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Milan Kočandrle

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DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS IN AFFINE SPACES WITH TENSOR STRUCTURE

MILAN KOČANDRLE, Praha

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1. INTRODUCTION

If V_n is a differentiable manifold of class C^ν, we may identify the tangent vector X_p at an arbitrary point p of V_n with a real valued operator (denoted by the same letter X_p) on the set of differentiable functions on V_n, the value of X_p at the function f being the number

$$X_p f = x^i \left(\frac{\partial f}{\partial u^i}\right)_p ;$$

here, x^i are the coordinates of the vector X_p and $u^1, \ldots, u^n$ are the local coordinates on V_n.

In what follows, the manifolds and mappings are of finite class C^ν. We do not mention this class explicitly, and we only suppose that it is always sufficiently high.

Throughout the paper, we use (without explicit citation) the fundamental facts from the theory of fibre bundles and connections in the sense of [1] and [2].

2. PROLONGATIONS OF A DIFFERENTIABLE MANIFOLD

Definition. a) The 1-vector at the point of p is a *tangent vector* at p, a differentiable field of 1-vectors on V_n is a *differentiable tangent vector field.*

b) The *k-vector* at the point p is an operator $X^p_{(k)}$ which may be written as a finite linear combination of operators of the type $X^{(1)}_p X^{(k-1)}$, $X^{(1)}_p$ being a 1-vector at p and $X^{(k-1)}$ being a differentiable field of $(k-1)$-vectors. The *differentiable field of k-vectors* is a rule associating to each point $p \in V_n$ a k-vector $X^{(k)}_p$ at p in such a way that the function $g(p) = X^{(k)}_p f$ is differentiable for each differentiable function f.

Theorem 1. *The k-vectors at the point $p \in V_n$ form a vector space with the basis*

$$(1) \quad \left(\frac{\partial^l}{(\partial u^1)^{i_1} \ldots (\partial u^n)^{i_n}}\right)_p , \quad i_1, \ldots, i_n \geqq 0 ; \quad i_1 + \ldots + i_n = l ; \quad l = 1, \ldots, k ,$$

where

$$\left(\frac{\partial^l}{(\partial u^1)^{i_1} \dots (\partial u^n)^{i_n}}\right)_p f = \left(\frac{\partial^l f}{(\partial u^1)^{i_1} \dots (\partial u^n)^{i_n}}\right)_p,$$

the expressions $(\partial u^j)^{i_j}$ *for* $i_j = 0$ *being omitted.*

Definition. The vector space of all k-vectors at p is called the *k-tangent space* and denoted by $T_p^{(k)}$.

Remark. The symbols $X_p^{(1)}$, X_p and $T_p^{(1)}$, T_p are thus equivalent.

Let us write $T^{(k)} = \bigcup_{p \in V} T_p^{(k)}$. Using Theorem 1, we can easily see that we may introduce local coordinates into the set $T^{(k)}$, and we can make it into a differentiable manifold.

Definition. The differentiable manifold $T^{(k)}$ is called the k-th *prolongation* of the manifold V_n.

Remark. If necessary, we shall write $T_p^{(k)}(V_n)$ and $T^{(k)}(V_n)$ instead of $T_p^{(k)}$ and $T^{(k)}$ resp.

Theorem 2. *Let* $\varphi : V_n \to V'_m$ *be a differentiable mapping.* $X_p^{(k)}$ *being a k-vector at the point* $p \in V_n$, *the operator* $X'^{(k)}_{\varphi(p)} = \varphi_p^{(k)} X_p^{(k)}$ *defined by*

$$X'^{(k)}_{\varphi(p)} f = X_p^{(k)}(f' \circ \varphi)$$

is a k-vector at the point $\varphi(p) \in V'_m$.

Proof. Using the local coordinates, the proof is evident.

Definition. The k-vector $X'^{(k)}_{\varphi(p)}$ is called the *image* of the k-vector $X_p^{(k)}$. The mapping $\varphi^{(k)}$ of the differentiable fields of k-vectors on V_n be defined by

$$(\varphi^{(k)} X^{(k)})_{\varphi(p)} = \varphi_p^{(k)} X_p^{(k)},$$

$X^{(k)}$ being an arbitrary differentiable field of k-vectors on V_n. The mappings $\varphi_p^{(k)}$ and $\varphi^{(k)}$ are the k-th *differentials* of φ; we shall write φ' instead of $\varphi^{(1)}$.

Lemma. *Let* V_n *be a differentiable submanifold of the manifold* V'_m, *and let us associate to each* $p \in V_n$ *a k-vector* $X_p^{(k)} \in T_p^{(k)}(V'_m)$ *in such a way that this mapping is differentiable. To each point* $q \in V_n$ *there exists a neighborhood* $U \subset V_n$ *of* q *and a differentiable field of k-vectors* $X'^{(k)}$ *on* V'_m *in such a way that we have* $X_p^{(k)} = X'^{(k)}_p$ *for each point* $p \in U$.

Proof. There is a neighborhood $U' \subset V'_m$ of the point q with local coordinates $u^1, \ldots, u^m$ such that V_n is given, in U', by $u^i = 0$ for $i = n + 1, \ldots, m$. The lemma now follows easily if we express the vectors $X^{(k)}_p$ by means of the bases introduced in Theorem 1.

Theorem 3. *Let V_n be a differentiable submanifold of the manifold V'_m, $X^{(k)}$ a differentiable field of k-vectors on V_n, $X_p \in T_p(V_n)$ and φ the injection of V_n into V'_m. We have*

$$(\varphi'_p X_p)\,(\varphi^{(k)} X^{(k)}) = \varphi^{(k+1)}_p (X_p X^{(k)}) \,. \tag{2}$$

where the left hand side means the application of the operator $\varphi'_p X_p$ to an arbitrary differentiable field of k-vectors on V'_m which coincides with $\varphi^{(k)} X^{(k)}$ on some neighborhood $U \subset V_n$ of the point p.

Proof. This is almost evident. Indeed, apply both sides of (2) to an arbitrary function f on V'_m, and observe that $f \circ \varphi$ is a restriction of f to V_n.

Lemma. *$X^{(k)}$ being a differentiable field of k-vectors on V_n such that $X^{(k)}_p = 0$ and $Y_p \in T_p(V_n)$, the $(k + 1)$-vector $Y_p X^{(k)}$ is a k-vector.*

Using this lemma, it is easy to prove the following

Theorem 4. *Let $X^{(k)}_p \in T^{(k)}_p(V_n)$, $Y_p \in T_p(V_n)$, V_n being a differentiable manifold, and let us choose an arbitrary differentiable field $X^{(k)}$ of k-vectors on V_n, its value at p being just the prescribed vector $X^{(k)}_p$. Then the mapping $(Y_p, X^{(k)}_p) \to Y_p X^{(k)}$ is a bilinear mapping of $T_p(V_n) \times T^{(k)}_p(V_n)$ into $T^{(k+1)}_p(V_n)/T^{(k)}_p(V_n)$.*

Theorem 5. *Let p be an arbitrary point of the differentiable manifold V_n. The assignment*

$$({}_1X_p, \ldots, {}_kX_p) \to ({}_1X, \ldots, {}_kX)_p \,, \tag{3}$$

${}_1X, \ldots, {}_kX$ being arbitrary differentiable vector fields on V_n with values ${}_iX_p$; $i = 1, \ldots, k$; at p, defines a symmetric mapping of $\times^k T_p(V_n) = T_p(V_n) \times \ldots \times T_p(V_n)$ (k-times) into $T^{(k)}_p(V_n)/T^{(k-1)}_p(V_n)$.

Proof. It is sufficient to use Theorem 4 and the induction with respect to k. The symmetry follows from the fact that after the replacement of two neighboring vectors in (3) the expression remains unaltered up to a $(k - 1)$-vector at p.

3. TENSORS ON A VECTOR SPACE

Be given a vector space W_m. All the automorphisms of this space form a Lie group G. Choosing a basis in W_m, every such automorphism is expressed by a matrix, this giving the isomorphism between G and the full linear group $GL(m)$. In the chosen basis, to each element A of the Lie algebra $\mathfrak{g}$ of the group G there corresponds a homomorphism $h(A)$ of W_m into itself. It is easy to see that the mapping $A \to h(A)$ does not depend on the basis in W_m, and it is 1-1.

Let us now suppose that, on W_m, there is given a tensor t covariant of degree r, i.e. a multilinear form $t(x_1, \ldots, x_r)$; $x_1, \ldots, x_r \in W_m$. Denote by ${}^iS$ the set of all vectors y such that

$$t(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_r) = 0$$

for arbitrary vectors $x_i, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r \in W_m$; ${}^iS$ is called the *i-th singular space of t*. Let $S = \bigcap_{i=1}^{r} {}^iS$ be the so-called *singular space of t*. If $S = \{O\}$, t is said to be *regular*.

The set of all automorphisms $g \in G$ preserving the tensor t, i.e. the set of all $g \in G$ such that

$$t(x_1, \ldots, x_r) = t(x_1 g, \ldots, x_r g)$$

for each $x_1, \ldots, x_r \in W_m$, is obviously a group.

Theorem 6. *The set of all automorphisms of a vector space W_m preserving a tensor t is a Lie subgroup G_0 of the group G, and its Lie algebra $\mathfrak{g}_0$ consists of all elements $A \in \mathfrak{g}$ such that*

$$\sum_{i=1}^{r} t(x_1, \ldots, x_{i-1}, x_i\, h(A), x_{i+1}, \ldots, x_r) = 0$$

for arbitrary vectors $x_1, \ldots, x_r \in W_m$.

Proof. We have $t \in \otimes^r \tilde{W}_m$; denote by P the natural representation of G on $\otimes^r \tilde{W}_m$. Thus our group is the set of all elements $g \in G$ such that $P(g)\, t = t$. The rest of the proof follows from [3], Corollary 2, p. 62.

4. AFFINE SPACE WITH A TENSOR STRUCTURE

Be given an affine space A_m, denote by W_m its vector space. It is easy to see that the manifold $T(A_m)$ may be identified with $A_m \times W_m$.

In A_m, let us choose a linear coordinate system. According to Theorem 1, this coordinate system determines a basis of the space $T_p^{(k)}(A_m)$. Consider the mapping $\mu^{(k)}$ associating to each k-vector $X^{(k)} \in T_p^{(k)}(A_m)$ its projection into $T_p(A_m)$ given by the mentioned bases of $T_p^{(k)}(A_m)$. (F being a vector space and $e_1, \ldots, e_h$ its basis, the projection of F into a space $F' \subset F$, $F' = \{e_1, \ldots, e_l\}$, $l \leqq h$, given by the basis $e_1, \ldots, e_h$ maps each vector $a = \sum_{i=1}^{h} a_i e_i \in F$ into a vector $\sum_{i=1}^{l} a_i e_i \in F'$.)

Theorem 7. *The mapping $\mu^{(k)}$ of the space $T_p^{(k)}(A_m)$ into $T_p(A_m)$ does not depend on the choice of the coordinate system.*

If X is a vector field on A_m, denote by $\lambda : A_m \to W_m$ the mapping defined by $\lambda(p) = X_p$.

Theorem 8. *Using the notation just introduced, we have*

$$\mu^{(2)}(Y_q X) = \lambda'_q Y_q , \tag{4}$$

$$\mu^{(k)}(Y_q X^{(k-1)}) = \mu^{(2)}(Y_q \mu^{(k-1)} X^{(k-1)}) , \tag{5}$$

where $Y_q \in T_q(A_m)$.

Proof. The first part is obvious, the second one follows by induction.

Let us now consider that there is given a tensor t on the vector space W_m of an affine space A_m. The set of all affine transformations of A_m inducing automorphisms of W_m preserving the tensor t is a Lie group H. Choosing a fixed point $p \in A_m$, each transformation $h \in H$ has a unique decomposition consisting of a translation and a transformation preserving p (these transformations being in 1-1 correspondence with the automorphisms of W_m).

Definition. The affine space A_m with the Lie group H is called the *affine space with the tensor structure.*

In what follows, we shall consider an affine space A_m with a fixed tensor structure given by a regular tensor t.

5. MANIFOLDS IN A_m

Suppose that V_n $(n < m)$ is a manifold in A_m. The injection $\varphi : V_n \to A_m$ is differentiable and regular, and according to Theorem 2 it induces a mapping $T_p^{(k)}(V_n) \to T_p^{(k)}(A_m)$. Accoridng to 4, we have a mapping $\mu^{(k)}$ of $T^{(k)}(A_m)$ into $T(A_m)$, i.e. a mapping into W_m. The mapping $\Phi^{(k)} = \mu^{(k)} \circ \varphi^{(k)}$ is thus a mapping of $T^{(k)}(V_n)$ into W_m. Denote by $\Phi_p^{(k)}$ the homomorphism of the space $T_p^{(k)}(V_m)$ into W_m given by $\Phi^{(k)}$. For $l < k$, the mappings $\Phi^{(k)}$ and $\Phi^{(l)}$ coincide on $T^{(l)}(V_n)$.

Definition. The vector space $\Phi_p^{(k)}(T_p^{(k)}(V_n))$ is called the *k-osculating space* of the manifold V_n at the point p.

Let $u^1, \ldots, u^n$ be local coordinates in a neighborhood $U \subset V_n$ and let the injection $\varphi : V_n \to A_m$ be given (locally) by

$$x^i = x^i(u^1, \ldots, u^n) ; \quad i = 1, \ldots, m ;$$

$x^1, \ldots, x^m$ being the linear coordinates in A_m. If $Z^{(k)}$ is one of the vectors (1) it is mapped by $\Phi^{(k)}$ on a vector of W_m with the coordinates $Z^{(k)} x^i$. This shows that the osculating space has the usual geometrical interpretation.

The mapping $\Phi^{(k)}$ determines the image $\Phi^{(k)}_* t = t^{(k)}_*$ of the tensor $t(x_1, \ldots, x_r)$ on W_m defined by

$$(\Phi^{(k)}_* t)\left({}_1X^{(k)}, \ldots, {}_rX^{(k)}\right) = t\left(\Phi^{(k)}{}_1X^{(k)}, \ldots, \Phi^{(k)}{}_rX^{(k)}\right)$$

for arbitrary differentiable fields of k-vectors ${}_1X^{(k)}, \ldots, {}_rX^{(k)}$. Obviously, the tensor $t^{(k)}_*$ is differentiable on V_n.

Of course, $t^{(k)}_*$ coincides with $t^{(l)}_*$ on $T^{(l)}(V_n)$. Thus it is possible to write simply t_*, and we have only to say what is the space in consideration.

Definition. The tensor t_* is called the *fundamental tensor* of the manifold V_n.

Theorem 9. *If ${}_1X^{(k)}, \ldots, {}_rX^{(k)}$ are differentiable fields of k-vectors on V_n and X_p is a tangent vector at the point p, we have*

$$X_p t_*\left({}_1X^{(k)}, \ldots, {}_rX^{(k)}\right) =$$
$$= \sum_{i=1}^{r} t_*\left({}_1X^{(k)}_p, \ldots, {}_{i-1}X^{(k)}_p, X_{p\,i}X^{(k)}, {}_{i+1}X^{(k)}_p, \ldots, {}_rX^{(k)}_p\right).$$

Proof. According to Theorems 8 and 9, we get

$$X_p t_*\left({}_1X^{(k)}, \ldots, {}_rX^{(k)}\right) =$$
$$= (\varphi' X_p)\, t\left(\Phi^{(k)}{}_1X^{(k)}, \ldots, \Phi^{(k)}{}_rX^{(k)}\right) = \sum_{i=1}^{r} t\left(\Phi^{(k)}{}_1X^{(k)}_p, \ldots, \Phi^{(k)}{}_{i-1}X^{(k)}_p,\right.$$
$$\left.\mu^{(2)}\left((\varphi' X_p)\left(\Phi^{(k)}{}_iX^{(k)}\right)\right),\quad \Phi^{(k)}{}_{i+1}X^{(k)}_p, \ldots, \Phi^{(k)}{}_rX^{(k)}_p\right) =$$
$$= \sum_{i=1}^{r} t\left(\Phi^{(k)}{}_1X^{(k)}_p, \ldots, \Phi^{(k)}{}_{i-1}X^{(k)}_p, \Phi^{(k+1)}\left(X_p\, {}_iX^{(k)}\right),\right.$$
$$\left.\Phi^{(k)}{}_{i+1}X^{(k)}_p, \ldots, \Phi^{(k)}{}_rX^{(k)}_p\right) =$$
$$= \sum_{i=1}^{r} t_*\left({}_1X^{(k)}_p, \ldots, {}_{i-1}X^{(k)}_p, X_{p\,i}X^{(k)}, {}_{i+1}X^{(k)}_p, \ldots, {}_rX^{(k)}_p\right).$$

From now on, we shall consider only such manifolds V_n in A_m that there is a number k_0 such that the k_0-osculating space of the manifold V_n (at each point $p \in V_n$) is just the space W_m.

Let $S^{(ko)}_p$ be the singular space of the tensor t_* in the space $T^{(ko)}_p(V_n)$. Evidently, $S^{(ko)}_p$ is the kernel of $\Phi^{(ko)}_p$, and $\Phi^{(ko)}_p$ is an isomorphism of $T^{(ko)}_p(V_n)/S^{(ko)}_p$ on W_n. The spaces $T^{(ko)}_p(V_n)/S^{(ko)}_p$ as well as the spaces $S^{(ko)}_p$ have constant dimension; the dimension of $S^{(ko)}_p$ be denoted by s_0. It follows that the assignment $p \to S^{(ko)}_p$ is differentiable in the following sense: to each point $p \in V_n$ there is a neighborhood U and differentiable fields of k_0-vectors ${}_1X^{(ko)}, \ldots, {}_{so}X^{(ko)}$ such that for each $q \in U$ the k_0-vectors ${}_1X^{(ko)}_q, \ldots, {}_{so}X^{(ko)}_q$ form a basis of $S^{(ko)}_q$.

The $(k_0 + 1)$-osculating space of V_n being equal to W_m as well, all previous considerations hold for $k_0 + 1$, $k_0 + 2$, etc. We have $S_p^{(k_0)} \subset S_p^{(k_0+1)}$, and each equivalence class of the factor space $T_p^{(k_0)}(V_n)/S_p^{(k_0)}$ is a subset of exactly one equivalence class of the space $T_p^{(k_0+1)}(V_n)/S_p^{(k_0+1)}$. The just introduced mapping is an isomorphism, and we may identify the spaces $T_p^{(k_0)}(V_n)/S_p^{(k_0)}$ and $T_p^{(k_0+1)}(V_n)/S_p^{(k_0+1)}$.

In what follows, denote by $\{X_p^{(k_0)}\}$ the element of the space $T_p^{(k_0)}(V_n)/S_p^{(k_0)}$ containing the k_0-vector $X_p^{(k_0)}$.

Theorem 10. *If $Y^{(k_0)}$ is a differentiable field of k_0-vectors and $Y_q^{(k_0)} \in S_q^{(k_0)}$ for each $q \in V_n$, we have $X_p Y^{(k_0)} \in S_p^{(k_0+1)}$ for each $X_p \in T_p(V_n)$.*

Proof. For arbitrary differentiable fields of k_0-vectors ${}_jX^{(k_0)}$; $j = 1, \ldots, r$; $j \neq i$; we have

$$t_*({}_1X^{(k_0)}, \ldots, {}_{i-1}X^{(k_0)}, Y^{(k_0)}, {}_{i+1}X^{(k_0)}, \ldots, {}_rX^{(k_0)}) = 0$$

for $i = 1, \ldots, r$. Applying the operator X_p, using Theorem 9 and the fact that $Y_p^{(k_0)} \in S_p^{(k_0)} \subset S_p^{(k_0+1)}$, we get

$$t_*({}_1X_p^{(k_0)}, \ldots, {}_{i-1}X_p^{(k_0)}, X_pY^{(k_0)}, {}_{i+1}X_p^{(k_0)}, \ldots, {}_rX_p^{(k_0)}) = 0\,;$$

the spaces $T_p^{(k_0)}/S_p^{(k_0)}$ and $T_p^{(k_0+1)}/S_p^{(k_0+1)}$ being isomorphic, the theorem is thereby proved.

The following theorem is an immediate consequence of the previous one.

Theorem 11. *If $X^{(k_0)}$ and $Y^{(k_0)}$ are differentiable fields of k_0-vectors, $X_p \in T_p(V_n)$ and $\{X_q^{(k_0)}\} = \{Y_q^{(k_0)}\}$ for each point $q \in V_n$, we have $\{X_pX^{(k_0)}\} = \{X_pY^{(k_0)}\}$.*

Denote by P_q the set of all bases of the space $T_q^{(k_0)}(V_n)/S_q^{(k_0)}$. It is easy to see that the space $P = \bigcup_{q \in V_n} P_q$ has the structure of the princial fibre bundle with the base manifold V_n and the structural group $GL(m)$. The set $E = \bigcup_{q \in V_n} T_q^{(k_0)}(V_n)/S_q^{(k_0)}$ is then the fibre bundle associated to P with the standard fibre $\mathbf{R}_m$.

Define the mapping $\Phi^{(k_0)} : E \to V_n \times W_m$ by

$$\Phi^{(k_0)}\{X_p^{(k_0)}\} = (p, \Phi_p^{(k_0)} X_p^{(k_0)})\,.$$

This mapping is differentiable, 1-1, induces an identity mapping on V_n and it is an isomorphism of the fibre of E over p on W_m. The regularity of $\Phi^{(k_0)}$ follows, and there is an inverse differentiable mapping. In W_m, choose a basis ϱ^w and consider the bases ϱ_q^v such that the vectors of these bases are mapped by $\Phi_q^{(k_0)}$ on the vectors of ϱ^w; these bases ϱ_q^v form a section ϱ^v of the principal fibre bundle P, and it is easy to see that the tensor t_* has constant components (independent on q) with respect to them.

Let $\bar{G}_0 \subset GL(m)$ be the group of all matrices g such that the mapping $\varrho^w \to \varrho^w g$ determines an automorphism of W_m preserving the tensor t, $\bar{G}_0$ is a Lie group, and the just constructed section ϱ^v determines a reduction P_0 of P to the group $\bar{G}_0$.

Denoting by Q the space of all bases of the vector space W_m, the basis ϱ^w determines the reductions $A_m \times Q_0$ and $V_n \times Q_0$ of the bundles $A_m \times Q$ and $V_n \times Q$ resp. to the group $\bar{G}_0$.

The space E may be considered as the fibre bundle associated to the bundle P_0 with the standard fibre $\mathbf{R}_m$. According to the definition of the group $\bar{G}_0$, the tensor t_* has the same components with respect to all bases which are the elements of the fibre bundle P_0.

Let us choose an arbitrary local section ${}^1\varrho^v$ of the fibre bundle P_0 over a neighborhood $U \subset V_n$; let $p \in U$. If the symbol $X_p{}^1\varrho^v$ denotes the m-tuple of vectors from $T_p^{(ko)}(V_n)/S_p^{(ko)}$ which we get applying successively the operator X_p on the vectors of ${}^1\varrho^v$ (according to Theorem 11, we may suppose that the vector $X_p \in T_p$ associates to each section of the fibre bundle E a vector from $T_p^{(ko)}(V_n)/S_p^{(ko)}$, i.e. a point of the fibre over p), we may write

$$X_p{}^1\varrho^v = {}^1\varrho^v\, \omega_p^1(X_p)\,, \tag{6}$$

$\omega_p^1(X_p)$ being an $(m \times m)$-matrix.

Theorem 12. *Let ω_p^1 be defined by (6). Then ω_p^1 is a linear $\bar{\mathfrak{g}}_0$-valued form on $T_p(V_n)$.*

Proof. It is clear that ω_p^1 is a linear form taking values in the vector space of all $(m \times m)$-matrices, and therefore it is sufficient to prove that $\omega_p^1(X_p) \in \bar{\mathfrak{g}}_0$ for each vector X_p. Let $X_p^{(ko)} \in T_p^{(ko)}(V_n)$ be an arbitrary k_0-vector. Let us extend the vector $\{X_p^{(ko)}\} \in T_p^{(ko)}(V_n)/S_p^{(ko)}$ to a differentiable field $\{X^{(ko)}\}$ on some neighborhood U in such a way that $\{X_q^{(ko)}\}$ has, with respect to the basis ϱ_q^v, the same coordinates as are those of the vector $\{X_p^{(ko)}\}$ with respect to the basis ϱ_p^v. Now, define the homomorphism h_x of the space $T_p^{(ko)}(V_n)/S_p^{(ko)}$ into itself by $\{X_p^{(ko)}\}\, h_x = X_p\{X^{(lo)}\}$. Suppose that $\{{}_1X^{(ko)}\}, \ldots, \{{}_rX^{(ko)}\}$ are differentiable vector fields which are the just defined extensions of arbitrary vectors $\{{}_1X_p^{(ko)}\}, \ldots, \{{}_rX_p^{(ko)}\} \in T_p^{(ko)}(V_n)/S_p^{(ko)}$. According to the definition of these vector fields, the tensor t_* has a constant value on them, and we have

$$X_p t_*(\{{}_1X^{(ko)}\}, \ldots, \{{}_rX^{(ko)}\}) = 0\,.$$

Applying Theorems 9 and 11, we get

$$\sum_{i=1}^{r} t_*(\{{}_1X_p^{(ko)}\}, \ldots, \{{}_{i-1}X_p^{(ko)}\}, X_p\{{}_iX^{(ko)}\}\,,$$
$$\{{}_{i+1}X_p^{(ko)}\}, \ldots, \{{}_rX_p^{(ko)}\}) = 0\,,$$

and Theorem 6 shows that the homomorphism h_x defines an element $A \in \mathfrak{g}_0$. Because the matrix $\omega_p^1(X_p)$ gives a transformation of the basis ϱ_p^v by means of the homomorphism h_x, we get $\omega_p^1(X_p) \in \bar{\mathfrak{g}}_0$.

Theorem 13. *Be given a covering $\{U_\alpha\}$ of the manifold V_n and a section of the principal fibre bundle P_0 over each U_α. The forms ω^i constructed to these sections by means of* (6) *determine a connection on P_0.*

Proof. It is sufficient to prove: Let $U \subset V_n$ be a neighborhood and ${}^1\varrho^v$, ${}^2\varrho^v$ two sections over it with ${}^2\varrho^v = {}^1\varrho^v g$, $g \in \bar{G}$, then we have

$$\omega^2 = \operatorname{ad}(g^{-1})\,\omega^1 + g^{-1}\,\mathrm{d}g$$

for the associated forms ω^1 and ω^2. Applying X to both sides of ${}^2\varrho^v = {}^1\varrho^v g$, we get

$$X\,{}^2\varrho^v = X({}^1\varrho^v g)$$

and

$$\begin{aligned} X\,{}^2\varrho^v &= (X\,{}^1\varrho^v)\,g + {}^1\varrho^v(Xg)\,, \\ {}^2\varrho^v\,\omega^2(X) &= {}^1\varrho^v\,\omega^1(X)\,g + {}^1\varrho^v\,\mathrm{d}g(X)\,, \\ {}^1\varrho^v\,g\,\omega^2(X) &= {}^1\varrho^v\,\omega^1(X)\,g + {}^1\varrho^v\,\mathrm{d}g(X)\,. \end{aligned}$$

Finally, we get

$$g\,\omega^2(X) = \omega^1(X)\,g + \mathrm{d}g(X)\,,$$

this being the desired equation.

Theorem 14. *The curvature of the connection constructed in Theorem* 13 *vanishes.*

Proof. For an arbitrary differentiable field of k-vectors $Z^{(k)}$ and two differentiable vector fields X and Y, we have

$$X(YZ^{(k)}) - Y(XZ^{(k)}) = [X, Y]\,Z^{(k)}$$

for each k. Thus we have

$$X(Y\,{}^1\varrho^v) - Y(X\,{}^1\varrho^v) = [X, Y]\,{}^1\varrho^v \tag{7}$$

for an arbitrary section ${}^1\varrho^v$ over some neighborhood U. Let ω^1 be the form associated to the section ${}^1\varrho^v$. From (7), we get successively

$$\begin{aligned} X({}^1\varrho^v\,\omega^1(Y)) - Y({}^1\varrho^v\,\omega^1(X)) &= {}^1\varrho^v\,\omega^1([X, Y])\,, \\ (X\,{}^1\varrho^v)\,\omega^1(Y) + {}^1\varrho^v X\,\omega^1(Y) - (Y\,{}^1\varrho^v)\,\omega^1(X) &- \\ - {}^1\varrho^v Y\,\omega^1(X) &= {}^1\varrho^v\,\omega^1([X, Y])\,, \\ {}^1\varrho^v\,\omega^1(X)\,\omega^1(Y) + {}^1\varrho^v X\,\omega^1(Y) - {}^1\varrho^v\,\omega^1(Y)\,\omega^1(X) &- \\ - {}^1\varrho^v Y\,\omega^1(X) &= {}^1\varrho^v\,\omega^1([X, Y]) \end{aligned}$$

$$\omega^1(X)\,\omega^1(Y) - \omega^1(Y)\,\omega^1(X) + X\,\omega^1(Y) - Y\,\omega^1(X) - \omega^1([X, Y]) = 0\,,$$

$$[\omega^1(X), \omega^1(Y)] + 2\,\mathrm{d}\omega^1(X, Y) = 0$$

and $\Omega = 0$.

6. THE EXISTENCE OF SUBMANIFOLDS IN A_m WITH A GIVEN FUNDAMENTAL TENSOR

Suppose now that on an abstract manifold V_n there is given a tensor t_*; we have to find out the conditions for the existence of a differentiable regular mapping of V_n into A_m such that the fundamental tensor of V_n induced by this mapping and the tensor t on A_m is the given tensor t_*.

Suppose that the following conditions are satisfied:

I. On V_n, there is given a differentiable tensor t_* covariant of degree r acting at each point $p \in V_n$ on the $(k_0 + 1)$-vectors from $T_p^{(k_0+1)}(V_n)$; k_0 is a fixed given number.

From I, we get the possibility of the construction of the singular spaces of t_* on $T_p^{(k_0)}(V_n)$ and $T_p^{(k_0+1)}(V_n)$ denoted by $S_p^{(k_0)}$ and $S_p^{(k_0+1)}$ resp.; the construction is the same as that for the embedded manifolds. In what follows, we shall denote all objects constructed from t_* by the same symbols as those constructed in analoguous way from the fundamental tensor of a submanifold in A_m. For a manifold V_n, we may use all theorems proved above, we have only to check if all conditions are satisfied. Further, let us suppose:

II. For any differentiable fields of k_0-vectors ${}_1X^{(k_0)}, \ldots, {}_rX^{(k_0)}$ and any vector $Y_p \in T_p(V_n)$, we have

$$Y_p t_*({}_1X^{(k_0)}, \ldots, {}_rX^{(k_0)}) =$$

$$= \sum_{i=1}^{r} t_*({}_1X_p^{(k_0)}, \ldots, {}_{i-1}X_p^{(k_0)}, Y_{p\,i}X^{(k_0)}, {}_{i+1}X_p^{(k_0)}, \ldots, {}_rX_p^{(k_0)}) .$$

III. $\dim T_p^{(k_0)}(V_n)/S_p^{(k_0)} = \dim T_p^{(k_0+1)}/S_p^{(k_0+1)} = m$ for each point $p \in V_n$, $S_p^{(k_0+1)} \cap \cap T_p(V_n) = \{O\}$ for each $p \in V_n$.

Let P denote the principal fibre bundle of all bases of the spaces $T_p^{(k_0)}(V_n)/S_p^{(k_0)}$, $p \in V_n$.

IV. To each point $p \in V_n$ there is a neighborhood $U \subset V_n$ of p and a local section ϱ^v of the fibre bundle P over U such that the components of t_* with respect to the basis ϱ_q^v are constant functions of q on U.

V. There is a point $p \in V_n$ such that the vector space W_m with the given tensor t is isomorphic to the space $T_p^{(k_0)}/S_p^{(k_0)}$ with the tensor t_*.

Now, it is easy to prove that V is satisfied at each point $q \in V_n$. Let us choose a fixed basis ϱ^w of the space W_m. It follows from IV and V that the set of all bases ϱ_p^v of $T_p^{(k_0)}(V_n)/S_p^{(k_0)}$, for all points $p \in V_n$, is identical to ϱ^w (i.e., the tensor t_* has at p the same coordinates with respect to ϱ_p^v as the tensor t with respect to ϱ^w) forms a reduction P_0 of the principal fibre bundle P to the group $\bar{G}$. At the same time, the basis ϱ^w determines a reduction $A_m \times Q_0$ of $A_m \times Q$ to the group $\bar{G}_0$.

Let us now investigate the set of all isomorphisms of the spaces $T_p^{(k_0)}/S_p^{(k_0)}$ into $T_x(A_m)$ for all couples $p \in V_n$ and $x \in A_m$. Consider the Cartesian product $C = P_0 \times$

$\times\ A_m \times Q_0$. The group $\bar{G}_0$ operates on this space to the right by the rule

$$({}^1\varrho_p^v, x, {}^1\varrho^w)\, g = ({}^1\varrho_p^v g, x, {}^1\varrho^w g)\,.$$

The factor space C' of C with respect to the equivalence relation given by $\bar{G}_0$ is the space of all investigated isomorphisms; more precisely, if $\{X_p^{(ko)}\} = {}^1\varrho_p^v\mathbf{x}$, $\mathbf{x}$ being an $(m \times 1)$ – matrix, the element $a = ({}^1\varrho_p^v, x, {}^1\varrho^w) \in C$ determines the isomorphism $\alpha \in C'$ given by

$$\alpha(\{X_p^{(ko)}\}) = (x, {}^1\varrho^w\mathbf{x})\,.$$

Denote by σ the natural projection $C \to C'$; we have $\alpha = \sigma(a)$. The space C' may be given the structure of a differentiable manifold.

Let us now construct a mapping ψ_a of the space $T_{1\varrho_p v}(P_0)$ into the space $T_a(C)$ where $a = ({}^1\varrho_p^v, x, {}^1\varrho^w)$: To the vector $\mathbf{X} \in T_{1\varrho_p v}(P_0)$, we associate the vector $\psi_a(\mathbf{X})$ given by

$$\psi_a(\mathbf{X}) = (\mathbf{X}, \alpha(\pi'\mathbf{X}), (\omega(\mathbf{X}))^{*Q_0})\,;$$

here, $\alpha \in C'$, $\alpha = \sigma(a)$, ω is the connection form given on P_0 by the tensor t_*, A^{*Q_0} is the vector of the fundamental vector field given on Q_0 by the element $A \in \mathfrak{g}_0$, π is the canonical projection $P_0 \to V_n$. If we construct, for each point $a \in C$, the image of the space $T_{1\varrho_p v}(P_0)$ in the mapping ψ_a, we get a differentiable distribution on C. Let us denote by π_C or $\pi_{C'}$ the projection of C or C' resp. into V_n, the projection π_C or $\pi_{C'}$ of $a = ({}^1\varrho_p^v, x, {}^1\varrho^w) \in C$ or $\alpha = \sigma(a)$ resp. being just the point $p \in V_n$.

Lemma. *To each $\alpha \in C'$ there is a mapping $\varkappa_\alpha$* of $T_p(V_n)$, $p = \pi_{C'}(\alpha)$, *into $T(C')$ such that*

$$\sigma'\,\psi_a(\mathbf{X}) = \varkappa_\alpha(\pi'\mathbf{X}) \quad \textit{for} \quad a = ({}^1\varrho_p^v, x, {}^1\varrho^w)\,, \quad \mathbf{X} \in T_{1\varrho_p v}(P_0)\,.$$

Proof. a) First of all, we shall prove: Choosing two vectors $\mathbf{X}, \mathbf{X}' \in T_{1\varrho_p v}(P_0)$ such that $\pi'(\mathbf{X}) = \pi'(\mathbf{X}') = X$, we have $\sigma'\,\psi_a(\mathbf{X}) = \sigma'\,\psi_a(\mathbf{X}')$. Evidently, we may write $\mathbf{X}' = \mathbf{X} + A^{*P_0}$, A^{*P_0} being the fundamental vector field on P_0 generated by $A \in \bar{\mathfrak{g}}_0$. We have $\omega(\mathbf{X}') = \omega(\mathbf{X}) + A$ and

$$\psi_a(X') = (X + A^{*P_0}, \alpha(X), (\omega(X))^{*Q_0} + A^{*Q_0}) = \psi_a(X) + A^{*C}\,.$$

b) Second, let us prove: If $b = ag$, $\mathbf{X}' = R_g^{P_0\prime}\mathbf{X}$, where $\mathbf{X} = T_{1\varrho_p v}(P_0)$ and $R_g^{P_0\prime}$ is the differential of the right translation (i.e. $R_g{}^1\varrho^v = {}^1\varrho^v g$), $a = ({}^1\varrho_p^v, x, {}^1\varrho^w)$, we have $\sigma'\,\psi_a(\mathbf{X}) = \sigma'\,\psi_b(\mathbf{X})$. In fact, we have

$$\sigma(a) = \sigma(b) = \alpha\,, \quad \pi'(\mathbf{X}) = \pi'(\mathbf{X}') = X\,,$$

$$\begin{aligned}\psi_b(\mathbf{X}') = (\mathbf{X}', \alpha(X), (\omega(\mathbf{X}'))^{*Q_0}) &= (R_g^{P_0\prime}\mathbf{X}, \alpha(X), (\omega(R_g^{P_0\prime}\mathbf{X}))^{*Q_0}) = \\ &= (R_g^{P_0\prime}\mathbf{X}, \alpha(X), (\operatorname{ad}(g^{-1})\,\omega(\mathbf{X}))^{*Q_0}) = \\ &= (R_g^{P_0\prime}\mathbf{X}, \alpha(X), R_g^{Q_0\prime}(\omega(\mathbf{X}))^{*Q_0}) = R_g^{C\prime}\,\psi_a(\mathbf{X})\,,\end{aligned}$$

$R_g^{C_0}$ and R_g^C being defined analogously to $R_g^{P_0}$.

The Lemma follows easily from a) and b).

Constructing, for all $\alpha \in C'$, the images $\varkappa_\alpha(T_{\beta}, (V_n))$, $\beta = \pi_{C'}(\alpha)$, we get a differentiable distribution on C', and we have $\pi'_{C'}\, \varkappa_\alpha(X) = X$ for each vector $X \in T_{\pi_{C'}(\alpha)}(V_n)$.

Let us suppose

VI. The curvature form of the connection ω generated on P_0 by the tensor t_* is equal to 0.

Lemma. *The just constructed distribution on C' is involutive.*

Proof. To each point $p \in V_n$ there is a neighborhood U and a section ${}^1\varrho^v$ of the fibre bundle P_0 over U such that the form ω' associated to this section is equal to 0 (this follows from VI). Introduce the mapping $\pi^{C'}_{-1}(U) \to U \times A_m \times Q_0$ as follows: if $\alpha \in C'$, choose $a \in C$ such that $\sigma(a) = \alpha$ and $a = ({}^1\varrho^v_p, x, {}^1\varrho^w)$, ${}^1\varrho^v_p$ being the point of the section ${}^1\varrho^v$ over $p \in V_n$; we set $\alpha \to (p, x, {}^1\varrho^w) \in U \times A_n \times Q_0$. The composition of $\varkappa_\alpha$ with the differential of this mapping gives the mapping($\bar{\varkappa}_\alpha$ given by $\bar{\varkappa}_\alpha(X) = (X, \alpha(X), 0)$ because of $\omega^1(X) = 0$ for each vector X.

Further, let us write $\bar{\varkappa}_\alpha(X) = X + \alpha(X)$; here, we denote $(X, 0, 0)$ as X and $(0, \alpha(X), 0)$ as $\alpha(X)$. To prove the involutiveness of our distribution we have to show that, for any differentiable vector fields X, Y on U, the bracket of the images of these vector fields (given, for each α, by the mappings $\bar{\varkappa}_\alpha$) into the space $U \times A_m \times Q_0$ is in our distribution. Because the third component of the vectors $\bar{\varkappa}_\alpha(X)$ is equal to 0, we may choose a fixed basis ${}^1\varrho^w$, and we may consider the distribution on the space $U \times A_m \times \{{}^1\varrho^w\}$ only. To each $p \in U$, we have thus associated a uniquely determined isomorphism α, and we have to prove

$$[\bar{\varkappa}_\alpha(X), \bar{\varkappa}_\alpha(Y)] = [X, Y] + \alpha([X, Y]),$$

this being equivalent to the involutiveness of our distribution. We have

$$[\bar{\varkappa}_\alpha(X), \bar{\varkappa}_\alpha(Y)] = [X, Y] + [X, \alpha(Y)] + [\alpha(X), Y] + [\alpha(X), \alpha(Y)].$$

Evidently, $[\alpha(X), \alpha(Y)] = 0$; choosing a linear coordinate system in A_m given by the point x_0 (where we have to calculate the relevant bracket) and expressing the vectors $\{X\}$ and $\{Y\}$ in the basis ${}^1\varrho^v$, an easy calculation leads to $[X, \alpha(Y)] + [\alpha(X), Y] = \alpha([X, Y])$. Q.E.D.

Consider a maximal integral manifold $\bar{V}_n$ of our distribution; it follows from $\pi'_{C'}\, \varkappa_\alpha(X) = X$ – see [4] – that the projection $\pi_{C'}$ is a local diffeomorphism of $\bar{V}_n$ onto V_n.

To each $\alpha \in C'$ with $\alpha = \sigma(a)$ and $a = ({}^1\varrho^v_p, x, {}^1\varrho^w)$, associate the point $x \in A_m$. V_n and $\bar{V}_n$ being locally diffeomorphic, there is, to each point $p \in V_n$, a neighborhood $U \subset V_n$ and a diffeomorphism of $U \subset V_n$ into a component of $\pi_{C'}^{-1}(U) \cap \bar{V}_n$. Composing this diffeomorphism with the just produced mapping of C' into A_m, we get a mapping ε such that $\varepsilon'(X_p) = \alpha(X_p)$ for $\pi_{C'}(\alpha) = p$. The isomorphism α being given by the assignment of the basis ${}^1\varrho^v_p$ (suppose that the section ${}^1\varrho^v$ determines the form $\omega^1 = 0)$ to the fixed basis ${}^1\varrho^w$ constant for all $\alpha \in \overline{V}_n$, we get $X\,{}^1\varrho^v = 0$ and, at the same time, $(\varepsilon' X)\,{}^1\varrho^w = 0$. The mapping ε and the considered mapping of $\overline{V}_n$ into A_m is regular, and the image of $\overline{V}_n$ is a manifold $V_n' \subset A_m$ which is locally diffeomorphic to V_n. Further, there is given a differentiable mapping of V_n' into V_n transforming the tensor t_*' induced on V_n' by the tensor t on W_m into the tensor t_* defined on V_n.

Theorem 15. *Be given differentiable manifolds V_n and V_n' in A_m and a differentiable mapping $\varphi : V_n \to V_n'$ which is onto. The necessary and sufficient condition for the existence of an affine collineation h of the space A_m onto itself such that $h(p) = \varphi(p)$ for each $p \in V_n$ is that the manifolds have the same fundamental tensor at the corresponding points.*

Let V_n be an arbitrary abstract manifold endowed by a tensor t_* satisfying the conditions I–VI. Then there is, in the affine space A_m, a manifold V_n' and its differentiable mapping on V_n which is locally diffeomorphic and which transforms the tensor t_*' induced on V_n' by the tensor structure of the affine space A_m into the tensor t_* on V_n.

Proof. Everything has been proved above or it is very easy; compare with [4].

Remark. The space A_m being equiaffine, the condition IV may be replaced by the assumption of the skew-symmetry of t.

The space A_m being Euclidean, we may construct, in each space $T_p^{(k)}(V_n)$, the space $N_p^{(k)}$ orthogonal to the space $T_p^{(k-1)}(V_n)$. If we define the function g^k by

$$g^k({}_1X_p, \ldots, {}_kX_p, {}_1Y_p, \ldots, {}_kY_p) = t_*(X_p^{(k)}, Y_p^{(k)}),$$

where

$${}_1X_p \ldots {}_kX_p - X_p^{(k)} \in T_p^{(k-1)}(V_n), \quad {}_1Y_p \ldots {}_kY_p - Y_p^{(k)} \in T_p^{(k-1)}(V_n); \quad X_p^{(k)}, Y_p^{(k)} \in N_p^{(k)}$$

(see Theorem 5), we have $g^k \in \widetilde{\otimes^{2k} T_p(V_n)}$. It is possible to prove that these tensors together with the condition II determine the tensor t_*.

Bibliography

[1] *K. Nomizu:* Lie groups and differential geometry. Math. Soc. of Japan, 1956.

[2] *A. Lichnerowicz:* Théorie globale des connexions et des groupes d'holonomie. Ed. Cremonese, Roma, 1955.

[3] *C. Chevalley:* Theory of Lie groups I–III. Princeton, 1946.

[4] *S. Sasaki:* A global formulation of the fundamental theorem of the theory of surfaces in three-dimensional Euclidean space. Nagoya Mat. J., *13*, 69–82, 1958.

[5] *A. Švec:* Global differential geometry of surfaces in affine space. Čas. pěst. mat., *89*, 340–346, 1964.

Author's address: Sokolovská 83, Praha 8 - Karlín, ČSSR (Matematicko-fyzikální fakulta Karlovy university).