mathbf{C}(I; G) \Rightarrow m \leqq \mathsf{T}\varphi \leqq M$$

Let $t_0 \in \langle \tau, \tau + \alpha \rangle$, *and let* $\varphi \mid \langle \tau, t_0)$ *be an* $\mathscr{F}$-*solution of* $(\mathscr{E})$ *on* $\langle \tau, t_0)$. *Then*

(4.2.2) $$\lim_{t \to t_0 -} \varphi(t) = \eta \quad \textit{exists}$$

(4.2.3) *if* $\lim_{t \to t_0 -} \varphi(t) = \eta \in G$, *then* $\varphi \mid \langle \tau, t_0) \oplus \hat{\eta} \mid \{t_0\}$ *is an* $\mathscr{F}$-*solution of* $(\mathscr{E})$ *on* $\langle \tau, t_0 \rangle$; *if moreover* $t_0 < \tau + \alpha$, *then there exists* $\Delta > 0$ *such that* φ *has an* $\mathscr{F}$-*extension on* $\langle \tau, t_0 + \Delta \rangle$

(4.2.4) *if* φ^* *is any non*-$\mathscr{F}$-*extendable extension of* φ, *then either* φ^* *is defined on* $\langle \tau, \tau + \alpha \rangle$, *or* φ^* *is defined on* $\langle \tau, \tau_1)$, *for some* $\tau_1 \leqq \tau + \alpha$, *and* $\lim_{t \to \tau_1 -} \varphi(t)$ *belongs to the boundary of G*.

Proof. For $\langle t_1, t_2 \rangle \subset \langle \tau, t_0)$ we have from (4.2.1)

(4.2.5) $$L(t_2) - L(t_1) \leqq \varphi(t_2) - \varphi(t_1) \leqq U(t_2) - U(t_1)$$

so that existence of η readily follows. The first assertion in (4.2.3) is simple; to prove the second one, let us consider the equation $x(t) = \eta + \int_{t_0}^{t} \mathsf{T}x \, \mathrm{d}p$. As a consequence of (4.2.1) and Theorem (3,4) we have $\Delta > 0$ such that there exists an $\mathscr{F}$-solution ψ of this equation on $\langle t_0, t_0 + \Delta \rangle$. The conclusion now follows from Lemma (4,1). (4.2.4) is a consequence of (4.2.2) and (4.2.3).

5. UNIQUENESS OF SOLUTIONS

In this section we prove the following result on uniqueness.

(5,1) Theorem. *Let* $[t, r] \to \psi(t, r)$ *be a finite nonnegative function defined on* $(\tau, \tau + \alpha \rangle \times \langle 0, \infty)$ *such that*

(5.1.1) *for each* $r \in \langle 0, \infty)$, $\psi(., r)$ *is measurable on* $(\tau, \tau + \alpha \rangle$

(5.1.2) *for each* $t \in (\tau, \tau + \alpha \rangle$, $\psi(t, .)$ *is continuous nondecreasing on* $\langle 0, \infty)$

(5.1.3) *for each* $R > 0$ *and each* $\gamma \in (\tau, \tau + \alpha \rangle$, $\int_{\gamma}^{\tau+\alpha} \psi(., R) \, \mathrm{d}p$ *converges*

(5.1.4) $\hat{0} \mid \langle \tau, \tau + \alpha \rangle$ *is the strongly* $\mathscr{L}$-*unique solution of* $\varrho(t) = \int_{\tau}^{t} \psi \circ \varrho \, \mathrm{d}p$ *on* $\langle \tau, \tau + \alpha \rangle$.

Let T *be a Carathéodory operator on* $\mathbf{C}(I)$ *such that*

(5.1.5) $$x, y \in \mathscr{R}^n \Rightarrow \left|\mathsf{T}\hat{x} - \mathsf{T}\hat{y}\right| \leqq \psi(., |x - y|)$$

Then there exists for each $\mathscr{F} \in \mathfrak{F}$ *and each* $\beta \in (0, \alpha \rangle$ *at most one* $\mathscr{F}$-*solution of* $(\mathscr{E})$ *on* $\langle \tau, \tau + \beta \rangle$ *which is strongly* $\{\mathscr{F}\}$-*unique there.*

Proof. It is sufficient to prove this for $\beta = \alpha$. Using Lemma (2,6) and (5.1.5) we get by simple considerations, the integration being $\mathscr{L}$,

$$\int_{t_1}^{t_2} |\mathrm{T}\varphi_1 - \mathrm{T}\varphi_2| \, \mathrm{d}p \leqq \int_{t_1}^{t_2} \psi \circ |\varphi_1 - \varphi_2| \, \mathrm{d}p \tag{5.1.6}$$

whenever $\tau < t_1 \leqq t_2 \leqq \tau + \alpha$, $\varphi_1, \varphi_2 \in \mathbf{C}(I; G)$. Suppose if possible that there exist $\mathscr{F}_1, \mathscr{F}_2 \in \mathfrak{F}$, $\mathscr{F}_1 \subset \mathscr{F}_2$ and $\mathscr{F}_j$-solution φ_j, $j = 1, 2$, of $(\mathscr{E})$ on I such that $\varphi_1 \neq \varphi_2$. Let q be defined by $q(t) = |\varphi_1(t) - \varphi_2(t)|$ on I, and let $\sigma \in (\tau, \tau + \alpha)$ be such that $q(\sigma) > 0$. Let us consider the equation

$$\varrho(t) = q(\sigma) + \int_\sigma^t \psi \circ \varrho \, \mathrm{d}p \tag{5.1.7}$$

In view of (5.1.2), (5.1.3) and Theorem (3.4), there exists an $\mathscr{L}$-solution of (5.1.7) on an interval $\langle \sigma - \vartheta, \sigma \rangle$, $\vartheta > 0$. It next will be shown that $\varrho(t) \leqq q(t)$ for each $t \in \langle \sigma - \vartheta, \sigma \rangle$. Otherwise there are ζ and h_0, where $\zeta \leqq \sigma$ and $h_0 > 0$ such that $\varrho(\zeta) = q(\zeta)$ and $t \in (\zeta - h_0, \zeta) \Rightarrow q(t) < \varrho(t)$. However, it holds $q(\zeta) = |\int_\tau^\zeta (\mathrm{T}\varphi_1 - \mathrm{T}\varphi_2) \, \mathrm{d}p|$, and similarly $q(\zeta - h) = |\int_\tau^{\zeta - h} (\mathrm{T}\varphi_1 - \mathrm{T}\varphi_2) \, \mathrm{d}p|$, for each $h \in \langle 0, h_0 \rangle$, the integration here being $\mathscr{F}_2$. By subtracting we get

$$q(\zeta) - q(\zeta - h) \leqq \left| \int_{\zeta - h}^{\zeta} (\mathrm{T}\varphi_1 - \mathrm{T}\varphi_2) \, \mathrm{d}p \right| \tag{5.1.8}$$

Hence, applying (5.1.6)

$$q(\zeta) - q(\zeta - h) \leqq \int_{\zeta - h}^{\zeta} \psi \circ q \, \mathrm{d}p \tag{5.1.9}$$

Further, it holds

$$q(\zeta) - \varrho(\zeta - h) = \varrho(\zeta) - \varrho(\zeta - h) = \int_{\zeta - h}^{\zeta} \psi \circ \varrho \, \mathrm{d}p \tag{5.1.10}$$

and from $t \in \langle \zeta - h_0, \zeta \rangle \Rightarrow \varrho(t) \geqq q(t) \Rightarrow \psi(t, \varrho(t)) \geqq \psi(t, q(t))$ we get for each $h \in \langle 0, h_0 \rangle$

$$\int_{\zeta - h}^{\zeta} \psi \circ \varrho \, \mathrm{d}p \geqq \int_{\zeta - h}^{\zeta} \psi \circ q \, \mathrm{d}p \tag{5.1.11}$$

In view of (5.1.11), (5.1.10) and (5.1.9) we now have $\varrho(\zeta - h) \leqq q(\zeta - h)$ for $h \in \langle 0, h_0 \rangle$; this is a contradiction. We have thus proved that $\varrho(t) \leqq q(t)$ for each $t \leqq \sigma$, where $\varrho(t)$ exists. Also, it holds $\varrho(t) > 0$; suppose if possible that there exists $\sigma_1 \in (\tau, \sigma)$ such that $\varrho(\sigma_1) = 0$. But, in view of Theorem (4,1), the function ϱ_1 defined by $\varrho_1 \mid \langle \tau, \sigma_1 \rangle = \hat{0}$, $\varrho_1 \mid \langle \sigma_1, \sigma \rangle = \varrho \mid \langle \sigma_1, \sigma \rangle$ would then be a non-zero solution of the equation in (5.1.4); this is a contradiction with (5.1.4). Hence $0 < \varrho(t) \leqq q(t)$, and using Theorem (4,2) we see that ϱ has an $\mathscr{L}$-extension over $(\tau, \sigma\rangle$. Thus, $\lim_{t \to \tau+} \varrho(t) = 0$; putting $\varrho(\tau) = 0$, we get that ϱ is an $\mathscr{L}$-solution of the equation in (5.1.4) on $\langle\tau, \sigma\rangle$, and this contradicts (5.1.4). Hence $q = \hat{0} \mid I$, which proves the theorem.

Let us show a class of "test" functions ψ of the preceding theorem.

(5,2) Theorem. *Let $\chi \in \mathscr{L}(I; p)$, χ finite nonnegative. Then the function $[t, r] \to \chi(t)\, r$, $[t, r] \in I \times \langle 0, \infty)$, satisfies the assumptions (5.1.1) to (5.1.4) of Theorem (5,1).*

Proof. It is clear that (5.1.1) to (5.1.3) are satisfied. Let us now consider the equation $\varrho(t) = \int_\tau^t \chi\varrho \, \mathrm{d}p$, which evidently has $\hat{0} \mid I$ as a solution. Suppose there exists another $\mathscr{L}$-solution ϱ on $\langle\tau, \tau + \beta\rangle$, $\beta \in (0, \alpha\rangle$ such that $\varrho(t_0) > 0$ for a $t_0 \in (\tau, \tau + \beta)$. Then there exists $\langle a, b\rangle \subset \langle\tau, \tau + \beta\rangle$ such that $\varrho(a) = 0$, $\varrho(t) > 0$ for each $t \in (a, b)$. Let $\langle t_1, t_2\rangle \subset \langle a, b\rangle$; it follows from $\varrho(t_2) = \int_a^{t_1} \chi\varrho \, \mathrm{d}p + \int_{t_1}^{t_2} \chi\varrho \, \mathrm{d}p$ that $\varrho(t_1) \leqq \varrho(t_2)$. Hence $\varrho(a + \varepsilon) = \int_a^{a+\varepsilon} \chi\varrho \, \mathrm{d}p \leqq \varrho(a + \varepsilon) \int_a^{a+\varepsilon} \chi \, \mathrm{d}p$ for each (small) $\varepsilon > 0$; but this gives a contradiction for $\varepsilon \to 0+$.

6. SUCCESSIVE APPROXIMATIONS

In Theorem (3,1), an essential rôle was played by (3.1.1). Here we show that another global existence theorem can be proved, using different assumptions. First, we prove a lemma.

(6,1) Lemma. *Let T be a Carathéodory operator on $\mathbf{C}(I; G)$, $\mathscr{F} \in \mathfrak{F}$, and suppose that*

(6.1.1) *G_1 is bounded*

(6.1.2) *there exists $\omega \in \mathbf{C}(I; G_1)$ such that $\mathsf{T}\omega \in \mathscr{F}(I; p)$*

(6.1.3) *there exists a finite $\varkappa \in \mathscr{L}(I; p)$ such that*

$$x, y \in G_1 \Rightarrow |\mathsf{T}\hat{x} - \mathsf{T}\hat{y}| \leqq \varkappa |x - y|$$

Then there exist $m, M \in \mathscr{F}(I; p)$ such that

$$\varphi \in \mathbf{C}(I; G_1) \Rightarrow m \leqq \mathsf{T}\varphi \leqq M \tag{6.1.4}$$

Proof. Let $k = \sup\{|x - y|;\ x, y \in G_1\}$. For each $\varphi \in \mathbf{C}(I; G_1)$, it holds $|\varphi - \omega| \leqq k$. Using Lemma (2,6), we obtain $|\mathsf{T}\varphi - \mathsf{T}\omega| \leqq k\varkappa$; hence

$$\mathsf{T}\omega - [k\varkappa] \leqq \mathsf{T}\varphi \leqq \mathsf{T}\omega + [k\varkappa] \tag{6.1.5}$$

whence the assertion.

(6,2) Theorem. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I)$, $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(6.2.1) *there exists* $\omega \in \mathbf{C}(I)$ *such that* $\mathsf{T}\omega \in \mathscr{F}(I; p)$

(6.2.2) *there exists a finite* $\varkappa \in \mathscr{L}(I; p)$ *such that*

$$x, y \in \mathscr{R}^n \Rightarrow |\mathsf{T}\hat{x} - \mathsf{T}\hat{y}| \leqq |x - y| \varkappa$$

Then $(\mathscr{E})$ *has an* $\mathscr{F}$*-solution* φ *on* I *which is strongly* $\{\mathscr{F}\}$*-unique there.*

Proof. Clearly $\varphi, \psi \in \mathbf{C}(I) \Rightarrow |\mathsf{T}\varphi - \mathsf{T}\psi| \leqq \varkappa|\varphi - \psi|$ and in view of (6.2.1), $\varphi \in \mathbf{C}(I) \Rightarrow \mathsf{T}\varphi \in \mathscr{F}(I; p)$. Now put $\varphi_0 = \hat{\xi}$, and for $i = 0, 1, 2, \ldots$ define $\varphi_{i+1} \mid I$ as follows:

$$\varphi_{i+1}(t) = \xi + \int_\tau^t \mathsf{T}\varphi_i \, \mathrm{d}p \tag{6.2.3}$$

Then $|\varphi_1 - \varphi_0| (t) \leqq \|\varphi_1 - \varphi_0\|$, $|\varphi_2 - \varphi_1| (t) = |\int_\tau^t (\mathsf{T}\varphi_1 - \mathsf{T}\varphi_0) \, \mathrm{d}p| \leqq \int_\tau^t |\mathsf{T}\varphi_1 - \mathsf{T}\varphi_0| \, \mathrm{d}p \leqq \int_\tau^t \varkappa|\varphi_1 - \varphi_0| \, \mathrm{d}p \leqq \|\varphi_1 - \varphi_0\| \, K(t)$ where $K(t) = \int_\tau^t \varkappa \, \mathrm{d}p$. It next will be shown that

$$i \in \mathscr{N} \Rightarrow |\varphi_{i+1} - \varphi_i| (t) \leqq \|\varphi_1 - \varphi_0\| \, (i!)^{-1} \, K^i(t) \tag{6.2.4}$$

This is true for $i = 1$. Suppose that (6.2.4) holds for $i = j - 1$; we have $|\varphi_{j+1} - \varphi_j| (t) = |\int_\tau^t (\mathsf{T}\varphi_j - \mathsf{T}\varphi_{j-1}) \, \mathrm{d}p| \leqq \int_\tau^t |\mathsf{T}\varphi_j - \mathsf{T}\varphi_{j-1}| \, \mathrm{d}p \leqq \int_\tau^t \varkappa|\varphi_j - \varphi_{j-1}| \, \mathrm{d}p \leqq ((j-1)!)^{-1} \|\varphi_1 - \varphi_0\| \int_\tau^t \varkappa K^{j-1} \, \mathrm{d}p \leqq \|\varphi_1 - \varphi_0\| \, (j!)^{-1} \, K^j(t)$ as we get using $\mathscr{L}$-integration by parts.

The series

$$\varphi_0 + (\varphi_1 - \varphi_0) + \ldots + (\varphi_{i+1} - \varphi_i) + \ldots \tag{6.2.5}$$

then converges uniformly on I to a $\varphi \in \mathbf{C}(I)$. We show that φ is the desired $\mathscr{F}$-solution of $(\mathscr{E})$. It follows from (6.2.5) that there exists a bounded region $G_1 \subset G$ such that $\varphi, \varphi_i \in \mathbf{C}(I; G_1)$, $i \in \mathscr{N}$. Using Lemma (6,1) and Theorem (1,8) we get $\lim \int_\tau^t \mathsf{T}\varphi_i \, \mathrm{d}p = \int_\tau^t \mathsf{T}\varphi \, \mathrm{d}p$ on I, and existence of an $\mathscr{F}$-solution of $(\mathscr{E})$ on I is thus proved. In view of (6.2.2) and Theorems (5,1), (5,2), the $\mathscr{F}$-solution φ is strongly $\{\mathscr{F}\}$-unique on I.

7. CONTINUOUS DEPENDENCE OF SOLUTIONS ON A PARAMETER

In this section, some theorems on continuous dependence on a parameter will be established. As above, we suppose that $G \neq \emptyset$ is a region in $\mathscr{R}^n$, $I = \langle \tau, \tau + \alpha \rangle$ and $\xi \in G$. Let further

(7.0.1) T_i, $i = 0, 1, 2, \ldots$ *be Carathéodory operators on* $\mathbf{C}(I; G)$.

We shall investigate equations

$$(\mathscr{E}_i) \qquad x(t) = \xi + \int_\tau^t \mathsf{T}_i x \, \mathrm{d}p\,, \quad i = 0, 1, 2, \dots$$

and show that if $\mathsf{T}_1, \mathsf{T}_2, \dots$ converge to T_0 in some sense, then under some additional assumptions the same holds for corresponding solutions φ_i of $(\mathscr{E}_i)$. First we prove the following general result.

(7,1) Theorem. *Let* (7.0.1) *hold,* $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(7.1.1) *for each* $i \in \mathscr{N}$ *and each compact* $K \subset G$, *there exist* $m_i = m_i(.\,, K)$, $M_i = M_i(.\,, K) \in \mathscr{F}(I; p)$ *such that*

$$\varphi \in \mathbf{C}(I; G)\,, \quad \varphi(I) \subset K \Rightarrow m_i \leqq \mathsf{T}_i\varphi \leqq M_i$$

(7.1.2) *if* $\varphi_i \mid \langle \tau, \tau_i \rangle$, $i \in \mathscr{N}$, *is an* $\mathscr{F}$-*solution of* $(\mathscr{E}_i)$ *on* $\langle \tau, \tau_i \rangle$ *and there exists a compact* $K \subset G$ *such that* $\varphi_i(\langle \tau, \tau_i \rangle) \subset K$ *for each* $i \in \mathscr{N}$, *then* $\{\varphi_i\}$, $i \in \mathscr{N}$, *form an equicontinuous system in the sense that, given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for each* $i \in \mathscr{N}$, $|t_1 - t_2| < \delta$,

$$t_1, t_2 \in \langle \tau, \tau_i \rangle \Rightarrow |\varphi_i(t_1) - \varphi_i(t_2)| < \varepsilon$$

(7.1.3) $x_i \in \mathbf{C}(I; G)$, $i = 0, 1, 2, \dots$, $\lim \|x_i - x_0\| = 0 \Rightarrow \lim \int_\tau^t \mathsf{T}_i x_i \,\mathrm{d}p = \int_\tau^t \mathsf{T}_0 x_0 \,\mathrm{d}p$ *for each* $t \in I$, *the integration being* $\mathscr{F}$

(7.1.4) $(\mathscr{E}_0)$ *has an* $\mathscr{F}$-*solution* φ_0 *on* I *which is strongly* $\mathscr{F}$-*unique there.*

Then

(7.1.5) *there exists* $i_0 \in \mathscr{N}$ *such that for each* $i > i_0$, *each non-*$\mathscr{F}$*-extendable* $\mathscr{F}$-*solution* φ_i *of* $(\mathscr{E}_i)$ *is defined on* I, *and* $\lim \|\varphi_i - \varphi_0\| = 0$.

Proof. First we prove that if φ_i is an $\mathscr{F}$-solution of $(\mathscr{E}_i)$ on I for each $i \in \mathscr{N}$ and if there exists a compact $K \subset G$ such that $\varphi_i(I) \subset K$, $i \in \mathscr{N}$, then $\lim \|\varphi_i - \varphi_0\| = 0$. Indeed, we have $\varphi_i(t) = \xi + \int_\tau^t \mathsf{T}_i\varphi_i \,\mathrm{d}p$ for each $i \in \mathscr{N}$ and $t \in I$. According to (7.1.2), there exist $i_1, i_2, \dots$ and $\psi \in \mathbf{C}(I; G)$ such that $\lim \|\varphi_{i_k} - \psi\| = 0$. In view of (7.1.3), $\lim \int_\tau^t \mathsf{T}_{i_k}\varphi_{i_k} \,\mathrm{d}p = \int_\tau^t \mathsf{T}_0\psi \,\mathrm{d}p$ for each $t \in I$; hence ψ is an $\mathscr{F}$-solution of $(\mathscr{E}_0)$ on I and $\psi = \varphi_0$, as we infer from (7.1.4). To prove that φ_i converge to φ_0 pointwise suppose that there exists $\tau' \in I$ such that $\varphi_i(\tau') \nrightarrow \varphi_0(\tau')$. As $\{\varphi_i(\tau')\}$ is bounded, there exist $l_1, l_2, \dots$ such that $\lim \varphi_{l_j}(\tau') \neq \varphi_0(\tau')$ and $\{\varphi_{l_j}\}$ converge uniformly on I; this gives clearly a contradiction. Uniform convergence of $\{\varphi_i\}$ follows now from pointwise convergence and equicontinuity.

Let $\varepsilon_0 > 0$ be such that the compact set

(7.1.6) $\Omega = \{x \in \mathscr{R}^n$; *the distance of* x *from* $\varphi_0(I)$ *is* $\leqq \varepsilon_0\}$

lies in G; hence Ω^0 is connected. Let $\varphi_i \mid J_i$ be any non-$\mathscr{F}$-extendable solution of $(\mathscr{E}_i)$, $i \in \mathscr{N}$, when T_i is considered on $\mathbf{C}(I; \Omega^0)$; these solutions exist in virtue of (7.1.1), the choice of Ω, and Theorem (4.2). If (7.1.5) were false, there would exist ε, $0 < \varepsilon < \varepsilon_0$, and $i_1 < i_2 < \dots$ such that $\sup\{|\varphi_{i_k}(t) - \varphi_0(t)|;\ t \in J_{i_k},\ k \in \mathcal{N}\} \geqq \varepsilon$. Put $\psi_k = \varphi_{i_k}$; now for each $k \in \mathcal{N}$, there exists unique $T_k \in (\tau, \tau + \alpha\rangle$ such that $|\psi_k(t) - \varphi_0(t)| < \varepsilon$ on $\langle\tau, T_k)$ while

$$(7.1.7) \qquad |\psi_k(T_k) - \varphi_0(T_k)| = \varepsilon$$

From (7.1.2) we infer using compactness of Ω that there exists $\eta > 0$ such that $T_k > \tau + \eta$, $k \in \mathcal{N}$. Let the above $i_1, i_2, \dots$ be such that $\lim T_k = T_0$ exists; then $T_0 \geqq \tau + \eta$ is fulfilled. In view of (7.1.2), to the above $\varepsilon > 0$ there exists $\delta > 0$ such that $\delta \leqq \eta$ and

$$(7.1.8) \qquad t_1, t_2 \in I\,, \quad |t_2 - t_1| < \delta \Rightarrow |\varphi_0(t_2) - \varphi_0(t_1)| < \frac{\varepsilon}{3}$$

and at the same time

$$(7.1.9) \quad \tau \leqq t_1 < t_2 \leqq T_k\,, \quad t_2 - t_1 < \delta\,, \quad k \in \mathcal{N} \Rightarrow |\psi_k(t_2) - \psi_k(t_1)| < \frac{\varepsilon}{3}$$

Further, there exists $j_0 \in \mathcal{N}$ such that

$$(7.1.10) \qquad k > j_0 \Rightarrow T_0 - \frac{\delta}{2} < T_k < T_0 + \frac{\delta}{2}$$

According to the first part of this proof applied, as it is allowed, to $\langle\tau, T_0 - \delta/2\rangle$, there exists $j_1 \in \mathcal{N}$ such that

$$(7.1.11) \qquad k > j_1 \Rightarrow \left|\psi_k\left(T_0 - \frac{\delta}{2}\right) - \varphi_0\left(T_0 - \frac{\delta}{2}\right)\right| < \frac{\varepsilon}{3}$$

Now for $k > \max(j_0, j_1)$, (7.1.8), (7.1.9) and (7.1.11) give clearly $|\psi_k(T_k) - \varphi_0(T_k)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$, which contradicts (7.1.7). This proves the theorem.

Conditions (7.1.2) and (7.1.3) of Theorem (7,1) are rather complicated and not easily provable in concrete cases. We are going to derive three more lucid consequences of it.

(7,2) Theorem. *Let* (7.0,1) *hold,* $\mathcal{F} \in \mathfrak{F}$, *and suppose that*

(7.2.1) *for each* $i = 0, 1, 2, \dots$ *and each compact* $K \subset G$, *there exist* $m_i = m_i(., K)$, $M_i = M_i(., K) \in \mathcal{F}(I; p)$ *such that* $\varphi \in \mathbf{C}(I; G)$, $\varphi(I) \subset K$, $i = 0, 1, 2, \dots \Rightarrow m_i \leqq \mathsf{T}_i\varphi \leqq M_i$

(7.2.2) *for each compact* $K \subset G$, $\operatorname{limas} m_i(., K) = m_0(., K)$, $\operatorname{limas} M_i(., K) = M_0(., K)$

(7.2.3) *if we put, for each* $i = 0, 1, 2, \ldots$, *each compact* $K \subset G$, *and* $t \in I$

$$L_i(t, K) = \int_\tau^t m_i(., K)\, \mathrm{d}p\,, \quad U_i(t, K) = \int_\tau^t M_i(., K)\, \mathrm{d}p$$

then the systems $\{L_i(., K)\}$, $\{U_i(., K)\}$ *are for each fixed* $K \subset G$ *equicontinuous on* I

(7.2.4) *for each fixed* $K \subset G$ *and each* $t \in I$, $\lim L_i(t, K) = L_0(t, K)$, $\lim U_i(t, K) = U_0(t, K)$

(7.2.5) $x_i \in \mathbf{C}(I; G)$, $i = 0, 1, 2, \ldots$, $\lim \|x_i - x_0\| = 0 \Rightarrow \lim \varrho(\mathsf{T}_i x_i, \mathsf{T}_0 x_0) = 0$

(7.2.6) $(\mathscr{E}_0)$ *has an* $\mathscr{F}$*-solution* φ_0 *on* I *which is strongly* $\mathscr{F}$*-unique there*

Then (7.1.5) *is fulfilled.*

Proof. Evidently, (7.2.1) ⇒ (7.1.1) and (7.2.6) ⇒ (7.1.4). To prove (7.1.2), it suffices to use an inequality like (4.2.5) and (7.2.3). Finally, (7.1.3) follows from Theorem (1,10) using (7.2.1), (7.2.2), (7.2.4), (7.2.5) and the fact that there exists a compact $K \subset G$ such that $x_i(I) \subset K$ for $i = 0, 1, 2, \ldots$; now, (7.1.5) is a consequence of Theorem (7,1).

As another application of Theorem (7,1), we prove

(7,3) Theorem. *Let* (7.0.1) *hold,* $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(7.3.1) *for each* $i = 0, 1, 2, \ldots$ *and each compact* $K \subset G$, *there exist* $m_i = m_i(., K)$, $M_i = M_i(., K) \in \mathscr{F}(I; p)$ *such that*

$$\varphi \in \mathbf{C}(I; G)\,, \quad \varphi(I) \subset K \Rightarrow m_i \leqq \mathsf{T}_i \varphi \leqq M_i$$

(7.3.2) *there exist* $d > 0$ *and a nondecreasing scalarvalued function* $\delta \to \psi(\delta)$, $\delta \in (0, d\rangle$ *such that* $\lim_{\delta \to 0+} \psi(\delta) = 0$, *and a scalarvalued* $\chi \in \mathscr{L}(I; p)$, $\chi \geqq \hat{1}$ *such that*

$$x_1, x_2 \in G\,, \quad |x_1 - x_2| \leqq d\,, \quad t \in I\,, \quad i = 0, 1, 2, \ldots \Rightarrow$$
$$\Rightarrow |\mathsf{T}_i \hat{x}_1 - \mathsf{T}_i \hat{x}_2| \leqq \psi(|x_1 - x_2|)\, \chi$$

(7.3.3) *for each compact* $K \subset G$, $\lim \int_\tau^t \mathsf{T}_i \hat{x}\, \mathrm{d}p = \int_\tau^t \mathsf{T}_0 \hat{x}\, \mathrm{d}p$, $t \in I$, $x \in K$, *uniformly on* $I \times K$, *the integration being* $\mathscr{F}$

(7.3.4) $(\mathscr{E}_0)$ *has an* $\mathscr{F}$*-solution* φ_0 *on* I *which is strongly* $\mathscr{F}$*-unique there*

Then (7.1.5) *is fulfilled.*

Proof. Evidently, (7.3.1) ⇒ (7.1.1). Note also that using Lemma (2,6) we get from (7.2.3)

(7.3.5) $$x, y \in \mathbf{C}^{\oplus}(I; G)\,, \quad i = 0, 1, 2, \ldots \Rightarrow |\mathsf{T}_i x - \mathsf{T}_i y| \leqq \psi(|x - y|)\, \chi$$

Before proving (7.1.3) we show that a similar assertion concerning step functions is true. Let $\tau = \tau_0 < \tau_1 < \ldots < \tau_k = \tau + \alpha$, and let $s \in \mathbf{C}^{\oplus}(I; G)$ be defined in the following way: $s(t) = e_j \in G$ for $\tau_{j-1} \leqq t < \tau_j$, $j = 1, \ldots, k$, and $s(\tau + \alpha) = e_{k+1} \in G$. Then we have for each $t \in (\tau, \tau + \alpha\rangle$, $i = 0, 1, 2, \ldots$ that $\int_\tau^t \mathrm{T}_i s \, \mathrm{d}p = \int_{\tau_0}^{\tau_1} \mathrm{T}_i e_1 \, \mathrm{d}p + \ldots + \int_{\tau_r}^t \mathrm{T}_i e_{r+1} \, \mathrm{d}p$, where $0 \leqq r < k$; $\mathscr{F}$-integrals exist according to (7.3.1). From (7.3.3) it follows that, given $\varepsilon > 0$, there exists i_0 such that $j = 1, \ldots, k$, $i > i_0$, $t \in I$ implies $\left|\int_\tau^t \mathrm{T}_i e_j \, \mathrm{d}p - \int_\tau^t \mathrm{T}_0 e_j \, \mathrm{d}p\right| < \varepsilon/2k$. However, it holds e.g. $\left|\int_{\tau_r}^t \mathrm{T}_i e_j \, \mathrm{d}p - \int_{\tau_r}^t \mathrm{T}_0 e_j \, \mathrm{d}p\right| \leqq \left|\int_\tau^t (\mathrm{T}_i e_j - \mathrm{T}_0 e_j) \, \mathrm{d}p - \int_\tau^{\tau_r} \ldots\right| < 2(\varepsilon/2k) = \varepsilon/k$. Hence $i > i_0$, $t \in I \Rightarrow \left|\int_\tau^t \mathrm{T}_i s \, \mathrm{d}p - \int_\tau^t \mathrm{T}_0 s \, \mathrm{d}p\right| < k(\varepsilon/k) = \varepsilon$; this proves our assertion.

We pass to the proof of (7.1.3). Let $x_i \in \mathbf{C}(I; G)$, $i = 0, 1, 2, \ldots$, $\lim \|x_i - x_0\| = 0$. Then there exists an open set Ω such that $\overline{\Omega}$ is compact, $\overline{\Omega} \subset G$ and that $x_i(I) \subset \Omega$ for $i = 0, 1, 2, \ldots$ According to (7.3.1), $\mathrm{T}_i x_i \in \mathscr{F}(I; p)$ for $i = 0, 1, 2, \ldots$ Further, given $\varepsilon > 0$, there exists in view of (7.3.2) a $\delta > 0$ such that $\psi(\delta) \int_I \chi \, \mathrm{d}p < \varepsilon/6$. Let s denote a step function on I such that $s(I) \subset \Omega$ and $\sup \{|s(t) - x_0(t)|; t \in I\} < \delta$. Let $i_0 \in \mathscr{N}$ be such that $i > i_0 \Rightarrow \|x_i - x_0\| < \delta$ and

$$(7.3.6) \qquad \left|\int_\tau^t (\mathrm{T}_i s - \mathrm{T}_0 s) \, \mathrm{d}p\right| < \frac{\varepsilon}{2} \quad \text{for each} \quad t \in I$$

Then, in virtue of (7.3.2) and (7.3.5), we have for $i > i_0$ and $t \in I$

$$(7.3.7) \qquad \int_\tau^t |\mathrm{T}_i x_i - \mathrm{T}_i x_0| \, \mathrm{d}p \leqq \psi(\delta) \int_I \chi \, \mathrm{d}p < \frac{\varepsilon}{6}$$

$$(7.3.8) \qquad \int_\tau^t |\mathrm{T}_i x_0 - \mathrm{T}_i s| \, \mathrm{d}p \leqq \psi(\delta) \int_I \chi \, \mathrm{d}p < \frac{\varepsilon}{6}$$

$$(7.3.9) \qquad \int_\tau^t |\mathrm{T}_0 s - \mathrm{T}_0 x_0| \, \mathrm{d}p \leqq \psi(\delta) \int_I \chi \, \mathrm{d}p < \frac{\varepsilon}{6}$$

As a consequence of (7.3.6) to (7.3.9) we have immediately $\left|\int_\tau^t (\mathrm{T}_i x_i - \mathrm{T}_0 x_0) \, \mathrm{d}p\right| < \varepsilon$ for each $i > i_0$ and $t \in I$ so that (7.1.3) is proved.

It remains to prove (7.1.2). For $i \in \mathscr{N}$, let φ_i be $\mathscr{F}$-solutions of $(\mathscr{E}_i)$ on $\langle\tau, \tau_i\rangle$, with $\varphi_i(\langle\tau, \tau_i\rangle) \subset K$, a suitable compact in G. For $\langle t_1, t_2\rangle \subset \langle\tau, \tau_i\rangle$, we have

$$\varphi_i(t_2) - \varphi_i(t_1) = \int_{t_1}^{t_2} \mathrm{T}_i \varphi_i \, \mathrm{d}p = \int_{t_1}^{t_2} \mathrm{T}_i \Phi_i \, \mathrm{d}p + \int_{t_1}^{t_2} (\mathrm{T}_i \varphi_i - \mathrm{T}_i \Phi_i) \, \mathrm{d}p$$

writing here $\Phi_i = \widehat{\varphi_i(t_1)}$ for simplicity of notations. Hence

$$(7.3.10) \qquad |\varphi_i(t_1) - \varphi_i(t_2)| \leqq \left|\int_{t_1}^{t_2} \mathrm{T}_i \Phi_i \, \mathrm{d}p\right| + \left|\int_{t_1}^{t_2} (\mathrm{T}_i \varphi_i - \mathrm{T}_i \Phi_i) \, \mathrm{d}p\right|$$

In virtue of (7.3.3), there exists a nonincreasing function $i \to A(i)$, $i \in \mathcal{N}$, such that $\lim A = 0$ and

$$(7.3.11) \quad \langle t_1, t_2 \rangle \subset I\,, \quad x \in K\,, \quad i \in \mathcal{N} \Rightarrow \int_{t_1}^{t_2} \mathsf{T}_i \hat{x}\, \mathrm{d}p = \int_{t_1}^{t_2} \mathsf{T}_0 \hat{x}\, \mathrm{d}p + R(t_1, t_2, x, i)\,,$$

where $|R(t_1, t_2, x, i)| \leqq A(i)$ for each $t_1, t_2 \in I$, $x \in K$, $i \in \mathcal{N}$. According to (7.3.1), $\int_{t_1}^{t_2} m_0(., K)\, \mathrm{d}p \leqq \int_{t_1}^{t_2} \mathsf{T}_0 \hat{x}\, \mathrm{d}p \leqq \int_{t_1}^{t_2} M_0(., K)\, \mathrm{d}p$, whenever $\langle t_1, t_2 \rangle \subset I$, $x \in K$. Put $\varrho(\delta) = \inf \{\int_{t_1}^{t_2} m_0\, \mathrm{d}p;\ \tau \leqq t_1 < t_2 \leqq \tau + \alpha,\ t_2 - t_1 \leqq \delta\}$, and similarly $\sigma(\delta) = \sup \{\int_{t_1}^{t_2} M_0\, \mathrm{d}p; \ldots\}$. Then $\lim_{\delta \to 0+} \varrho(\delta) = \lim_{\delta \to 0+} \sigma(\delta) = 0$, and $\varrho \leqq \hat{0} \leqq \sigma$. Further, it holds $\varrho(t_2 - t_1) \leqq \int_{t_1}^{t_2} \mathsf{T}_0 \hat{x}\, \mathrm{d}p \leqq \sigma(t_2 - t_1)$, whenever $\langle t_1, t_2 \rangle \subset I$, $x \in K$. Putting $c = \max(-\varrho, \sigma)$, we have under these conditions $|\int_{t_1}^{t_2} \mathsf{T}_0 \hat{x}\, \mathrm{d}p| \leqq c(t_2 - t_1)$. Using (7.3.11), we get

$$(7.3.12) \qquad \left| \int_{t_1}^{t_2} \mathsf{T}_i \hat{x}\, \mathrm{d}p \right| \leqq c(t_2 - t_1) + A(i)$$

whenever $i \in \mathcal{N}$, $\langle t_1, t_2 \rangle \subset I$, and $x \in K$.

Let $0 < \varepsilon \leqq d$. We show that there exist $i_0 \in \mathcal{N}$ and $\delta_0 > 0$ such that $t_1, t_2 \in \langle \tau, \tau_i \rangle$, $0 < t_2 - t_1 < \delta_0$, $i > i_0 \Rightarrow |\varphi_i(t_2) - \varphi_i(t_1)| < \varepsilon$. Choose $i_0 \in \mathcal{N}$ such that $i > i_0 \Rightarrow A(i) < \varepsilon/3$; further, let $\delta_0 > 0$ be such that $c(\delta_0) < \varepsilon/3$ and $\nu(\delta_0)\, \psi(\varepsilon) < \varepsilon/3$, where ν is defined by $\nu(\delta) = \sup \{\int_{t_1}^{t_2} \chi\, \mathrm{d}p;\ 0 \leqq t_2 - t_1 \leqq \delta,\ t_1, t_2 \in I\}$. Now, let $i > i_0$, $0 < t_2 - t_1 < \delta_0$, $\langle t_1, t_2 \rangle \subset \langle \tau_1, \tau_i \rangle$; supposing that $|\varphi_i(t_2) - \varphi_i(t_1)| \geqq \varepsilon$ for some $i > i_0$, there would exist $t_3 \in (t_1, t_2 \rangle$ such that $|\varphi_i(t_3) - \varphi_i(t_1)| = \varepsilon$, but $|\varphi_i(t) - \varphi_i(t_1)| < \varepsilon$ for each $t \in \langle t_1, t_3)$. Thence we get in virtue of (7.5.3)

$$(7.3.13) \quad \int_{t_1}^{t_3} |\mathsf{T}_i \varphi_i - \mathsf{T}_i \Phi_i|\, \mathrm{d}p \leqq \int_{t_1}^{t_3} \chi(s)\, \psi(|\varphi_i(s) - \varphi_i(t_1)|)\, \mathrm{d}p\,(s) \leqq \psi(\varepsilon)\, \nu(\delta_0)$$

This, however, gives a contradiction, for using (7.3.12), we get

$$\varepsilon = |\varphi_i(t_3) - \varphi_i(t_1)| \leqq \left| \int_{t_1}^{t_3} \mathsf{T}_i \Phi_i\, \mathrm{d}p \right| + \left| \int_{t_1}^{t_3} (\mathsf{T}_i \varphi_i - \mathsf{T}_i \Phi_i)\, \mathrm{d}p \right| \leqq$$

$$\leqq c(\delta_0) + A(i) + \psi(\varepsilon)\, \nu(\delta_0) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Theorem (7,3) is thus proved.

In the last two theorems it was supposed that minor and major functions $m_0, M_0 \in \mathcal{F}$ exist; note that this wasn't required in Theorem (7,1). Before proving a theorem of this kind, we introduce a new notion.

(7,4) Definition. Let $\{m_i\}$, $\{M_i\}$ be sequences of functions on I. We say that the pair $\{\{m_i\}, \{M_i\}\}$ has the property (Vi) with respect to an $\mathcal{F}$-integration, or shortly has the property $(Vi, \mathscr{F})$ iff

(7.4.1) $i \in \mathscr{N} \Rightarrow m_i,\ M_i \in \mathscr{F}(I; p),\ m_i \leqq M_i$ a.e. on I

(7.4.2) the sequences $\{\int_\tau^t m_i \, dp\}$, $\{\int_\tau^t M_i d\, p\}$ are equicontinuous on I

(7.4.3) $m_i \leqq f_i \leqq M_i,\ f_i$ measurable on $I,\ i \in \mathscr{N},\ t \in I$, $\operatorname{limas} f_i = f_0$ on $I \Rightarrow$
$\Rightarrow f_0 \in \mathscr{F}(I; p)$ and $\lim \int_\tau^t f_i \, dp = \int_\tau^t f_0 \, dp$

(7,5) Theorem. *Let* (7.0.1) *hold,* $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(7.5.1) *for each compact* $K \subset G$, *there exists a pair* $\{\{m_i(.,K)\}, \{M_i(.,K)\}\}$ *having the property* $(Vi, \mathscr{F})$ *such that* $i \in \mathscr{N},\ \varphi \in \mathbf{C}(I; G),\ \varphi(I) \subset K \Rightarrow$
$\Rightarrow m_i \leqq \mathsf{T}_i\varphi \leqq M_i$

(7.5.2) $x_i \in \mathbf{C}(I; G),\ i = 0, 1, 2, \ldots,\ \lim \|x_i - x_0\| = 0 \Rightarrow \lim \varrho(\mathsf{T}_i x_i, \mathsf{T}_0 x_0) = 0$

(7.5.3) $(\mathscr{E}_0)$ *has an* $\mathscr{F}$-*solution* φ_0 *on* I *which is strongly* $\mathscr{F}$-*unique there*

Then (7.1.5) *is fulfilled.*

Proof. (7.5.1) $\Rightarrow$ (7.1.1), (7.5.3) $\Rightarrow$ (7.1.4), and (7.1.2) follows from (7.5.1), as a result of (7.4.2) and an inequality like (4.2.5). Finally, to prove (7.1.3), suppose that $x_i \in \mathbf{C}(I; G),\ i = 0, 1, 2, \ldots,\ \lim \|x_i - x_0\| = 0$. Hence there exists a compact $K \subset G$ such that $\bigcup_{i=1}^{\infty} x_i(I) \subset K$. From (7.5.1) we infer that there exists $\{\{m_i(.,K)\}, \{M_i(.,K)\}\}$ with the property $(Vi, \mathscr{F})$ such that $m_i \leqq \mathsf{T}_i x_i \leqq M_i$ for each $i \in \mathscr{N}$. From (7.5.2) and (7.4.3) we infer that $\int_\tau^t \mathsf{T}_0 x_0 \, dp = \lim \int_\tau^t \mathsf{T}_i x_i \, dp$ for each $t \in I$.

We are going to show three examples of pairs with the property (Vi).

(7,6) Theorem. *Let* $\mathscr{F} \in \mathfrak{F}$, *and suppose that* $m_i, M_i,\ i = 0, 1, 2, \ldots$ *are such that*

(7.6.1) $$m_i, M_i \in \mathscr{F}(I; p)\,, \quad \textit{for each} \quad i = 0, 1, 2, \ldots$$

(7.6.2) $$m_i \leqq M_i \quad \textit{a.e. on } I, \textit{for each} \quad i \in \mathscr{N}$$

(7.6.3) $$\operatorname{limas} m_i = m_0\,, \quad \operatorname{limas} M_i = M_0$$

(7.6.4) *the sequences* $\{\int_\tau^t m_i \, dp\}$, $\{\int_\tau^t M_i \, dp\}$ *are equicontinuous on* I

(7.6.5) $t \in I \Rightarrow \lim \int_\tau^t m_i \, dp = \int_\tau^t m_0 \, dp,\ \lim \int_\tau^t M_i \, dp = \int_\tau^t M_0 \, dp.$

Then the pair $\{\{m_i\}, \{M_i\}\}$, $i \in \mathscr{N}$, *has the property* $(Vi, \mathscr{F})$.

Proof. This is a direct consequence of Theorem (1,10).

In two further theorems we take $p = \lambda$ for simplicity. Let F be a continuous function on I, and let $E \subset I$. We say that F is AC on E iff, given $\varepsilon > 0$, there exists $\delta > 0$ such that

(7.6.6) $$a_j, b_j \in E\,,\ j = 1, \ldots, r\,,\ a_1 < b_1 \leqq a_2 < b_2 \leqq \ldots \leqq a_n < b_n\,,$$
$$\sum_{j=1}^{r} (b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{r} |F(b_j) - F(a_j)| < \varepsilon$$

Further, we say that a sequence $\{F_i\}$ of functions continuous on I is equi-AC on $E \subset I$ iff, given $\varepsilon > 0$, there exists $\delta > 0$ such that (7.6.6) holds independantly of $i \in \mathcal{N}$.

(7,7) Theorem. *Let m_i, M_i, $i \in \mathcal{N}$ defined on I be such that*

(7.7.1) $\quad i \in \mathcal{N} \Rightarrow m_i\,,\quad M_i \in \mathscr{L}(I; \lambda)\,,\quad m_i \leqq M_i \quad$ *a.e. on* I

(7.7.2) *the sequences* $\{\int_\tau^t m_i \,\mathrm{d}\lambda\}$, $\{\int_\tau^t M_i \,\mathrm{d}\lambda\}$, $t \in I$, *are equi-AC on* I

Then the pair $\{\{m_i\}, \{M_i\}\}$ *has the property* $(Vi, \mathscr{L})$.

Proof. (7.4.1) follows from (7.7.1), (7.4.2) is a consequence of (7.4.2). To prove (7.4.3), it is sufficient to note that (7.7.2) gives that $\{\int_\tau^t f_i \,\mathrm{d}\lambda\}$ is equi-AC, and use the Lebesgue-Vitali theorem.

There exists a generalization of the Lebesgue-Vitali theorem to $\mathscr{D}$-integration, which enables us to prove a corresponding analogue of Theorem (7,7).

(7,8) First, we say that F is ACG on I iff F is continuous on I and there exist E_i, $i \in \mathcal{N}$, such that $\bigcup_{i=1}^{\infty} E_i = I$ and F is AC on each E_i. Further, we say that a sequence $\{F_j\}$ of functions continuous on I is equi-ACG on I iff there exist $E_i \subset I$, $i \in \mathcal{N}$, such that $\bigcup_{i=1}^{\infty} E_i = I$ and $\{F_j\}$ is equi-AC on each E_i.

Now, there is the following theorem of Džvaršejšvili [5]: *Let $f_i \in \mathscr{D}(I; \lambda)$, $i \in \mathcal{N}$, and let f be such that* $\lim \varrho(f_i, f) = 0$. *Put* $F_i(t) = \int_\tau^t f_i \,\mathrm{d}\lambda$. *If $\{F_i\}$ is equi-ACG and equicontinuous on I, then $f \in \mathscr{D}(I; \lambda)$ and* $\lim \int_I f_i \,\mathrm{d}\lambda = \int_I f \,\mathrm{d}\lambda$.

A similar theorem concerning $\mathscr{D}_*$-integration is proved in the same article, too.

A second non-trivial example of systems with the property (Vi) may now be stated.

(7,9) Theorem. *Let m_i, M_i, $i \in \mathcal{N}$, defined on I be such that*

(7.9.1) $\quad i \in \mathcal{N} \Rightarrow m_i,\ M_i \in \mathscr{D}(I; \lambda),\ m_i \leqq M_i$ *a.e. on* I

(7.9.2) *the sequences* $\{\int_\tau^t m_i \,\mathrm{d}\lambda\}$, $\{\int_\tau^t M_i \,\mathrm{d}\lambda\}$ *are equi-ACG and equicontinuous on* I.

Then the pair $\{\{m_i\}, \{M_i\}\}$ *has the property* $(Vi, \mathscr{D})$.

Proof. As it is similar to the preceding reasoning, we note that the main feature consists in the proof that $\{\int_\tau^t f_i \,\mathrm{d}\lambda\}$ is equi-ACG on I. As $\{\int_\tau^t m_i \,\mathrm{d}\lambda\}$ resp. $\{\int_\tau^t M_i \,\mathrm{d}\lambda\}$ is equi-ACG on I, there exist e_i resp. E_i, $i \in \mathcal{N}$, such that $\bigcup_{i=1}^{\infty} e_i = \bigcup_{i=1}^{\infty} E_i = I$, and that $\{\int_\tau^t m_i \,\mathrm{d}\lambda\}$ resp. $\{\int_\tau^t M_i \,\mathrm{d}\lambda\}$ es equi-AC on each e_i resp. E_i. The end of the proof is now based on the identity $(\bigcup e_i) \cap (\bigcup E_j) = \bigcup(e_i \cap E_j)$.

In this order of ideas it is appropriate to state the following local existence theorem.

(7,10) Theorem. *Let* (7.0.1) *hold*, $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(7.10.1) *for each compact* $K \subset G$, *there exists a pair* $\{\{m_i(.,K)\}, \{M_i(.,K)\}\}$ *having the property* $(Vi, \mathscr{F})$ *such that* $i \in \mathscr{N}$, $\varphi \in \mathbf{C}(I; G)$, $\varphi(I) \subset K \Rightarrow m_i \leqq \leqq \mathsf{T}_i\varphi \leqq M_i$

(7.10.2) $$x_i \in \mathbf{C}(I; G)\,, \quad i = 0, 1, 2, \dots\,, \quad \lim \|x_i - x_0\| = = 0 \Rightarrow \lim \varrho(\mathsf{T}_i x_i, \mathsf{T}_0 x_0) = 0$$

Then there exists $\beta \in (0, \alpha\rangle$ *such that* $(\mathscr{E}_0)$ *has an* $\mathscr{F}$-*solution* φ_0 *on* $\langle\tau, \tau + \beta\rangle$. *Further, if*

(7.10.3) $(\mathscr{E}_0)$ *has at most one* $\mathscr{F}$-*solution on* $\langle\tau, \tau + \beta\rangle$, *then* $\lim \varphi_i = \varphi_0$ *uniformly on* $\langle\tau, \tau + \beta\rangle$; *here* φ_i *denotes an* $\mathscr{F}$-*solution of* $(\mathscr{E}_i)$ *on* $\langle\tau, \tau + \beta\rangle$.

Proof. Choose a cube $K \subset G$ with centre ξ, and using (7.10.1) a pair $\{\{m_i\}, \{M_i\}\}$ with the property $(Vi, \mathscr{F})$, according to K. From equicontinuity of $\{\int_\tau^t m_i(.,K)\,\mathrm{d}p\}$, $\{\int_\tau^t M_i(.,K)\,\mathrm{d}p\}$, there exists $\beta \in (0, \alpha\rangle$ such that $\xi + \int_\tau^t m_i\,\mathrm{d}p$ and $\xi + \int_\tau^t M_i\,\mathrm{d}p$ lies in the interior of K for each $t \in \langle\tau, \tau + \beta\rangle$. Theorem (3,1) now yields solutions φ_i of $(\mathscr{E}_i)$ on $\langle\tau, \tau + \beta\rangle$ for each $i \in \mathscr{N}$; as $\{\varphi_i\}$ fulfil the conditions of Ascoli's theorem, there exist $i_1, i_2, \dots$ and $\psi \in \mathbf{C}(\langle\tau, \tau + \beta\rangle; G)$ such that φ_{i_k} converge to ψ uniformly on $\langle\tau, \tau + \beta\rangle$. From (7.10.1) and (7.10.2) we get that $\lim \int_\tau^t \mathsf{T}_{i_k}\varphi_{i_k}\,\mathrm{d}p = \int_\tau^t \mathsf{T}_0\psi\,\mathrm{d}p$ for each $t \in \langle\tau, \tau + \beta\rangle$ so that ψ is an $\mathscr{F}$-solution of $(\mathscr{E}_0)$ on $\langle\tau, \tau + \beta\rangle$. The remaining part of the proof is now clear.

8. LINEAR EQUATIONS

The starting point of this section is the following theorem on a characterization of linear Carathéodory operators, i.e. Carathéodory operators T on $\mathbf{C}(I)$ having the property $\mathsf{T}(\lambda x + \mu y) = \lambda \mathsf{T}x + \mu \mathsf{T}y$, for each $\lambda, \mu \in \mathscr{R}$, and $x, y \in \mathbf{C}(I)$.

(8,1) Theorem. *Let* T *be a linear Carathéodory operator on* $\mathbf{C}(I)$. *Then there exists an* $n \times n$ *matrix* A *consisting of finite measurable functions on* I *such that* $\mathsf{T}\varphi = [A\varphi]$ *for each* $\varphi \in \mathbf{C}(I)$. *Any other matrix satisfying the same relation differs from* A *at most a.e. on* I.

Proof. For each characteristic function χ_J of an interval $J \subset I$, the relation $\mathsf{T}(\chi_J\varphi) = \chi_J\mathsf{T}\varphi$ is evidently valid. Let e_i denote the vector function $(\hat{0}, \dots, \hat{1}, \dots, \hat{0})$ on I, with $\hat{1}$ in the j-th component, and zero otherwise. Let us define the j-th column of the matrix A as an arbitrary vector function $A_j \in \mathbf{S}(I; p)$ such that $[A_j] = \mathsf{T}e_j$, $j = 1, \dots, n$. Thus, $Ae_j = A_j$, and we prove that $\mathsf{T}s = [As]$ for each step function s on I. First, let $\zeta = \zeta_1 e_1 + \dots + \zeta_n e_n$, where $\zeta_j \in \mathscr{R}$ for $j = 1, \dots, n$. Then

(8.1.1) $$\mathsf{T}\zeta = \sum \zeta_j \mathsf{T}e_j = \sum \zeta_j [A_j] = \sum [\zeta_j A e_j] = \sum [A\zeta_j e_j] = [A\zeta]$$

Now let s, a step function on I, be of the form $s = \chi_{J_1}\zeta_{J_1} + \dots + \chi_{J_r}\zeta_{J_r}$, where ζ's are defined as above, and χ's denote the characteristic functions of intervals $J_1, \dots$ $\dots, J_r \subset I$. Then, by (8.1.1) and linearity,

$$(8.1.2) \qquad \mathsf{T}s = \sum \mathsf{T}\chi_{J_k}\zeta_{J_k} = \sum \chi_{J_k}[A\zeta_{J_k}] = \sum [A\chi_{J_k}\zeta_{J_k}] = As$$

Let $\varphi \in \mathbf{C}(I)$, and let step functions s_i, $i \in \mathscr{N}$, converge to φ uniformly on I. Then, in virtue of Lemma (2,6), $\lim \varrho(\mathsf{T}s_i, \mathsf{T}\varphi) = 0$. On the other hand, As_i converge to $A\varphi$ pointwise on I so that $\mathsf{T}\varphi = [A\varphi]$. From the proof we infer the "uniqueness" of A.

(8,2) Corollary. *Let* T *be a Carathéodory operator on* $\mathbf{C}(I)$ *such that the mapping* U *defined by* $\mathsf{U}\varphi = \mathsf{T}\varphi - \mathsf{T}\hat{0}$ *is linear. Then there exist an* $n \times n$ *matrix* A *consisting of finite measurable functions on* I, *and a vector function* $b \in \mathbf{S}(I; p)$ *such that* $\mathsf{T}\varphi = [A\varphi + b]$.

The preceding sections give us some theorems concerning the linear equation

$$(\mathscr{L}\mathscr{E}) \qquad x(t) = \xi + \int_\tau^t (Ax + b)\,\mathrm{d}p$$

where $\xi \in \mathscr{R}^n$ and A, b are described above. In what follows, matrices are considered as n^2-dimensional vectors.

(8,3) Theorem. *Let* $A = \|a_{jk}\|$ *be such that* $A \in \mathscr{L}(I; p)$. *Let* $\mathscr{F} \in \mathfrak{F}$ *and* $b \in \mathscr{F}(I; p)$. *Then* $(\mathscr{L}\mathscr{E})$ *has an* $\mathscr{F}$*-solution* φ *on* I, *which is strongly* $\{\mathscr{F}\}$*-unique there.*

Proof. This is a straightforward consequence of Theorem (6,2). It suffices to take $\omega = \hat{0}$, and define $\varkappa$ as $|\!|A|\!|$, where

$$(8.3.1) \qquad |\!|A|\!|(t) = \max\left(\sum_{k=1}^n |a_{1k}(t)|, \dots, \sum_{k=1}^n |a_{nk}(t)|\right)$$

We are going to show the effect of Theorems (7,2) and (7,3).

Let Π denote a metrical space of parameters with metric $|\mu - \nu|$, and let $\mu_0 \in \Pi$. In what follows we suppose that

(8.3.2) to each $\mu \in \Pi$, there corresponds an $n \times n$ matrix $A(.,\mu) \in \mathscr{L}(I; p)$, and a vector function $b(.,\mu) \in \mathbf{S}(I; p)$.

We shall consider the equations

$$(\mathscr{L}\mathscr{E}_\mu) \qquad x(t) = \xi + \int_\tau^t [A(.,\mu)\,x + b(.,\mu)]\,\mathrm{d}p$$

and prove two theorems on continuous dependence of solutions of $(\mathscr{L}\mathscr{E}_\mu)$ on μ.

(8,4) Theorem. *Let* (8.3.2) *hold,* $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(8.4.1) $$b(.,\mu) \in \mathscr{F}(I; p) \quad \textit{for each} \quad \mu \in \Pi$$

(8.4.2) *the mappings* $\mu \to A(.,\mu)$, $\mu \to b(.,\mu)$, *from* Π *to* $\mathbf{S}(I; p)$, *are continuous at* μ_0

(8.4.3) *the sequence* $\{\int_\tau^t \|A(.,\mu_i)\| \,\mathrm{d}p\}$ *is equi-AC on I, whenever* $\lim \mu_i = \mu_0$

(8.4.4) *the sequence* $\{\int_\tau^t b(.,\mu_i)\,\mathrm{d}p\}$ *is equicontinuous on I, whenever* $\lim \mu_i = \mu_0$

(8.4.5) $\lim \int_\tau^t b(.,\mu_i)\,\mathrm{d}p = \int_\tau^t b(.,\mu_0)\,\mathrm{d}p$ *for each* $t \in I$, *whenever* $\lim \mu_i = \mu_0$.

(8.4.6) *For each* $\mu \in \Pi$, *let* φ_μ *denote the* $\mathscr{F}$*-solution of* $(\mathscr{L}\mathscr{E}_\mu)$ *on I, existing in virtue of Theorem* (8,3). *Then, given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\mu \in \Pi$, $|\mu - \mu_0| < \delta \Rightarrow \|\varphi_\mu - \varphi_{\mu_0}\| < \varepsilon$.

Proof. It is clearly sufficient to prove this for any $\Pi' \subset \Pi$, $\Pi' = \{\mu_0, \mu_1, \mu_2, \ldots\}$, where $\lim \mu_i = \mu_0$, $\mu_i \neq \mu_0$, $i \in \mathscr{N}$. We show that the assumptions of Theorem (7,2) are satisfied. For simplicity of notations, we write i instead of μ_i, and similarly. For a non-empty compact $K \subset \mathscr{R}^n$, put $\|K\| = \sup\{|x|; x \in K\}$. To prove (7.2.1), it is sufficient to put $m_i = -\|A(.,i)\| \,.\, \|\vec{K}\| + b(.,i)$, $M_i = \|A(.,i)\| \,.\, \|\vec{K}\| + b(.,i)$. (7.2.2) follows from (8.4.2), and (7.2.3) follows from (8.4.3) and (8.4.4). (7.2.4) is a consequence of (8.4.2), (8.4.3) and (8.4.5), using the Lebesgue-Vitali convergence theorem and the simple fact that limas $A_i = A_0 \Rightarrow$ limas $\|A_i\| = \|A_0\|$. To prove (7.2.5), it is sufficient to note that if for finite a_i, x_i, $i = 0, 1, 2, \ldots$ defined on I we have that limas $a_i = a_0$, $\lim x_i = x_0$ pointwise, then limas $a_i x_i = a_0 x_0$. Finally, (7.2.6) is true in virtue of Theorem (8,3).

(8,5) Theorem. *Let* (8.3.2) *hold,* $\mathscr{F} \in \mathfrak{F}$, *and suppose that*

(8.5.1) $$b(.,\mu) \in \mathscr{F}(I; p) \quad \textit{for each} \quad \mu \in \Pi$$

(8.5.2) $$\sup \|A(.,\mu_i)\| \in \mathscr{L}(I; p)\,, \quad \textit{whenever} \quad \lim \mu_i = \mu_0$$

(8.5.3) $$\lim \int_\tau^t A(.,\mu_i)\,\mathrm{d}p = \int_\tau^t A(.,\mu_0)\,\mathrm{d}p \quad \textit{for each} \quad t \in I\,,$$

whenever $\lim \mu_i = \mu_0$

(8.5.4) $$\lim \int_\tau^t b(.,\mu_i)\,\mathrm{d}p = \int_\tau^t b(.,\mu_0)\,\mathrm{d}p \quad \textit{for each} \quad t \in I\,,$$

whenever $\lim \mu_i = \mu_0$.

Then (8.4.6) *is fulfilled.*

Proof. We use again the simplifications indicated in the beginning of the preceding proof. (7.3.1) follows as in the preceding theorem. To prove (7.3.2), note that $|\mathsf{T}_i\hat{x}_1 - \mathsf{T}_i\hat{x}_2| = [|A(.,i)(x_1 - x_2)|] \leqq \|A(.,i)\| \, |x_1 - x_2|$; now we put $d = 1$, $\psi(\delta) = \delta$, $\chi = \max\{\hat{1}, \sup \|A(.,i)\|\}$ so that the result follows from (8.5.2). Finally, (7.3.3) follows from (8.5.3) and (8.5.4), and (7.3.4) is true in virtue of Theorem (8,3).

(8,6) Example. It is of interest to compare the applicability of both theorems. While Theorem (8,5) is seemingly more general, we show a counter-example to this hypothesis.

Let $\langle \tau, \tau + \alpha \rangle = \langle 0, 1 \rangle$, $J_i = (2^{-i}, 2^{-i+1})$, $i \in \mathcal{N}$, and let $\{\zeta_i\}$ be a sequence of positive numbers such that

(8.6.1)
$$\sum_{i=1}^{\infty} \zeta_i 2^{-i} = \infty$$

(8.6.2)
$$\lim \zeta_i 2^{-i} = 0$$

Let χ_{J_i} denote the characteristic function of J_i, and put $a_i = \hat{\zeta}_i \chi_{J_i}$, $i \in \mathcal{N}$. Put further $a_0 = \hat{0}$. Let us consider the equations

(8.6.3)
$$x(t) = 1 + \int_0^t a_i x \, \mathrm{d}\lambda \,, \quad i = 0, 1, 2, \ldots$$

According to Theorem (8,3), the function $\varphi_i(t) = \exp \int_0^t a_i \, \mathrm{d}\lambda$, $i = 0, 1, 2, \ldots$ is the strongly $\{\mathcal{L}\}$-unique solution of the i-th equation in (8.6.3). Here evidently $\lim \|\varphi_i - \varphi_0\| = 0$. This phenomenon can be discovered without knowledge of the explicit formulae giving the solutions, when we use Theorem (8,4). Indeed, (8.4.1), (8.4.4) and (8.4.5) are fulfilled trivially, (8.4.2) follows from pointwise convergence of a_i to a_0, and (8.4.3) holds in virtue of (8.4.2) and (8.6.2), as it follows from the Lebesgue-Vitali theorem.

However, Theorem (8,5) is non-applicable here, for in view of (8.6.1), the assumption (8.5.2) is not fulfilled.

(8,7) Remark. The theory described in this paper, although seemingly general, has a rather limited use. Indeed, existence of minorants and majorants $m(., K)$ and $M(., K)$ plays clearly a dominant rôle in the results concerning existence of solutions and continuous dependence on a parameter. Let, for example, $[t, x] \to f(t, x)$, $[t, x] \in I \times G$ be such that (2.4.1) and (2.4.2) are fulfilled, and that there exist $\mathcal{F} \in \mathfrak{F}$ and $m, M \in \mathcal{F}(I; p)$ such that

(8.7.1)
$$[t, x] \in I \times G \Rightarrow m(t) \leqq f(t, x) \leqq M(t)$$

Then $0 \leqq f(t, x) - m(t) \leqq M(t) - m(t)$. Thus, there exists g satisfying the usual Carathéodory conditions such that $f(t, x) = m(t) + g(t, x)$; for $M - m \in \mathcal{L}(I; p)$, as a result of (8.7.1).

Let us show that this theory is of no use for the simple linear equation

(8.7.2)
$$x(t) = \xi + \int_\tau^t Ax \, \mathrm{d}\lambda \,,$$

where $A \in \mathcal{D}_*(I; \lambda) - \mathcal{L}(I; \lambda)$, $n = 1$.

Let $\beta > 0$, and put $\xi_1 = \xi - \beta$, $\xi_2 = \xi + \beta$. We prove that there exist no minorants and majorants, corresponding to $K = \langle \xi_1, \xi_2 \rangle$. We put $P = \{t \in I; A(t) \geqq 0\}$, $N = \{t \in I; A(t) < 0\}$. Then we have for each $x \in \langle \xi_1, \xi_2 \rangle$

$$(8.7.3) \qquad t \in P \Rightarrow A(t)\,\xi_1 \leqq A(t)\,x \leqq A(t)\,\xi_2$$

$$t \in N \Rightarrow A(t)\,\xi_2 \leqq A(t)\,x \leqq A(t)\,\xi_1$$

Hence, the minimal function $M \mid I$ satisfying the condition $[t, x] \in I \times \langle \xi_1, \xi_2 \rangle \Rightarrow$ $\Rightarrow M(t) \geqq A(t)\,x$, is given by $M(t) = A(t)\,\xi_2$ for $t \in P$, $M(t) = A(t)\,\xi_1$ for $t \in N$. However, $M \notin \mathscr{F}(I; \lambda)$ for any $\mathscr{F} \in \mathfrak{F}$ whatsoever, as it is easy to see, and all the more no $M_1 \geqq M$ is $\mathscr{F}$-integrable.

This example also shows that existence of minorants and majorants is unnecessary for a solution to exist. Indeed, the function φ defined by $\varphi(t) = \xi \exp \int_\tau^t A \,\mathrm{d}\lambda$ is a $\mathscr{D}_*$-solution of (8.7.2) on I, as a simple consequence of Lemma (1,14). Also, it is of interest that Theorem (5,1) on uniqueness is of no use here; for if $|A(t)\,x - A(t)\,y| \leqq$ $\leqq \psi(t, |x - y|)$ a.e. on I, then we see from (5.1.3) that $\int_\gamma^{\tau+\alpha} |A| \,\mathrm{d}\lambda$ converges for each $\gamma \in (\tau, \tau + \alpha\rangle$.

9. APPLICATIONS TO DIFFERENTIAL EQUATIONS

The results of the preceding sections give some new theorems in the theory of differential equations; we only intend to state here a result on continuous dependence on a parameter, generalizing thus theorem 4 of [8].

Under the $[t, x]$-space we mean $\mathscr{R}^{n+1}$, the points of which will be denoted $[t, x_1, \ldots, x_n]$, or shortly $[t, x]$. Let D be a non-empty region in $[t, x]$-space; for $x \in \mathscr{R}^n$, let $D^{[\cdot, x]} = \{t \in \mathscr{R}; [t, x] \in D\}$, and similarly for $D^{[t, \cdot]}$. Further, let $\mathrm{proj}_t\, D =$ $= \bigcup \{D^{[\cdot, x]};\ x \in \mathscr{R}^n\}$, and similarly for $\mathrm{proj}_x\, D$. In what follows, measurability notions refer to λ.

Let there be given a mapping f from D to $\mathscr{R}^n$. We say that a continuous vector function $\varphi \mid J$, where J is an interval in $\mathrm{proj}_t\, D$ such that $J^0 \neq \emptyset$, is a solution of the differential equation

$$(\mathscr{D}\mathscr{E}) \qquad x' = f(t, x)$$

iff there exists $\mathscr{F} \in \mathfrak{F}$ such that for each closed interval $I' \subset J$ and each region $G' \subset \mathscr{R}^n$ such that $I' \times G' \subset D$, the mapping $\psi \to [(f \mid I' \times G') \circ \psi]$, $\psi \in \mathbf{C}(I'; G')$, is a Carathéodory operator on $\mathbf{C}(I'; G')$, and $\varphi \mid I'$ is an $\mathscr{F}$-solution of $x(t) =$ $= \varphi(\tau') + \int_{\tau'}^t (f \mid I' \times G') \circ x \,\mathrm{d}\lambda$ on I', for a $\tau' \in I'$. The function $\varphi \mid J$ is then called an $\mathscr{F}$-solution of $(\mathscr{D}\mathscr{E})$.

As in section 8, Π denotes a metrical space of parameters with metric $|\mu - \nu|$. Let $\mu_0 \in \Pi$, and for any $\vartheta > 0$ put $\Pi(\vartheta) = \{\mu \in \Pi; |\mu - \mu_0| < \vartheta\}$.

(9,1) Theorem. *Let D be a region of $[t, x]$-space, $\langle a, b\rangle \subset \operatorname{proj}_t D$, $N \subset \langle a, b\rangle$, $\lambda(N) = 0$, $\tau \in \langle a, b\rangle$. Let ξ be a mapping from Π to $\operatorname{proj}_x D$ continuous at μ_0; put $\xi_0 = \xi(\mu_0)$ and suppose that $[\tau, \xi_0] \in D$. Let $[t, x, \mu] \to f(t, x, \mu)$ be a function on $D \times \Pi$ with values in $\mathscr{R}^n$ such that*

(9.1.1) *for each $x \in \operatorname{proj}_x D$ and each $\mu \in \Pi$, $f(., x, \mu)$ is measurable on $D^{[., x]}$*

(9.1.2) *for each $t \in \operatorname{proj}_t D - N$ and each $\mu \in \Pi$, $f(t, ., \mu)$ is continuous on $D^{[t, .]}$*

(9.1.3) *for each $t \in \langle a, b\rangle - N$ and each $x \in G$, the function $f(t, ., .)$ is continuous at $[x, \mu_0]$.*

Further, let $\mathscr{F} \in \bar{\mathfrak{F}}$ and suppose that

(9.1.4) *the equation*

$$(\mathscr{E}_{\mu_0}) \qquad x' = f(t, x, \mu_0), \quad x(\tau) = \xi_0$$

has an $\mathscr{F}$-solution $\varphi_0 \mid \langle a, b\rangle$; this solution is unique in the sense that if $\psi \mid \langle c, d\rangle$, where $\langle c, d\rangle \subset \langle a, b\rangle$, is an $\mathscr{F}$-solution of $x' = f(t, x, \mu_0)$ passing through a point of the graph of φ_0, then $\varphi_0 \mid \langle c, d\rangle = \psi$

(9.1.5) *for each $\mu \in \Pi$ and each compact $K \subset \operatorname{proj}_x D$, there exist functions $m(., \mu; K)$, $M(., \mu; K) \in \mathscr{F}(\langle a, b\rangle; \lambda)$ such that $t \in \langle a, b\rangle$, $x \in K$, $[t, x, \mu] \in D \times \Pi \Rightarrow m(t, \mu; K) \leqq f(t, x, \mu) \leqq M(t, \mu; K)$*

(9.1.6) limas $m(., \mu_i; K) = m(., \mu_0; K)$, limas $M(., \mu_i; K) = M(., \mu_0; K)$, *whenever* $\lim \mu_i = \mu_0$

(9.1.7) *the sequences $\{L(., \mu_i; K)\}$, $\{U(., \mu_i; K)\}$, (where e.g. $L(t, \mu; K) = \int_a^t m(., \mu; K)\, d\lambda$), are equicontinuous on $\langle a, b\rangle$, whenever $\lim \mu_i = \mu_0$*

(9.1.8) *for each $t \in \langle a, b\rangle$, $L(t, .; K)$, $U(t, .; K)$ are continuous at μ_0.*

Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $\mu \in \Pi(\delta)$, every $\mathscr{F}$-solution φ_μ of

$$(\mathscr{D}\mathscr{E}_\mu) \qquad x' = f(t, x, \mu), \quad x(\tau) = \xi(\mu)$$

exists over $\langle a, b\rangle$, and $|\varphi_\mu(t) - \varphi_0(t)| < \varepsilon$ for each $t \in \langle a, b\rangle$.

Proof. Let $\varepsilon > 0$ be such that $G = \{[t, x] \in \mathscr{R}^{n+1}$; *the distance of $[t, x]$ from the graph of φ_0 is* $< 3\varepsilon\}$ lies in D. Let η be such that $0 < \eta \leqq \varepsilon$ and $\sup\{|\varphi_0(t) - \varphi_0(t')|;\ t, t' \in \langle a, b\rangle,\ |t - t'| \leqq \eta\} < \varepsilon$. Suppose for simplicity $\tau = a$, and let $\tau = a < \tau_1 < \tau_2 < \ldots < \tau_r = b$, $\max\{\tau_j - \tau_{j-1};\ j = 1, \ldots, r\} < \eta$. Put $Q_j = \langle \tau_{j-1}, \tau_j\rangle \times (\varphi_0(\tau_i) - 2\vec{\varepsilon},\ \varphi_0(\tau_i) + 2\vec{\varepsilon})$, the meaning of the last symbol being obvious. Then $\bar{Q}_j \subset G$ for $j = 1, \ldots, r$, as is easy to see.

It follows from Theorem (7,2) that we have $\vartheta_r > 0$ and $\varepsilon_r > 0$ such that $\mu \in \Pi(\vartheta_r)$, $|\xi_r - \varphi_0(\tau_{r-1})| < \varepsilon_r$ implies: every $\mathscr{F}$-solution φ of $x' = f(t, x, \mu)$, $x(\tau_{r-1}) = \xi_r$, exists over $\langle \tau_{r-1}, \tau_r\rangle$, and $|\varphi(t) - \varphi_0(t)| < \varepsilon$ holds there; the only point here is to take into consideration the variability of ξ, and this is proved using a simple device of introducing the function $f(t, x - \xi, \mu)$. Similarly, there exist $\vartheta_{r-1} \leqq \vartheta_r$ and $\varepsilon_{r-1} \leqq \varepsilon_r$ such that $\mu \in \Pi(\vartheta_{r-1})$, $|\xi_{r-1} - \varphi_0(\tau_{r-2})| < \varepsilon_{r-1}$ implies that every solution φ of $x' = f(t, x, \mu)$, $x(\tau_{r-2}) = \xi_{r-1}$ exists over $\langle \tau_{r-2}, \tau_r \rangle$, and $|\varphi(t) - \varphi_0(t)| < \varepsilon$ there; etc. Finally, there exist ϑ_1 and ε_1 such that supposing $\mu \in \Pi(\vartheta_1)$ and $|\xi_1 - \varphi_0(\tau)| < \varepsilon_1$, it follows that every solution φ of $x' = f(t, x, \mu)$, $x(\tau) = \xi_1$ exists over $\langle \tau, \tau_r \rangle = \langle a, b \rangle$, and $|\varphi(t) - \varphi_0(t)| < \varepsilon$ for each $t \in \langle a, b \rangle$, whence the conclusion. The case $a \neq \tau$ may be dealt with in a similar way.

10. REMARKS AND PROBLEMS

Here, we include some examples relevant to the previous definitions, and collect some problems.

(10,1) In (1,2), classical examples of $\mathscr{F}$-integrations have been given. As a recent contribution, the $\mathscr{Q}$-integration defined by Ka. Iseki in [6] should be mentioned. For an example of $*$-integration having approximately continuous antiderivatives, see the approximately continuous Perron integral of J. C. Burkill [2]; that paper was the starting point to many other researches in this direction.

Let us still show quite a simple example of an $*$-integration, called here $\mathscr{M}_s^2$, such that $\mathscr{M}_s^2$-antiderivatives needn't even be approximately continuous.

(10,1,1) Let $\langle a, b \rangle \subset \mathscr{R}$, and let $\langle c, d \rangle \subset \langle a, b \rangle$. We say that $f \in \mathscr{M}_s^2(c, d; \lambda)$ iff

(10.1.1.1) f is $\mathscr{M}^2$-integrable over $\langle c, d \rangle$ (see [14], Vol. II, Chapter 11)

(10.1.1.2) the second indefinite integral F of f has a finite derivative at each point $t \in \langle c, d \rangle$.

Then we put $(\mathscr{M}_s^2) \int_c^d f \,\mathrm{d}\lambda = F'(d) - F'(c)$. Using known properties of $\mathscr{M}^2$-integration we see immediately that $\mathscr{M}_s^2$ is an $*$-integration, which is more general than that of Perron. To show that the corresponding antiderivatives needn't be approximately continuous, it is sufficient to put $a = -1$, $b = 1$, $F(x) = x^2 \sin(1/x)$, $F(0) = 0$, $f = F''$ for $x \neq 0$, $f(0) = 0$.

(10,1,2) Let us show that there exist at least two maximal elements of the set $(\mathfrak{F}, \subset)$. To this end we define $\mathscr{A}_c \in \mathfrak{F}$ as follows.

First, recall that $f \in \mathbf{S}(a, b; \lambda)$ is $\mathscr{A}$-integrable on $\langle a, b \rangle$ iff

($A1$) f is measurable on $\langle a, b \rangle$ and $\lim i\lambda\{t \in \langle a, b \rangle; |f(t)| > i\} = 0$

($A2$) $\lim (\mathscr{L}) \int_a^b [f]^i \,\mathrm{d}\lambda$, where $[f]^i(t) = f(t)$ resp. 0 according to $|f(t)| \leqq i$ resp. $|f(t)| > i$, exists. This last limit is called the $\mathscr{A}$-integral of f over $\langle a, b \rangle$, and is denoted by $(\mathscr{A}) \int_a^b f \,\mathrm{d}\lambda$.

As already mentioned in (1,2), $\mathscr{A} \notin \mathfrak{F}^*$; for, as it is known from [12], if f is $\mathscr{A}$-integrable on $\langle a, b\rangle$, then f needn't be $\mathscr{A}$-integrable on each $\langle c, d\rangle \subset \langle a, b\rangle$.

We say that $f \in \mathscr{A}_c(a, b; \lambda)$ iff

(10.1.2.1) for each $t \in (a, b\rangle$, f is $\mathscr{A}$-integrable on $\langle a, t\rangle$

(10.1.2.2) $(\mathscr{A}) \int_a^t f \, d\lambda$ is continuous on $\langle a, b\rangle$.

Using known properties of $\mathscr{A}$-integration, it is easy to prove that $\mathscr{A}_c \in \mathfrak{F}$. Now, existence of two maximal elements of $(\mathfrak{F}, \subset)$ readily follows, when we use a result of I. A. Vinogradova [13], p. 136: There exists a function f on $\langle 0, 1\rangle$ such that f is both $\mathscr{A}_c$- and $\mathscr{D}_*$-integrable, but $(\mathscr{A}_c) \int_0^1 f \, d\lambda \neq (\mathscr{D}_*) \int_0^1 f \, d\lambda$.

This example also shows that there exist more than one integration corresponding to the class $\mathscr{A}_c \cap \mathscr{D}_*$ and the Lebesgue measure λ. It would then be preferrable to write instead of "$\mathscr{F}$-integration" e.g. "$(\mathscr{F}, ?)$-integration", with ? standing for the function $f \to (?) \int f$, $f \in \mathscr{F}$. However, following the usage, we use only one letter to distinguish integrations.

(10,1,3) It is of interest to inquire into the possibility of refinding f supposing an $\mathscr{F}$-antiderivative of f is given. More precisely: Given an integration $\mathscr{F} \in \mathfrak{F}$, does there exists a filter $\mathfrak{A}$ converging to 0 such that, for each $f \in \mathscr{F}(I; \lambda)$, $\lim_{\mathfrak{A}} h^{-1}(\mathscr{F}) \,.\, \int_t^{t+h} f \, d\lambda = f(t)$ a.e. on I? We show that the answer to this question is "no". To this end, it is sufficient to construct an integration $\dot{\mathscr{F}}(I; \lambda)$ such that there exists a function $g \in \dot{\mathscr{F}}$ such that $(\dot{\mathscr{F}}) \int_J g \, d\lambda = 0$ for each subinterval J of I, without $f = \hat{0}$ a.e. on I.

Let g be a finite λ-measurable function on I such that $\int_J \max(g, 0) \, d\lambda = -\int_J \min(g, 0) \, d\lambda = \infty$ for each interval $J \subset I$. Let $\langle c, d\rangle \subset I$, and let s be any step function on $\langle c, d\rangle$. Now define $\dot{\mathscr{F}}(c, d; \lambda)$ as the set of all functions of the form $f + s \,.\, g \mid \langle c, d\rangle$, with $f \in \mathscr{L}(c, d; \lambda)$, and put $(\dot{\mathscr{F}}) \int_c^d (f + sg) \, d\lambda = (\mathscr{L}) \int_c^d f \, d\lambda$. It is easy to see that the integration $\dot{\mathscr{F}} \in \mathfrak{F}$ solves our problem.

Let us still add two problems on integrations.

Problem A. Give a direct definition of a maximal integral in $(\mathfrak{F}, \subset)$.

Problem B. Does there exist $\mathscr{F} \in \mathfrak{F}$, $f \in \mathscr{F}(I; \lambda)$ and φ which is AC on I such that $f\varphi \notin \mathscr{F}(I; \lambda)$?

(10,2) Passing to Carathéodory operators, it is perhaps of interest to note that the set of all Carathéodory operators on $\mathbf{C}(I; G)$ may be given the structure of a complete linear metric space. Here, the metric dist may be defined as follows: $\operatorname{dist}(\mathsf{T}_1, \mathsf{T}_2) = \sup \{\varrho(\mathsf{T}_1\varphi, \mathsf{T}_2\varphi);\ \varphi \in \mathbf{C}(I; G)\}$. Of course, the fundamental question is the following one on representation:

Problem C. Is each Carathéodory operator of the classical type?

(10,2,1) The notion of a solution of $(\mathscr{E})$ was based here on the concept of integration. If we restrict ourselves to classical Carathéodory operators, we may introduce the following kind of a "formal" solution of $(\mathscr{E})$:

Problem D. Let f fulfil the assumptions of Theorem (2,4). Does there exist a continuous vector function $\varphi \mid \langle\tau, \tau + \delta\rangle$, $0 < \delta \leqq \alpha$, such that $\varphi'(t) = f(t, \varphi(t))$ a.e. on $\langle\tau, \tau + \delta\rangle$? (This problem was also published in Časopis pro pěstování matematiky, 91 (1966), p. 104.)

(10,2,2) In connection with the notion of the strong uniqueness of solutions, it is useful to note that if we put $\tau = 0$, $\alpha = \pi$, $G = \mathscr{R}$, $f(t, x) = \sin^{-2} tx^2$ for $[t, x] \in (0, \pi) \times \mathscr{R}$ and $f(0, x)$, $f(\pi, x)$ define in an arbitrary way, then $0 \mid \langle 0, \pi\rangle$ is the $\mathscr{L}$-unique solution of $x(t) = \int_0^t \sin^{-2} s\, x(s)\, d\lambda$ on $\langle 0, \pi\rangle$, which is not strongly $\mathscr{L}$-unique there.

We have the following question on uniqueness:

Problem E. Does there exist a Carathéodory operator T such that a corresponding equation $(\mathscr{E})$ has an $\mathscr{L}$-unique solution on $\langle\tau, \tau + \alpha\rangle$, which is not $\mathscr{F}$-unique there for some $\mathscr{F} \supset \mathscr{L}$?

(10,2,3) Example. In connection with the previous problem, it may be of interest to investigate a particular equation.

Put $A(0) = 0$, $A(t) = t^2 \sin t^{-6}$ for $t \in (0, 1\rangle$. Then $a(t) = A'(t) \in \mathscr{R}$ exists for each $t \in \langle 0, 1\rangle$; hence A is ACG_* on $\langle 0, 1\rangle$ [11]. Moreover, A^3 is ACG_* on $\langle 0, 1\rangle$ without being AC there. The first assertion is a consequence of Lemma (1,14); to prove that A^3 is not AC on $\langle 0, 1\rangle$, put $t_i = 2^{1/6}[(4i + 1)\pi]^{-1/6}$, $i \in \mathscr{N}$. There is a first point t_i' to the left of t_i such that $A(t_i') = 0$. Now we have $\sum_{i=1}^{\infty} (A(t_i) - A(t_i')) = (2/\pi) \sum_{i=1}^{\infty} (4i + 1)^{-1} = \infty$, whence easily the assertion.

Let us consider the equation

$$x(t) = \int_0^t a(s)\, x^{2/3}(s)\, d\lambda \tag{10.2.3.1}$$

It is clear that $\hat{0} \mid \langle 0, 1\rangle$ is an $\mathscr{L}$-solution of (10.2.3.1). On the other hand, $3^{-3}A^3$ is a $\mathscr{P}$-solution of this equation on $\langle 0, 1\rangle$, as we easily prove using differentiation; however, this is not an $\mathscr{L}$-solution, as we proved above. Further, it is clear that the functions $3^{-3}A^3\chi_E$, where χ_E is the characteristic function of a suitable subset $E \subset \langle 0, 1\rangle$, give new $\mathscr{P}$-solutions of (10.2.3.1); some of them are even $\mathscr{L}$-solutions. The question of other $\mathscr{F}$-solutions of (10.2.3.1) remains open.

(10,3) Another proof of Theorem (3,1) can be given using Schauder fixed-point theorem. Let $\mathbf{C}_1 = \{\varphi \in \mathbf{C}(I; G);\ t \in I \Rightarrow \{L(t) \leqq \varphi(t) - \xi \leqq U(t)\}$. Then $\mathbf{C}_1$ is a closed convex subset of $\mathbf{C}(I)$. Define $\tilde{\mathrm{T}} \mid \mathbf{C}_1$ as follows: $(\tilde{\mathrm{T}}\varphi)(t) = \xi + \int_\tau^t \mathrm{T}\varphi \,\mathrm{d}p$. Then $\tilde{\mathrm{T}}$ is a continuous map from $\mathbf{C}_1$ into itself such that $\tilde{\mathrm{T}}\mathbf{C}_1$ is relatively compact.

Problem F. It would be of interest to state an existence theorem modelled on Theorem (3,1), for $*$-integrals with approximately continuous antiderivatives. Here, a substitute for Ascoli's theorem is needed.

(10,4) Using Theorem (4,2) we are able to complete Theorem (3,1) as follows. Suppose that the $\mathscr{F}$-solution φ of $(\mathscr{E})$ obtained there is $\mathscr{F}$-unique on I. Then it is strongly $\mathscr{F}$-unique there. Otherwise there would exist an $\mathscr{F}$-solution of $(\mathscr{E})$ on $\langle\tau, \tau + \beta\rangle$, $\beta \in (0, \alpha)$, not of the form $\varphi \mid \langle\tau, \tau + \beta\rangle$. However, in view of (3.1.2), (3.1.3) and Theorem (4,2), this solution may be extended over I to yield an $\mathscr{F}$-solution ψ_1 on I different from φ. In the same way, $\{\mathscr{F}\}$-uniqueness of φ implies its strong $\{\mathscr{F}\}$-uniqueness.

(10,5) A new feature in Theorem (5,1), which itself is a modification of a known result of Kamke, consists in the fact that there, each $\mathscr{F}$-solution is even $\{\mathscr{F}\}$-unique.

Let us still remark that Theorem (5,2) may be generalized as follows.

(10,5,1) Theorem. Let $\chi \in \mathscr{L}(I; p)$, χ finite nonnegative. Let Θ be a positive non-decreasing function on $(0, \infty)$ such that

$$\int_0^1 \Theta^{-1}\,\mathrm{d}\lambda = \infty \tag{10.5.1.1}$$

and let $\Theta(0) = 0$.

Then the function $[t, r] \to \chi(t)\,\Theta(r)$, $[t, r] \in I \times \langle 0, \infty)$, satisfies the assumptions (5.1.1) to (5.1.4) of Theorem (5,1).

Proof. To prove (5.1.4), let ϱ be a non-zero $\mathscr{L}$-solution of

$$\varrho(t) = \int_\tau^t \chi(s)\,\Theta(\varrho(s))\,\mathrm{d}p \tag{10.5.1.2}$$

Hence there exists $(a, b\rangle \subset I$ such that $\varrho(t) > 0$ for each $t \in (a, b\rangle$, $\varrho(a) = 0$. In view of (10.5.1.1), there exist $\tau_0 = 0 < \tau_1 < \ldots < \tau_k = \varrho(b)$ such that

$$\sum_{j=1}^{k} (\tau_j - \tau_{j-1})\,\Theta^{-1}(\tau_j) > \int_I \chi\,\mathrm{d}p \tag{10.5.1.3}$$

Let $t_j \in \langle a, b\rangle$ be such that $\varrho(t_j) = \tau_j$, $j = 0, 1, \ldots, k$; then $\varrho(t_j) - \varrho(t_{j-1}) = \int_{t_{j-1}}^{t_j} \chi(s)\,\Theta(\varrho(s))\,\mathrm{d}p \leqq \Theta(\varrho(t_j)) \int_{t_{j-1}}^{t_j} \chi\,\mathrm{d}p$.

Hence $\sum_{j=1}^{K}(\tau_j - \tau_{j-1})\,\Theta^{-1}(\tau_j) \leqq \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\chi\,\mathrm{d}p \leqq \int_I \chi\,\mathrm{d}p$, in contradiction with (10.5.1.3).

(10,6) Theorem (6,2) is a generalized result of Exercise 1, Chapter III of [4], which gives a global existence theorem for linear differential equations with $\mathscr{L}$-integrable coefficients.

(10,7) Besides general theorems on limit passages in integration, this section was influenced by [10]; thus, Theorem (7,3) is a direct generalization of theorem 2 of that article. The motivation for applying compacts K there lies in effort to use these theorems in the theory of linear differential equations.

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