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SOME ESTIMATES IN THE THEORY OF NON-NEGATIVE MATRICES

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Dedicated to academician VOJTĚCH JARNÍK *on the occation of his seventieth birthday on December 22, 1967*

This paper presupposes the knowledge of some results proved in [1] and [2].

We briefly recall the notions necessary in the following which have been introduced in [1] and [2].

Denote $N = \{1, 2, \ldots, n\}$ and consider the set S of all "$n \times n$ matrix units e_{ij}" together with a zero 0 adjoint: $S = \{e_{ij} \mid i, j \in N\} \cup \{0\}$. Define in S a multiplication by

$$e_{ij}e_{ml} = \begin{cases} e_{il} & \text{for } j = m, \\ 0 & \text{for } j \neq m, \end{cases}$$

the zero element 0 having the usual properties of a multiplicative zero. The set S becomes then a 0-simple semigroup.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support C_A of A we shall mean the subset of S containing 0 and all e_{ij} for which $a_{ij} > 0$.

The sequence of powers

$$C_A, C_A^2, C_A^3, \ldots \tag{1}$$

contains only a finite number of different elements (subsets of S). Let $k = k(A) \geqq 1$ be the least integer for which C_A^k appears in (1) more than once and let $d = d(A)$ be the least integer $\geqq 1$ for which $C_A^k = C_A^{k+d}$ holds. Then the sequence (1) is of the form

$$C_A, C_A^2, \ldots, C_A^{k-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid C_A^k, \ldots$$

and contains exactly $k + d - 1$ different elements. The system of sets $\{C_A^k, \ldots \ldots, C_A^{k+d-1}\}$ forms with respect to the multiplication of subsets a cyclic group of order d with the unit element C_A^{ϱ}, where $\varrho = \varrho(A)$ is a uniquely defined multiple of d, say τd, satisfying $k \leqq \varrho = \tau d \leqq k + d - 1$.

Denote $S_i = \{0, e_{i1}, e_{i2}, \ldots, e_{in}\}$, so that $S_1 \cup S_2 \cup \ldots \cup S_n = S$. For a given A denote $F_i = F_i(A) = S_i \cap C_A$, so that F_i is the "support" of the i-th row of A.

Consider the sequence

$$F_i, F_iC_A, F_iC_A^2, \ldots \tag{2}$$

and define $F_iC_A^0 = F_i$. Clearly $F_iC_A^t = S_i \cap C_A^{t+1}$ for any integer $t \geqq 0$. The sequence (2) contains again only a finite number of different elements. Denote $k_i = k_i(A)$ the least integer k_i such that $F_iC_A^{k_i-1}$ occurs in (2) more than once. Let further $d_i = d_i(A)$ be the least integer $\geqq 1$ such that $F_iC_A^{k_i-1} = F_iC_A^{k_i+d_i-1}$. Then the sequence (2) is of the form

$$F_i, F_iC_A, \ldots, F_iC_A^{k_i-2} \mid F_iC_A^{k_i-1}, \ldots, F_iC_A^{k_i+d_i-2} \mid F_iC_A^{k_i-1}, \ldots .$$

Clearly $F_iC_A^{k_i-1} = F_iC_A^{k_i-1+\tau} (\tau \geqq 0)$ holds if and only if d_i/τ. Since $C_A^k = C_A^{k+d}$, we have $S_i \cap C_A^k = S_i \cap C_A^{k+d}$; hence $F_iC_A^{k-1} = F_iC_A^{k+d-1}$. Therefore $k_i \leqq k$ and $d_i \leqq d$. In [2] we have proved the following more precise result:

Lemma 1. *For any non-negative $n \times n$ matrix A we have*:

a) $k(A) = \max_{i=1,\ldots,n} k_i(A)$;

b) $d(A) =$ *the least common multiple of* $d_1, d_2, \ldots, d_n$.

The main purpose of this paper is to find estimates for the number $k(A)$, whereby we restrict ourselves to the case of an *irreducible* $n \times n$ matrix.

For an irreducible matrix A we have proved in [1] that d is the index of imprimitivity of A and we always have $1 \leqq d \leqq n$. Also A is irreducible if and only if

(3) $$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S .$$

Here the summands on the left are quasidisjoint, i.e. the intersection of any two of them contains only the zero element 0. More generally: Any d consecutive powers $C_A^t, C_A^{t+1}, C_A^{t+2}, \ldots, C_A^{t+d-1}$, $t \geqq 1$, are quasidisjoint. This implies that for an irreducible matrix A any d consecutive members in the sequence (2) are quasidisjoint.

If A is irreducible, then (3) implies

$$(C_A^k \cap S_i) \cup \ldots \cup (C_A^{k+d-1} \cap S_i) = S \cap S_i ,$$

i.e.

(4) $$F_iC_A^{k-1} \cup F_iC_A^k \cup \ldots \cup F_iC_A^{k+d-2} = S_i .$$

It should be noted, by the way, that for an irreducible matrix A each member in the sequence (2) contains at least one non-zero element $\in S_i$. (For if there were $F_iC_A^\tau = \{0\}$ for some $\tau \geqq 0$, we would have $F_iC_A^t = \{0\}$ for all $t \geqq \tau$, a contradiction with (4).)

The relation (4) implies $\bigcup_{j=0}^{\infty} F_iC_A^j = S_i$ or, which is the same,

$$F_i \cup F_iC_A \cup \ldots \cup F_iC_A^{k_i-1} \cup \ldots \cup F_iC_A^{k_i+d_i-2} = S_i .$$

Since an irreducible matrix contains in each column at least one non-zero element, we have $S_iC_A = S_i$. Hence, multiplying the last relation by $C_A^{k_i-1}$ from the right we get

(5) $$F_iC_A^{k_i-1} \cup F_iC_A^{k_i} \cup \dots \cup F_iC_A^{k_i+d_i-2} = S_i\,.$$

Since any set on the left has at least one non-zero element and any d consecutive members in (2) are quasidisjoint, we necessarily have $d_i \geqq d$. Therefore, in the case of an irreducible matrix, $d_i = d$. We summarize:

Lemma 2. *Let A be an $n \times n$ non-negative irreducible matrix. Then*:

a) $d_1(A) = d_2(A) = \dots = d_n(A) = d(A)$;

b) $F_iC_A^{k_i-1} \cup F_iC_A^{k_i} \cup \dots \cup F_iC_A^{k_i+d-2} = S_i$;

c) *the summands on the left are quasidisjoint.*

Lemma 3. *If A is irreducible and there are two integers $\varkappa \geqq 1$, $\delta \geqq 1$ such that*

$$F_iC_A^{\varkappa-1} \cup F_iC_A^{\varkappa} \cup \dots \cup F_iC_A^{\varkappa+\delta-2} = S_i\,,$$

then $\delta \geqq d$.

Proof. Suppose that $\delta < d$. Multiplying by a sufficiently high power C_A^t we get

$$F_iC_A^{\varkappa-1+t} \cup F_iC_A^{\varkappa+t} \cup \dots \cup F_iC_A^{\varkappa+\delta-2+t} = S_iC_A^t = S_i\,.$$

The δ summands on the left, which are equal to some of the quasidisjoint sets $F_iC_A^{k_i-1}, \dots, F_iC_A^{k_i+d-2}$, cannot exhaust the whole set S_i. This proves our assertion.

For further purposes we introduce a positive integer h_i associated with the "row F_i" in the following way:

By h_i we shall denote the least integer $\geqq 1$ such that $F_i \subset F_iC_A^{h_i}$.

In [2] we have proved that $1 \leqq h_i \leqq n$.

Lemma 4. *For $i = 1, 2, \dots, n$, we have d/h_i.*

Proof. We have $F_iC_A^{k_i-1} \subset F_iC_A^{h_i}C_A^{k_i-1} = F_iC_A^{k_i+h_i-1}$ and

$$S_i = F_iC_A^{k_i-1} \cup \dots \cup F_iC_A^{k_i+d-2} \subset F_iC_A^{k_i+h_i-1} \cup \dots \cup F_iC_A^{h_i+k_i+d-2}\,.$$

Hence,

$$S_i = F_iC_A^{k_i+h_i-1} \cup \dots \cup F_iC_A^{h_i+k_i+d-2}\,.$$

Since the sets on the right are quasidisjoint, we have $F_iC_A^{k_i-1} = F_iC_A^{k_i-1+h_i}$. This proves d/h_i.

Lemma 5. *Let A be irreducible. If for some $t \geqq 1$ we have*

(6) $$F_iC_A^{t-1} \cup F_iC_A^{t} \cup \dots \cup F_iC_A^{t+d-2} = S_i\,,$$

then $t \geqq k_i$.

Proof. As in the proof of Lemma 4 we have

$$F_iC_A^{t-1} \subset F_iC_A^{h_i}C_A^{t-1} = F_iC_A^{t+h_i-1},$$

and consequently

$$F_iC_A^{t-1+u} \subset F_iC_A^{t+h_i+u-1}$$

for $u = 0, 1, \ldots, d-1$. This implies

$$S_i = F_iC_A^{t-1} \cup \ldots \cup F_iC_A^{t+d-2} \subset F_iC_A^{t+h_i-1} \cup \ldots \cup F_iC_A^{t+d+h_i-2} = S_i .$$

Since the summands are quasidisjoint, we have $F_iC_A^{t-1} = F_iC_A^{t+h_i-1}$. Hence, $F_iC_A^{t-1}$ appears in the sequence (2) more than once. Since k_i is the least exponent with this property, we have $t \geqq k_i$, q.e.d.

We shall also need the following

Lemma 6. *Let A be irreducible. If for some $t \geqq 1$ we have*

$$C_A^t \cup C_A^{t+1} \cup \ldots \cup C_A^{t+d-1} = S,$$

then $t \geqq k(A)$.

Proof. In [1] we have proved that there exists a power $C_A^{\varrho_1}$, $\varrho_1 \leqq \varrho(A)$, such that $E = \{e_{11}, e_{22}, \ldots, e_{nn}\} \subset C_A^{\varrho_1}$. We have $C_A^t = C_A^t E \subset C_A^{t+\varrho_1}$, and consequently $C_A^{t+1} \subset C_A^{t+\varrho_1+1}, \ldots$. Now

$$S = C_A^t \cup \ldots \cup C_A^{t+d-1} \subset C_A^{t+\varrho_1} \cup \ldots \cup C_A^{t+d-1+\varrho_1} = S .$$

Since the d summands on the right are quasidisjoint, we have $C_A^t = C_A^{t+\varrho_1}, \ldots$ $\ldots, C_A^{t+d-1} = C_A^{t+d-1+\varrho_1}$. In particular, the first of these equalities implies $k(A) \leqq t$, q.e.d.

We now give a series of theorems concerning $k = k(A)$ all being consequences of the following Theorem 1.

Theorem 1. *If A is an $n \times n$ non-negative irreducible matrix, then*

$$k(A) \leqq n - 1 + \min_{i=1,\ldots,n} k_i(A) .$$

Proof. Let $e_{i\alpha}$ be any element $\in S_i$. Take $j \neq i$ and write $e_{i\alpha} = e_{ij}e_{j\alpha}$. By Lemma 2 in [2] we have $e_{ij} \in F_iC_A^t$ with $t = t(i, j)$ satisfying $0 \leqq t \leqq n - 2$. By Lemma 2 we have (for any α)

$$e_{j\alpha} \in S_j = F_jC_A^{k_j-1} \cup \ldots \cup F_jC_A^{k_j+d-2} .$$

Hence,

$$S_i = \{0, e_{i1}, e_{i2}, \ldots, e_{in}\} \subset F_iC_A^t\{F_jC_A^{k_j-1} \cup \ldots \cup F_jC_A^{k_j+d-2}\} \subset$$
$$\subset F_iC_A^{t+k_j} \cup \ldots \cup F_iC_A^{t+k_j+d-1} .$$

Therefore

$$S_i = F_iC_A^{t+k_j} \cup F_iC_A^{t+k_j+1} \cup \ldots \cup F_iC_A^{t+k_j+d-1} . \tag{7}$$

By Lemma 5 we have $k_i \leqq t + k_j + 1$. Since j is arbitrary, we have $k_i \leqq n - 2 + \min_j k_j + 1$. By Lemma 1 we finally obtain $k(A) \leqq n - 1 + \min_j k_j$, q.e.d.

In [2] we have proved: If F_i contains g_i non-zero elements, then $k_i \leqq (n - g_i)^2 + (n - g_i) + 1 = 1 + (n - g_i)(n - g_i + 1)$. This implies:

Theorem 2. *If A is irreducible and the i-th row of A contains $g_i \geqq 1$ non-zero elements, then*

$$k(A) \leqq n + \min_i (n - g_i)(n - g_i + 1) .$$

Remark. If A is primitive and $n \geqq 2$, A contains at least one row with $g_i \geqq 2$, so that $k(A) \leqq n + (n - 2)^2 + (n - 2) = n^2 - 2n + 2$. It is known that this result is sharp in the class of matrices with $C_A = \{0, e_{12}, e_{23}, \dots, e_{n-1,n}, e_{n1}, e_{n2}\}$.

In [2] (see Theorem 1) we have proved: If A is irreducible, then $k_i \leqq (n - g_i) h_i + 1$. This implies by Theorem 1:

Theorem 3. *If A is irreducible, g_i and h_i have the meaning introduced above, then*

$$k(A) \leqq n + \min_i (n - g_i) h_i .$$

Corollary 1. *If A is irreducible and if it contains a row with $F_i \subset F_i C_A$, then A is primitive and $k(A) \leqq 2n - 1$.*

This follows from $h_i = 1$, d/h_i and $g_i \geqq 1$.

Corollary 2. *If A is irreducible and if it contains a non-zero element in the main diagonal, then A is primitive and $k(A) \leqq 2n - 2$.*

Proof. The row containing e_{ii} contains at least one other non-zero element (since otherwise A would not be irreducible). Hence $g_i \geqq 2$. Further $F_i = e_{ii} C_A \subset F_i C_A$, hence $h_i = 1$. Therefore $d = 1$, and our statement follows from Theorem 3.

Theorems 4 through 6 below give information concerning the relations between $k(A)$ and $d(A)$.

It is known: If A is irreducible with index of imprimitivity $d \geqq 1$, then there is a permutation matrix P such that $P^{-1}AP$ is of the form

$$P^{-1}AP = \begin{pmatrix} 0 & M_1 & 0 & \dots & 0 \\ 0 & 0 & M_2 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & M_{d-1} \\ M_d & 0 & 0 & \dots & 0 \end{pmatrix},$$

where in the diagonal there are square blocks of orders $n_1, n_2, \dots, n_d$ such that $n_1 + n_2 + \dots + n_d = n$. Also, A^d is completely reducible into primitive matrices and

$$P^{-1}A^dP = \operatorname{diag}(A_1, A_2, \ldots, A_d)\,, \tag{8}$$

where $A_i = M_iM_{i+1} \ldots M_dM_1 \ldots M_{i-1}$ is of order n_i.

Clearly, $k(P^{-1}AP) = k(A)$ and $d(P^{-1}AP) = d(A)$.

We now prove:

Theorem 4. *Let A be irreducible with the index of imprimitivity $d \geqq 1$. Write $n = rd + s$ with the integers r, s satisfying $r \geqq 1$, $0 \leqq s \leqq d - 1$. Then*

$$k(A) \leqq \begin{cases} (n^2/d) - 2n + 3d - s - 1\,, & \text{if} \quad r \geqq 2\,, \\ s + 1\,, & \text{if} \quad r = 1\,. \end{cases}$$

Proof. Consider the matrix A_1. Since it is primitive, it contains in the case $n_1 \geqq 2$ at least one row with two non-zero elements. Hence

$$\min_{i=1,2,\ldots,n_1} k_i(A_1) \leqq (n_1 - 2)^2 + (n_1 - 2) + 1 = n_1^2 - 3n_1 + 3\,. \tag{9}$$

This result holds also in the case $n_1 = 1$. This implies that

$$\min_{i=1,2,\ldots,n_1} k_i(P^{-1}AP) \leqq d(n_1^2 - 3n_1 + 3)\,.$$

Since the same holds for $A_2, A_3, \ldots, A_d$, we have with respect to Theorem 1,

$$k(A) \leqq n - 1 + d \min_{j=1,2,\ldots,d} (n_j^2 - 3n_j + 3)\,. \tag{10}$$

Note for further purposes that at least one of the numbers $n_1, n_2, \ldots, n_d$ is $\leqq r$.[1]) We shall consider two cases:

a) If $r \geqq 2$, we have

$$k(A) \leqq n - 1 + d(r^2 - 3r + 3) = dr + s - 1 + d(r^2 - 3r + 3) = \\ = d(r^2 - 2r + 3) + s - 1\,.$$

To get a result in terms of n we write

$$k(A) \leqq d\left[\left(\frac{n-s}{d}\right)^2 - 2\,.\,\frac{n-s}{d} + 3\right] + s - 1 = \\ = \frac{n^2}{d} - 2n + 3d - 1 - \frac{s}{d}(2n - s - 3d)\,.$$

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[1]) For if there were $n_j \geqq r + 1$ for all j, we would have $\sum_j n_j \geqq d(r + 1) > dr + s = n$, a contradiction.

Now

$$\frac{s}{d}(2n - s - 3d) = \frac{s}{d}[2(dr + s) - s - 3d] = s(2r - 3) + \frac{s^2}{d} \geqq s + \frac{s^2}{d} \geqq s .$$

Therefore

$$k(A) \leqq \frac{n^2}{d} - 2n + 3d - s - 1 .$$

(This is slightly better than a corresponding result in [3].) [2])

b) If $r = 1$, i.e. $n = d + s, 0 \leqq s \leqq d - 1$, we have from (10), $k(A) \leqq n - 1 + d$. This result can be strengthened. Since A is irreducible, we have $C_A \cup C_A^2 \cup \ldots \cup C_A^n = S$. Now it follows from the results in [1] that the non-zero idempotents $\in S$, i.e. the elements of the set $E = \{e_{11}, e_{22}, \ldots, e_{nn}\}$ can be contained only in the powers $C_A^d, C_A^{2d}, C_A^{3d}, \ldots$ Hence (in our case) we necessarily have $E \subset C_A^d$. This implies $C_A = C_A E \subset C_A^{d+1}$, $C_A^2 \subset C_A^{d+2}, \ldots, C_A^{n-d} \subset C_A^n$; hence $C_A^{n-d+1} \cup C_A^{n-d+2} \cup \ldots \cup C_A^n = S$. By Lemma 6 we have $k(A) \leqq n - d + 1 = s + 1$. This completes the proof of Theorem 4.

In the "extreme case" $s = d - 1$ we have $k(A) \leqq n^2/d - 2n + 2d$ for $r \geqq 2$, and $k(A) \leqq d$ for $r = 1$. Since $d \leqq (n - d)^2/d + d = n^2/d - 2n + 2d$, we have $k(A) \leqq n^2/d - 2n + 2d$ in both cases. We next show by modifying our argument that the same is true in the second "extreme case", namely $s = 0$.

Theorem 5. *If A is irreducible and d/n, then*

$$k(A) \leqq \frac{n^2}{d} - 2n + 2d .$$

Proof. There are two possibilities. Either all matrices $A_1, A_2, \ldots, A_d$ are of order $r = n/d$, or there is at least one matrix, say A_1, such that $n_1 \leqq r - 1$.

A) The first case. Since $A_1, \ldots, A_d$ are primitive of order r, we have $k(A_j) \leqq r^2 - 2r + 2$ for $j = 1, 2, \ldots, d$, and $k_i(A) \leqq d(r^2 - 2r + 2)$ for $i = 1, 2, \ldots, n$. By Lemma 1

$$k(A) = \max_i k_i(A) \leqq d(r^2 - 2r + 2) = \frac{n^2}{d} - 2n + 2d .$$

B) In the second case we have to distinguish the following possibilities.

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[2]) In the meantime the paper [4] appeared in which even a slightly better result than our is proved, namely $k(A) \leqq d(r^2 - 2r + 2) + 2s$.

a) $2 \leqq n_1 \leqq r - 1$. Then $r \geqq 3$. We have from (10):

$$k(A) \leqq n - 1 + d[(r-1)^2 - 3(r-1) + 3] = \frac{n^2}{d} - 4n + 7d - 1 =$$
$$= \frac{n^2}{d} - 2n + 2d + (5d - 2n - 1).$$

But for $r \geqq 3$ we have $5d - 2n - 1 = 5d - 2rd - 1 = (5 - 2r)\, d - 1 \leqq -d - 1$, so that $k(A) \leqq n^2/d - 2n + d - 1 < n^2/d - 2n + 2d$.

b) $1 = n_1 \leqq r - 1$. Here $r \geqq 2$. By (10) we have $k(A) \leqq n - 1 + d = (r + 1)\, d - 1$.

α) If $r > 2$, then $(r + 1)\, d - 1 < d(r^2 - 2r + 2)$, so that our statement holds.

β) If $r = 2$ (i.e. $n = 2d$) our result $k(A) \leqq 3d - 1$ is not sufficient for the proof of our statement. It can be strengthened in the following way.

Since $n_1 = 1$, we have in (8) $A_1 = I_1\alpha$ (I_1 = the unit matrix of order 1, α is a positive number) and $I_1\alpha = M_1M_2 \dots M_d$. Therefore $A_i^2 = (M_i \dots M_dM_1 \dots M_{i-1}) \cdot (M_i \dots M_dM_1 \dots M_{i-1}) = (M_i \dots M_d)\, I_1\alpha(M_1 \dots M_{i-1})$. Now $M_iM_{i+1} \dots M_dI_1$ is an $n_i \times 1$ positive matrix, since the existence of a zero row in $M_iM_{i+1} \dots M_d$ would imply the existence of a zero row in A_i^2, contrary to the fact that A_i is primitive. Analogously, $I_1M_1M_2 \dots M_{i-1}$ is a $1 \times n_i$ positive matrix. Hence A_i^2 is positive. Therefore $k_i(A) \leqq 2d$ for $i = 1, 2, \dots, d$, so that $k(A) \leqq 2d = n^2/d - 2n + 2d$. This completes the proof of Theorem 5.

Remark 1. The result of Theorem 5 is sharp in the sense that to any n and d, d/n, there exists a matrix A for which $k(A) = n^2/d - 2n + 2d$ holds. (See [3].)

Remark 2. The fact that in the two extreme cases, i.e. $s = d - 1$ and $s = 0$, we have $k(A) \leqq n^2/d - 2n + 2d$ leads to the *conjecture* that the last inequality holds in all cases. However, at this time I am unable to prove or disprove this conjecture.

The next Theorem 6 gives a result which in the special case $d = 1$ goes back to Frobenius.

Lemma 7. *Suppose that* $e_{ii} \in F_iC_A^{d-1}$. *Then*

$$k_i \begin{cases} = 1, & \text{if } d = n, \\ \leqq n - d, & \text{if } d < n. \end{cases}$$

Proof. a) If $d = n$, then

$$F_i \cup F_iC_A \cup \dots \cup F_iC_A^{n-1} = S_i \tag{11}$$

implies (with respect to Lemma 5) $k_i \leqq 1$; hence $k_i = 1$.

b) If $d < n$, we have (see [2], Lemma 2)

$$F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-2} = S_i . \tag{12}$$

Further

$$F_i = e_{ii} C_A \subset F_i C_A^d , \quad F_i C_A \subset F_i C_A^{d+1}, \ldots, F_i C_A^{n-d-2} \subset F_i C_A^{n-2} .$$

Hence, (12) can be written in the form

$$F_i C_A^{n-d-1} \cup \ldots \cup F_i C_A^{n-2} = S_i .$$

Lemma 5 implies $k_i \leqq n - d$, q.e.d.

Suppose now that C_A^d contains at least one element $\in E = \{e_{11}, e_{22}, \ldots, e_{nn}\}$. Then, if $d = n$, $C_A \cup \ldots \cup C_A^d = S$ implies (by Lemma 6) $k(A) = 1$. If $d < n$ we may use Theorem 1 and Lemma 7 by which

$$k(A) \leqq n - 1 + \min_i k_i(A) \leqq n - 1 + (n - d) = 2n - d - 1 .$$

We have proved:

Theorem 6. *Let A be irreducible with the index of imprimitivity $d \geqq 1$. Then if A^d contains at least one non-zero element in the main diagonal, we have $k(A) \leqq$ $\leqq 2n - d - 1$.*

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Резюме

НЕКОТОРЫЕ ОЦЕНКИ В ТЕОРИИ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦ

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Пусть C_A – носитель неотрицательной матрицы A (в смысле работы [1]). Пусть C_A^k – самая низкая степень, которая встречается в последовательности (1) более чем один раз. Цель статьи – доказательство некоторых теорем, касающихся оценки числа $k = k(A)$.