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A closure space is a set P and a mapping u of the family of all subsets of P into itself such that the following axioms are satisfied:

(C_0) $u\emptyset = \emptyset$.
(C_1) $A \subset uA$ for each $A \subset P$.
(C_2) $u(A \cup B) = uA \cup uB$ for each $A \subset P$ and $B \subset P$.

The mapping u will be called a closure topology.

A topological space is a closure space (P, u) in which the following axiom is satisfied:

(F) $u(uA) = uA$ for each $A \subset P$;

the mapping u will then be called a topology.

Let L be a set and let $\mathfrak{L}$ be a set of pairs $(\{x_n\}, x)$, where $\{x_n\}$ is a sequence of points $x_n \in L$, $n \in N$, and $x \in L$. The set $\mathfrak{L}$ is called a multivalued convergence on L if the following axioms are satisfied:

$(\mathscr{L}_1)$ If $x_n = x$, $n \in N$, then $(\{x_n\}, x) \in \mathfrak{L}$.
$(\mathscr{L}_2)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $n_i < n_{i+1}$, $i \in N$, then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

A convergence $\mathfrak{L}$ on L is a multivalued convergence $\mathfrak{L}$ on L such that the following axiom is satisfied:

$(\mathscr{L}_0)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$, then $x = y$.

Let $\mathfrak{L}$ be a (multivalued) convergence on a set L. Define a mapping λ on the family of all subsets of L into itself as follows: If $A \subset L$ and $x \in L$, then $x \in \lambda A$ if there is a sequence $\{x_n\}$ such that $(\{x_n\}, x) \in \mathfrak{L}$ and $\bigcup_{n=1}^{\infty} x_n \subset A$ [1]. The mapping λ is a closure topology for L. The closure space (L, λ) will be called a (multivalued) convergence

[1]) The set $\bigcup_{n=1}^{\infty} (x_n)$ will be denoted simply by $\bigcup_{n=1}^{\infty} x_n$.

space and denoted by $(L, \mathfrak{L}, \lambda)$. The closure topology λ will be called a (multivalued) convergence topology. Every convergence space is a T_1-closure space.

If $(L, \mathfrak{L}, \lambda)$ is a convergence space, then there exists exactly one convergence on L such that the corresponding convergence topology is identical with λ and such that it satisfies the following axiom:

$(\mathscr{L}_3)$ If each subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ contains a subsequence $\{x_{n_{i_j}}\}$ converging to a point x, then the sequence $\{x_n\}$ itself converges to x. This convergence is called the largest convergence and it will be denoted by $\mathfrak{L}^*$.

Throughout this paper the family of all continuous functions on a convergence space $(L, \mathfrak{L}, \lambda)$ to the closed interval $\langle 0, 1\rangle$ will be denoted by $\mathfrak{F}(L)$.

The general theory of closure spaces is developed in [1] and some basic concepts are mentioned in [9]. In both cases the closure space is called a topological space and the topological space in the usual sense is called an F-space.

The exposition of the theory of convergence spaces is given in [7] while the same for multivalued convergence spaces is contained in [5]. In these papers the needed concepts of the theory of closure spaces can also be found. The knowledge of [7] is assumed in the following.

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The notion of sequential regularity was introduced in [3]. A convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular if for each point $x \in L$ and each sequence $\{x_n\}$ of points $x_n \in L$ no subsequence of which converges to x there is a function $f \in \mathfrak{F}(L)$ such that the sequence $\{f(x_n)\}$ does not converge to $f(x)$ (cf. [7]).

Now we are going to characterize sequential regularity in terms of the convergence topology.

Lemma 1. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space and let $\mathfrak{F}(L) = \{f_\alpha : \alpha \in I\}$. Let $A \subset L$ and let $x \in \lambda^{\omega_1}A$. Then, for each $\alpha \in I$, there is a sequence $\{x_n^\alpha\}$ such that $\bigcup_{n=1}^{\infty} x_n^\alpha \subset A$ and $\lim f_\alpha(x_n^\alpha) = f_\alpha(x)$.*

Proof. Since f_α is continuous on $(L, \mathfrak{L}, \lambda)$, it is also continuous on (L, λ^{ω_1}). Since the topology of $\langle 0, 1\rangle$ is a convergence topology, the assertion follows.

Theorem 1. *A convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular if and only if for each countably infinite set $S \subset L$ and for each point $x_0 \in L - \lambda S$ there is a function $f \in \mathfrak{F}(L)$ and an infinite set $T \subset S$ such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in T$.*

Proof. I. Suppose that $x_0 \in L$ and $\{x_n\}$ is a sequence of points $x_n \in L$ such that $(\{x_{n_i}\}, x_0) \notin \mathfrak{L}$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We have to prove that there is a function $f \in \mathfrak{F}(L)$ such that the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$.

Let $S = \bigcup_{n=1}^{\infty} x_n$; clearly $x_0 \in L - \lambda S$.

1. If the set S is infinite, then the assumptions of the theorem imply that there is a function $f \in \mathfrak{F}(L)$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $f(x_0) = 0$ and $f(x_{n_i}) = 1$ for $i \in N$. Hence the assertion follows.

2. Suppose that S is finite. Then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} = y$, $i \in N$, for some point $y \neq x_0$. If y is isolated, let $f(y) = 1$ and $f(x) = 0$ for $x \neq y$; clearly $f \in \mathfrak{F}(L)$ and the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$. If y is not isolated, then there is a one-to-one sequence $\{y_n\}$ such that $(\{y_n\}, y) \in \mathfrak{L}$ and $y_n \neq x_0$ for $n \in N$. Let $S' = \bigcup_{n=1}^{\infty} y_n$; clearly $x_0 \in L - \lambda S'$. It follows from the assumptions of the theorem that there is a function $f \in \mathfrak{F}(L)$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $f(x_0) = 0$ and $f(y_{n_i}) = 1$ for $i \in N$. Since clearly $f(y) = 1$ and $x_{n_i} = y$ for $i \in N$, the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$.

II. To prove the converse suppose that $S \subset L$ is a countably infinite set and $x_0 \in \in L - \lambda S$. Arrange the points of S into a one-to-one sequence $\{x_n\}$. It follows from the assumption of sequential regularity that there is a function $g \in \mathfrak{F}(L)$ such that the sequence $\{g(x_n)\}$ does not converge to $g(x_0)$. Then there is $\delta > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $|g(x_{n_i}) - g(x_0)| > \delta$, $i \in N$. Let $T = \bigcup_{i=1}^{\infty} x_{n_i}$ and let $f = \min [(1/\delta) |g - g(x_0)|, 1]$. The set T and the function f clearly have the desired properties.

Note 1. Theorem 1 clearly also holds if we assume that the set S is infinite and that the set T is countably infinite. It follows from Lemma 1 that if $x_0 \in \lambda^{\omega_1} S - \lambda S$, then the set $S - T$ is always infinite.

Now let us turn to the relation between convergence spaces and sequentially regular spaces.

Lemma 2. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular convergence space. Then $(\{x_n\}, x) \in \mathfrak{L}^*$ if and only if the sequence $\{f(x_n)\}$ converges to $f(x)$ for each $f \in \mathfrak{F}(L)$.*

The easy proof is omitted.

Lemma 3. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Let $\mathfrak{M}$ be the set of all pairs $(\{x_n\}, x)$ such that the sequence $\{f(x_n)\}$ converges to $f(x)$ whenever $f \in \mathfrak{F}(L)$. Then:*

(a) *$(L, \mathfrak{M}, \mu)$ is a multivalued convergence space.*

(b) *$(L, \mathfrak{M}, \mu)$ is a convergence space if and only if the space $(L, \mathfrak{L}, \lambda)$ has the following property:*

(P) If $x \neq y$, then there is a function $f \in \mathfrak{F}(L)$ such that $f(x) \neq f(y)$.

Proof. The easy proof of (a) is omitted. To prove (b) observe that $(\{x_n\}, x) \in \mathfrak{M}$ and $(\{x_n\}, y) \in \mathfrak{M}$ if and only if $f(x) = \lim f(x_n) = f(y)$ for each $f \in \mathfrak{F}(L)$.

The multivalued convergence space $(L, \mathfrak{M}, \mu)$ will be said to be generated by $\mathfrak{F}(L)$.

Example 1. Let $L = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn} \cup (x) \cup (y)$. Let $(\{z\}, z) \in \mathfrak{L}$ for each $z \in L$, $(\{x_{mn_i}\}, x) \in \mathfrak{L}$ for each $m \in N$ and each subsequence $\{n_i\}$ of $\{n\}$, and $(\{x_{m_i n}\}, y) \in \mathfrak{L}$ for each $n \in N$ and each subsequence $\{m_i\}$ of $\{n\}$. This is a well-known example of a convergence space which is not separated. The multivalued convergence space $(L, \mathfrak{M}, \mu)$ generated by $\mathfrak{F}(L)$ is not a convergence space.

Note 2. M. DOLCHER has defined in [2] several successively weaker forms of the axiom $(\mathscr{L}_0) : (\mathscr{L}_0) = FKT_2,\ FKT_1,\ FKT_0'',\ FKT_0'$. The multivalued convergence space generated by $\mathfrak{F}(L)$ either satisfies $(\mathscr{L}_0)$ or does not satisfy even the weakest axiom FKT_0': If $x \neq y$, then there is a sequence $\{x_n\}$ such that it does not converge to both points x and y.

Definition 1. Let $(L, \mathfrak{L}, \lambda)$ be a convergence space which has the property (P). The convergence space generated by $\mathfrak{F}(L)$ will be denoted by $(L, \hat{\mathfrak{L}}, \hat{\lambda})$ and the convergence topology $\hat{\lambda}$ will be called a *sequentially regular modification* of λ.

Theorem 2. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space which has the property (P) and let $(L, \hat{\mathfrak{L}}, \hat{\lambda})$ be the convergence space generated by $\mathfrak{F}(L)$. Then:*

(a) $\mathfrak{L} \subset \hat{\mathfrak{L}}$; *consequently* $\lambda < \hat{\lambda}$.

(b) $\mathfrak{F}(L) = \hat{\mathfrak{F}}(L)$.

(c) *The space $(L, \hat{\mathfrak{L}}, \hat{\lambda})$ is sequentially regular.*

(d) $\hat{\mathfrak{L}} = \mathfrak{L}^*$ *and* $\hat{\lambda} = \lambda$ *if and only if $(L, \mathfrak{L}, \lambda)$ is sequentially regular.*

Proof. The easy proof of (a) and (b) is omitted. Since $(\{x_n\}, x) \in \hat{\mathfrak{L}}$ if and only if $\lim f(x_n) = f(x)$ for each $f \in \mathfrak{F}(L)$, it follows that $(\{x_n\}, x) \notin \hat{\mathfrak{L}}$ implies the existence of a function $f \in \hat{\mathfrak{F}}(L)$ such that $\{f(x_n)\}$ does not converge to $f(x)$. Hence (c) holds. $\lambda = \hat{\lambda}$ implies the sequential regularity of $(L, \mathfrak{L}, \lambda)$ in view of (c). Conversely, if the space $(L, \mathfrak{L}, \lambda)$ is sequentially regular, then $\hat{\mathfrak{L}} = \mathfrak{L}^*$, i.e. $\hat{\lambda} = \lambda$, by Lemma 2.

Corollary 1. *The sequentially regular modification $\hat{\lambda}$ is the weakest of all sequentially regular convergence topologies stronger than λ.*

Example 2. Let $L = \langle 0, 1 \rangle$. Define $\mathfrak{M}$ as follows: $(\{x_n\}, x) \in \mathfrak{M}$ whenever $\lim |x_n - x| = 0$. Let $\mathfrak{L}$ be the set of all pairs $(\{x_n\}, x) \in \mathfrak{M}$ such that $\{x_n\}$ does not contain a subsequence of $\{1/n\}$ and let μ and λ be the corresponding convergence topologies. Then $(L, \mathfrak{M}, \mu)$ is a sequentially regular space, $\hat{\mathfrak{L}} = \mathfrak{M}$, $\hat{\lambda} = \mu$, and $\lambda \neq \hat{\lambda}$.

Lemma 4. *Let $(L, \mathfrak{L}_1, \lambda_1)$ and $(L, \mathfrak{L}_2, \lambda_2)$ be convergence spaces which have the property (P). Then $\hat{\lambda}_1 = \hat{\lambda}_2$ if and only if $\mathfrak{F}((L, \mathfrak{L}_1, \lambda_1)) = \mathfrak{F}((L, \mathfrak{L}_2, \lambda_2))$.*

The easy proof is omitted.

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Now we shall consider the relation between sequentially regular convergence spaces and completely regular spaces. The following definition is based on a suggestion made by Prof. M. KATĚTOV.

Definition 2. Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. The weakest of all completely regular topologies[2]) for L which are stronger than λ will be called a *completely regular modification* of λ and denoted by $\tilde{\lambda}$.

Theorem 3. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. The completely regular modification $\tilde{\lambda}$ of λ exists if and only if the space $(L, \mathfrak{L}, \lambda)$ has the property (P). Furthermore*:

(a) A function f on $(L, \tilde{\lambda})$ to $\langle 0, 1 \rangle$ is continuous if and only if $f \in \mathfrak{F}(L)$.
(b) If $A \subset L$, then $\tilde{\lambda}A = \{y\colon \text{if } f \in \mathfrak{F}(L) \text{ and } f(x) = 0 \text{ for } x \in A, \text{ then } f(y) = 0\}$.

Proof. If the convergence space $(L, \mathfrak{L}, \lambda)$ has the property (P), then it follows from [1] (the proof of theorem 8.4.4.[3])) that there is a completely regular topology u for L; the system of all sets of the form $f^{-1}(I)$, where $f \in \mathfrak{F}(L)$ and I is an open interval, is a base for u. Since each set $f^{-1}(I)$ is open in $(L, \mathfrak{L}, \lambda)$ it follows that $\lambda < u$. Hence, if f is a continuous function on (L, u) to $\langle 0, 1 \rangle$, then $f \in \mathfrak{F}(L)$. Conversely, if $f \in \mathfrak{F}(L)$, then f is continuous on (L, u) because of the definition of the base for u. Hence u has property (a). Now suppose that v is a completely regular topology for L and that $\lambda < v$. Let $A \subset L$ and $x_0 \in uA$. It follows from property (a) of u that, if f is a continuous function on (L, v) to $\langle 0, 1 \rangle$ and such that $f(x) = 1$ for $x \in A$, then $f(x_0) = 1$. Since v is completely regular, we have $x_0 \in vA$. Consequently $u < v$ and $\tilde{\lambda} = u$. Hence $\tilde{\lambda}$ has property (a). Property (a) together with the complete regularity of $\tilde{\lambda}$ imply (b).

If the convergence space $(L, \mathfrak{L}, \lambda)$ has not property (P), then it follows that no completely regular topology for L is stronger than λ.

Corollary 2. *Each sequentially regular convergence topology has a completely regular modification.*

Note 3. If $(L, \mathfrak{L}, \lambda)$ is a sequentially regular space, then $\lambda < \lambda^{\omega_1} < \tilde{\lambda}$. The example of a space for which $\lambda \neq \lambda^{\omega_1} \neq \tilde{\lambda}$ will be mentioned later (Example 6.).

Definition 3. Let (P, u) be a separated topological space. Define a convergence $\mathfrak{P}$ on P as follows: $(\{x_n\}, x) \in \mathfrak{P}$ if each neighbourhood of x contains nearly all[4]) points

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[2]) In this paper a completely regular topology is always understood to be a separated topology.

[3]) **Theorem 8.4.4.** Let P be any closure space. Then there is a completely regular topological space P_1 and a continuous mapping ϱ on P into P_1 such that each continuous function on P is a composition of the mapping ϱ and a continuous function on P_1.

[4]) A proposition is true for nearly all $n \in N$ if there is $n_0 \in N$ such that the proposition is true for all $n \geqq n_0$.

of $\{x_n\}$. Denote π the corresponding convergence topology. The convergence space $(P, \mathfrak{P}, \pi)$ will be called a convergence space *associated* with the space (P, u).

It is well known that $\mathfrak{P} = \mathfrak{P}^*$ and $\pi < u$.

Theorem 4. *A convergence space $(P, \mathfrak{P}, \pi)$ associated with a completely regular space (P, u) is sequentially regular.*

Proof. Suppose that $(\{x_n\}, x) \notin \mathfrak{P}$. Then there is a u-neighbourhood U of x and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\bigcup_{i=1}^{\infty} x_{n_i} \subset P - U$. It follows from the complete regularity of u that there is a continuous function f on (P, u) to $\langle 0, 1\rangle$ such that the sequence $\{f(x_n)\}$ does not converge to $f(x)$. Since $\pi < u$, we have $f \in \mathfrak{F}(P)$ and the proof is complete.

Lemma 5. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space which has property (P). Let $\hat{\lambda}$ be the sequentially regular modification of λ. Then $\tilde{\hat{\lambda}} = \tilde{\lambda}$.*

Proof. The assertion follows directly from the statement (b) of Theorem 2.

Theorem 5. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space which has property (P). Let $\hat{\lambda}$ $(\tilde{\lambda})$ be the sequentially regular (completely regular) modification of λ. Let $(L, \mathfrak{M}, \mu)$ be the convergence space associated with $(L, \tilde{\lambda})$. Then $\mathfrak{M} = \hat{\mathfrak{L}}$ and therefore $\mu = \hat{\lambda}$.*

Proof. Let $(\{x_n\}, x) \in \mathfrak{M}$ and let $f \in \mathfrak{F}(L)$. Suppose that the sequence $\{f(x_n)\}$ does not converge to $f(x)$. Then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and an open interval I such that $f(x) \in I$ and $\bigcup_{i=1}^{\infty} x_{n_i} \subset L - f^{-1}(I)$. This contradicts the definition of $\mathfrak{M}$. Therefore $\lim f(x_n) = f(x)$ and hence $(\{x_n\}, x) \in \hat{\mathfrak{L}}$. To prove the converse suppose that $(\{y_n\}, y) \in \hat{\mathfrak{L}}$ and let U be any $\tilde{\lambda}$-neighbourhood of y. Then there is a continuous function f on $(L, \tilde{\lambda})$ into $\langle 0, 1\rangle$ and an open interval I such that $y \in f^{-1}(I) \subset U$. By Theorem 3 we have $f \in \mathfrak{F}(L)$ and therefore $\lim f(y_n) = f(y)$. It follows that $(\{y_n\}, y) \in \mathfrak{M}$.

The statement (d) of Theorem 2 and Theorem 5 imply the following

Corollary 3. *Let $(L, \mathfrak{L}_1, \lambda_1)$ and $(L, \mathfrak{L}_2, \lambda_2)$ be sequentially regular spaces and let $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ be the corresponding completely regular modifications of λ_1 and λ_2, respectively. Then $\tilde{\lambda}_1 = \tilde{\lambda}_2$ if and only if $\lambda_1 = \lambda_2$.*

In view of statement (d) of Theorem 2, Theorem 5 and Theorem 4 we have

Corollary 4. *The class of sequentially regular spaces whose convergences are largest is exactly the class of convergence spaces associated with completely regular spaces.*

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The class of all completely regular spaces whose topology is a completely regular modification of some convergence topology will be denoted by **P**. It is natural to ask whether every completely regular space is a member of **P**. The negative answer is supplied by the following

Example 3. Let $P = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn} \cup (x_0)$. The points x_{mn}, $m \in N$, $n \in N$, are isolated. The complete collection of neighbourhoods of the point x_0 is the family of sets $\bigcup_{m=k}^{\infty} \bigcup_{n=r(m)}^{\infty} x_{mn} \cup (x_0)$ where $k \in N$ and r is any mapping of N into itself. The space (P, u) is clearly completely regular and the space $(P, \mathfrak{P}, \pi)$ associated with (P, u) is discrete. Therefore (P, π) is completely regular and hence $\tilde{\pi} = \pi$. According to Lemma 5 and Theorem 5 there does not exist a convergence topology λ for P such that $\tilde{\lambda} = u$. Consequently (P, u) is not a member of the class **P**.

This example shows that a sequentially regular space $(P, \mathfrak{P}, \pi)$ can be associated simultaneously with completely regular spaces (P, u_1) and (P, u_2) while $u_1 \neq u_2$. The situation is different when the spaces (P, u_1) and (P, u_2) are members of **P**.

Lemma 6. *If a completely regular space (P, u) is a member of* **P** *and if the convergence space $(P, \mathfrak{P}, \pi)$ is associated with (P, u), then $u = \tilde{\pi}$.*

Proof. The assertion follows immediately from Theorem 5 and Corollary 3.

Corollary 5. *Let (P, u_1) and (P, u_2) be members of the class* **P** *and let $(P, \mathfrak{P}_1, \pi_1)$ and $(P, \mathfrak{P}_2, \pi_2)$ be the corresponding convergence spaces associated with them. Then $\pi_1 = \pi_2$ if and only if $u_1 = u_2$.*

Theorem 6. *Let (P, u) be a completely regular space. The space (P, u) is a member of the class* **P** *if and only if the following condition is satisfied:*

A function f on (P, u) to $\langle 0, 1 \rangle$ is continuous if and only if $\lim f(x_n) = f(x)$ whenever for each neighbourhood U of x we have $x_n \in U$ for nearly all $n \in N$.

Proof. Denote $(P, \mathfrak{P}, \pi)$ the convergence space associated with (P, u).

If (P, u) is a member of **P**, then according to Lemma 6 we have $u = \tilde{\pi}$ and the assertion follows by Theorem 3.

To prove the converse observe that, according to Theorem 3, the family of sets $f^{-1}(I)$, where $f \in \mathfrak{F}(P)$ and I is an open interval, is a base for $\tilde{\pi}$ and u simultaneously. Consequently $u = \tilde{\pi}$ and (P, u) is a member of **P**.

If a space (P, u) is a member of **P** and at the same time a convergence space, then any subspace $(Q, u \mid Q)$ of (P, u) is again a member of **P**. This is not true in general as it is shown in the following

Example 4. Let $P = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn} \cup \bigcup_{m=1}^{\infty} x_m \cup (x_0)$. The points x_{mn}, $m \in N$, $n \in N$, are isolated. The complete collection of neighbourhoods of the point x_m, $m \in N$, is the family of sets $\bigcup_{n=k}^{\infty} x_{mn} \cup (x_m)$, $k \in N$. The complete collection of neighbourhoods of the point x_0 is the family of sets $\bigcup_{m=l}^{\infty} \bigcup_{n=r(m)}^{\infty} x_{mn} \cup \bigcup_{m=l}^{\infty} x_m \cup (x_0)$ where $l \in N$ and r is any mapping of N into itself. The space (P, u) is completely regular. Denote $(P, \mathfrak{P}, \pi)$ the convergence space associated with (P, u). Then $u = \pi^2 = \pi^{\omega_1}$ and therefore $u = \tilde{\pi}$. Consequently the space (P, u) is a member of **P**. Let $Q = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn} \cup (x_0)$. It was shown in Example 3 that the space $(Q, u \mid Q)$ is not a member of **P**.

In considering whether a topological product of two members of **P** is a member of **P** we shall restrict our attention to convergence spaces. Let $(L_1, \mathfrak{L}_1, \lambda_1)$ and $(L_2, \mathfrak{L}_2, \lambda_2)$ be completely regular convergence spaces. Denote $(L, \mathfrak{L}, \lambda)$ their convergence product and (L, w) their topological product. It was shown in [8] that, if $(L_1, \mathfrak{L}_1, \lambda_1)$ satisfies the first axiom of countability and $(L_2, \mathfrak{L}_2, \lambda_2)$ does not contain a ϱ-point (in particular, if it satisfies the first axiom of countability), then $\lambda = w$ and hence $\tilde{\lambda} = w$. Consequently, (L, w) is a member of **P**.

Now we are going to present an example of two normal convergence spaces (one of which satisfies the second axiom of countability) whose topological product is not a member of **P**.

Example 5. Let $L_1 = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn} \cup (x_0)$. Define convergence $\mathfrak{L}_1$:

$(\{s_i\}, s) \in \mathfrak{L}_1$ if either $s_i = s \in L_1$, $i \in N$, or $s_i = x_{mn_i}$, $i \in N$, and $s = x_0$, where $\{n_i\}$ is any subsequence of $\{n\}$.

Let $L_2 = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} y_{mn} \cup (y_0)$. Define convergence $\mathfrak{L}_2$:

$(\{t_i\}, t) \in \mathfrak{L}_2$ if either $t_i = t \in L_2$, $i \in N$, or $t_i = y_{m_i r(m_i)}$, $i \in N$, and $t = y_0$, where $\{m_i\}$ is any subsequence of $\{n\}$ and r is any mapping of the set $\bigcup_{i=1}^{\infty} m_i$ into N.

Denote λ_1 and λ_2 the corresponding convergence topologies.

The spaces $(L_1, \mathfrak{L}_1, \lambda_1)$ and $(L_2, \mathfrak{L}_2, \lambda_2)$ are clearly normal topological spaces and the space $(L_2, \mathfrak{L}_2, \lambda_2)$ satisfies the second axiom of countability.

Let $(L, w) = (L_1, \lambda_1) \times (L_2, \lambda_2)$ be the topological product and let $(L, \mathfrak{L}, \lambda) = (L_1, \mathfrak{L}_1, \lambda_1) \times (L_2, \mathfrak{L}_2, \lambda_2)$ be the convergence product of the two spaces.

Let $A = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (x_{mn}, y_{mn}) \subset L$ and define a function f on L as follows: $f(z) = 0$, $z \in A$, $f(z) = 1$, $z \in L - A$. It is easy to prove that f is continuous on $(L, \mathfrak{L}, \lambda)$. On the other hand, note that the family of sets $\bigcup_{m=1}^{\infty} \bigcup_{n=r(m)}^{\infty} x_{mn} \cup (x_0) \times \bigcup_{m=k}^{\infty} \bigcup_{n=1}^{\infty} y_{mn} \cup (y_0)$, where $k \in N$ and r is any mapping of N into itself, is a complete collection of w-

neighbourhoods of (x_0, y_0). Hence f is not continuous on (L, w). By Theorem 3 it follows that $\tilde{\lambda} \neq w$ and, by Theorem 5, (L, w) is not a member of $\boldsymbol{P}$ since $(L, \mathfrak{L}, \lambda)$ is clearly associated with (L, w).

Note 4. J. NOVÁK asked in [6] whether the following two definitions of continuity of functions on the topological product (L, w) of convergence spaces $(L_1, \mathfrak{L}_1, \lambda_1)$ and $(L_2, \mathfrak{L}_2, \lambda_2)$ are equivalent:

(D_1) A function f is continuous on (L, w) if for each $(x, y) \in L$ and $\varepsilon > 0$ there are λ_1-neighbourhood U of x and λ_2-neighbourhood V of y such that $f(U \times V) \subset \subset (f(x, y) - \varepsilon, f(x, y) + \varepsilon)$.

(D_2) A function f is continuous on (L, w) if $\lim f(x_n, y_n) = f(x, y)$ whenever $\mathfrak{L}_1 - \lim x_n = x$ and $\mathfrak{L}_2 - \lim y_n = y$.

The example 5 shows that the answer is negative since the function f is continuous in the sense of (D_2) but is not continuous in the sense of (D_1) [5]).

4

J. Novák defined the notion of the sequential envelope $\sigma(L)$ of a sequentially regular convergence space L in [4].

The sequentially regular convergence space $(S, \mathfrak{S}, \sigma)$ is a sequential envelope of a convergence space $(L, \mathfrak{L}, \lambda)$ if the following conditions are satisfied:

(σ_0) $(L, \mathfrak{L}, \lambda)$ is a subspace of $(S, \mathfrak{S}, \sigma)$.

(σ_1) $S = \sigma^{\omega_1} L$.

(σ_2) Each function $f \in \mathfrak{F}(L)$ has an extension $\tilde{f} \in \mathfrak{F}(S)$.

(σ_3) There is no sequentially regular space $(S', \mathfrak{S}', \sigma')$ containing $(S, \mathfrak{S}, \sigma)$ as a proper subspace and fulfilling (σ_1) and (σ_2) relative to L and S'.

The sequential envelope of a sequentially regular space can be directly obtained by successive adjoining of "ideal points" to the given space. The following definition was suggested to me by Prof. J. Novák.

Definition 4. A sequence $\{x_n\}$ of points of a sequentially regular space $(L, \mathfrak{L}, \lambda)$ will be called *remarkable* if the sequence $\{f(x_n)\}$ is convergent for each $f \in \mathfrak{F}(L)$.

Lemma 7. *Every remarkable sequence in a sequentially regular space $(L, \mathfrak{L}, \lambda)$ is either $\mathfrak{L}^*$-convergent or totally $\mathfrak{L}^*$-divergent.*

Proof. If a remarkable sequence $\{x_n\}$ of points $x_n \in L$ is not totally $\mathfrak{L}^*$-divergent, then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $x \in L$ such that $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

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[5]) To find necessary and sufficient conditions under which the definitions (D_1) and (D_2) are equivalent remains an open problem.

Hence $\lim f(x_{n_i}) = f(x)$ for each $f \in \mathfrak{F}(L)$. Therefore $\lim f(x_n) = f(x)$ for each $f \in \mathfrak{F}(L)$ and the assertion follows by Lemma 2.

We define an equivalence relation in the set of all remarkable sequences in a sequentially regular space $(L, \mathfrak{L}, \lambda)$ as follows: $\{x_n\} \sim \{y_n\}$ whenever $\lim f(x_n) = \lim f(y_n)$ for each $f \in \mathfrak{F}(L)$. Denote $\mathscr{A}$ the family of all equivalence classes $[\{x_n\}]$ of remarkable sequences.

Lemma 8. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space, let $\{y_n\} \in [\{x_n\}]$ and let $(\{y_n\}, y) \in \mathfrak{L}^*$. Then $[\{x_n\}]$ is the set of all sequences $\{z_n\}$ such that $(\{z_n\}, y) \in \mathfrak{L}^*$.*

The easy proof is omitted.

Corollary 6. *The family $\mathscr{A}$ is the union of two disjoint families $\mathscr{B}$ and $\mathscr{C}$ where $\mathscr{B}$ is the family of those equivalence classes which contain exactly one constant sequence and $\mathscr{C}$ is the family of those classes which contain only totally $\mathfrak{L}^*$-divergent sequences.*

Theorem 7. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space. For each ordinal $\xi \leqq \omega_1$ there is a convergence space $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$ with the following properties:*

(a) $\mathfrak{L}_\xi = \mathfrak{L}_\xi^*$ *for each* $\xi \geqq 1$.
(b) $\mathfrak{L}_\eta \subset \mathfrak{L}_\xi$ *for each* $\eta \leqq \xi$.
(c) $(L_\eta, \mathfrak{L}_\eta, \lambda_\eta)$ *is a subspace of* $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$ *for each* $\eta \leqq \xi$.
(d) $L_\xi = \lambda_\xi^\xi L_0$.
(e) *For each $\eta \leqq \xi$ there is a one-to-one mapping h_η on $\mathfrak{F}(L_\xi)$ onto $\mathfrak{F}(L_\eta)$ such that $h_\eta(f) = f \mid L_\eta$ for each $f \in \mathfrak{F}(L_\xi)$.*
(f) *The space $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$ is sequentially regular.*

Proof. Let $(L_0, \mathfrak{L}_0, \lambda_0) = (L, \mathfrak{L}, \lambda)$. The conditions (b) through (f) are clearly satisfied for $\xi = 0$. Suppose that the spaces $(L_\eta, \mathfrak{L}_\eta, \lambda_\eta)$ with required properties are already defined for each $\eta < \xi \leqq \omega_1$.

I. Let $\xi = \zeta + 1$. The space $(L_\zeta, \mathfrak{L}_\zeta, \lambda_\zeta)$ is sequentially regular by (f). Let $\mathscr{C}_\zeta$ be the family of all equivalence classes of remarkable sequences in L_ζ which contain only totally $\mathfrak{L}_\zeta^*$-divergent sequences. Let $L_\xi = L_\zeta \cup \mathscr{C}_\zeta$. Let $f \in \mathfrak{F}(L_\zeta)$. Define the extension $\tilde{f}$ of f on L_ξ as follows: $\tilde{f}(x) = f(x)$ for $x \in L_\zeta$, $\tilde{f}(x) = \lim f(x_n)$ for $x \in \mathscr{C}_\zeta$, $x = [\{x_n\}]$. Let $\tilde{\mathfrak{F}}$ be the family of all extensions of $f \in \mathfrak{F}(L_\zeta)$. Define the convergence $\mathfrak{L}_\xi$ on L_ξ as follows: $(\{x_n\}, x) \in \mathfrak{L}_\xi$ if $\lim \tilde{f}(x_n) = \tilde{f}(x)$ for each $\tilde{f} \in \tilde{\mathfrak{F}}$ (if $(\{x_n\}, y) \in \mathfrak{L}_\xi$ and $(\{x_n\}, z) \in \mathfrak{L}_\xi$, then $\tilde{f}(y) = \tilde{f}(z)$ for each $\tilde{f} \in \tilde{\mathfrak{F}}$ and therefore $y = z$; hence $\mathfrak{L}_\xi$ satisfies $(\mathscr{L}_0)$). Denote λ_ξ the corresponding convergence topology. It is easy to prove that $\tilde{\mathfrak{F}} = \mathfrak{F}(L_\xi)$ and that the space $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$ satisfies conditions (a) through (f).

II. Let ξ be a limiting ordinal. Let $L_\xi = \bigcup_{\eta < \xi} L_\eta$. Let $f \in \mathfrak{F}(L_0)$ and $x \in L_\xi$. Then there is the least $\zeta < \xi$ such that $x \in L_\zeta$. By (e) there is a unique function $g \in \mathfrak{F}(L_\zeta)$ such that $f = g \mid L_0$. Let $\tilde{f}(x) = g(x)$. Thus to each $f \in \mathfrak{F}(L_0)$ we have defined a unique extension $\tilde{f}$ on L_ξ. The convergence $\mathfrak{L}_\xi$ and the convergence topology λ_ξ are defined in the same way as in the case of an isolated ordinal, and again it is easy to see that the required conditions are satisfied.

Denote $(S, \mathfrak{S}, \sigma)$ the convergence space $(L_{\omega_1}, \mathfrak{L}_{\omega_1}, \lambda_{\omega_1})$.

Theorem 8. *The convergence space $(S, \mathfrak{S}, \sigma)$ is a sequential envelope of the space $(L, \mathfrak{L}, \lambda)$.*

Proof. The space $(S, \mathfrak{S}, \sigma)$ is sequentially regular by (f). Condition (σ_0) (cf. p. 240) is implied by (c), condition (σ_1) is implied by (d), and condition (σ_2) is implied by (e).

Suppose that condition (σ_3) is not satisfied. Then there is a sequentially regular space $(M, \mathfrak{M}, \mu)$ containing $(S, \mathfrak{S}, \sigma)$ as a subspace and satisfying conditions (σ_1) and (σ_2) while $M - S \neq \emptyset$. By (σ_1) we have $M = \mu^{\omega_1}L$. Let ξ be the least ordinal such that $\mu^\xi L - S \neq \emptyset$. Since $L \subset S$ we have $\xi > 0$. Let $a \in \mu^\xi L - S$. Then $a \in \mu^\xi L - \mu^{\xi-1}L$ and there is a sequence $\{x_n\}$ of points $x_n \in \mu^{\xi-1}L$, $n \in N$, such that $(\{x_n\}, a) \in \mathfrak{M}$. Since $\xi - 1 < \xi$, we have $x_n \in S$, $n \in N$, and there is $\eta < \omega_1$ such that $x_n \in L_\eta$, $n \in N$. It is easy to see that $\{x_n\}$ is a remarkable sequence in L_η and therefore there is $b \in L_{\eta+1}$ such that $(\{x_n\}, b) \in \mathfrak{L}_{\eta+1}$. It follows that $(\{x_n\}, b) \in \mathfrak{S}$. This is a contradiction since clearly $a \neq b$. Therefore condition (σ_3) is also satisfied.

Lemma 9. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space and let $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$, $0 \leqq \xi \leqq \omega_1$, be the sequentially regular spaces defined in the proof of Theorem 7. For any ξ the space $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$ is a sequential envelope of the space $(L, \mathfrak{L}, \lambda)$ if and only if $\mathscr{C}_\xi = \emptyset$, i.e. if all remarkable sequences in L_ξ are $\mathfrak{L}_\xi^*$-convergent.*

Proof. The assertion follows from the construction of spaces $(L_\xi, \mathfrak{L}_\xi, \lambda_\xi)$ and from Theorem 8.

Definition 5. A sequentially regular space $(L, \mathfrak{L}, \lambda)$ will be called *$\mathscr{L}$-complete* if every remarkable sequence in L is $\mathfrak{L}^*$-convergent.

Theorem 9. *A sequentially regular space $(L, \mathfrak{L}, \lambda)$ is $\mathscr{L}$-complete if and only if $\sigma(L) = L$.*

Proof. The theorem follows from Lemma 9 if we consider the case of $\xi = 0$.

Corollary 7. *If a sequentially regular space $(L, \mathfrak{L}, \lambda)$ is either isolated or countably compact, then $\sigma(L) = L$.*

Theorem 10. *A sequentially regular space $(S, \mathfrak{S}, \sigma)$ is a sequential envelope of a convergence space $(L, \mathfrak{L}, \lambda)$ if and only if the following conditions are satisfied:*

(σ_0) *$(L, \mathfrak{L}, \lambda)$ is a subspace of $(S, \mathfrak{S}, \sigma)$.*

(σ_1) *$S = \sigma^{\omega_1}L$.*

(σ_2) *Each function $f \in \mathfrak{F}(L)$ has an extension $\tilde{f} \in \mathfrak{F}(S)$.*

(σ_3^*) *The space $(S, \mathfrak{S}, \sigma)$ is $\mathscr{L}$-complete.*

Proof. I. Suppose that the space $(S, \mathfrak{S}, \sigma)$ satisfies conditions (σ_0) through (σ_3^*) but that it is not a sequential envelope of $(L, \mathfrak{L}, \lambda)$. Then, by (σ_3), there is a sequentially regular space $(M, \mathfrak{M}, \mu)$ which contains the space $(S, \mathfrak{S}, \sigma)$ as a proper subspace and satisfies conditions (σ_1) and (σ_2) in respect to L. Condition (σ_1) implies that $M = \mu^{\omega_1}L$. Let ξ be the least ordinal such that $\mu^\xi L - S \neq \emptyset$; clearly $\xi > 0$. Let $a \in \mu^\xi L - S$. Then there is a sequence $\{x_n\}$ of points $x_n \in S \cap \mu^{\xi-1}L$, $n \in N$, such that $(\{x_n\}, a) \in \mathfrak{M}$. Let $f \in \mathfrak{F}(S)$. It follows, by (σ_2) for M, that there is an extension $\bar{f} \in \mathfrak{F}(M)$. Since $(\{x_n\}, a) \in \mathfrak{M}$, the sequence $\{\bar{f}(x_n)\}$ and therefore also the sequence $\{f(x_n)\}$ are convergent. Consequently $\{x_n\}$ is a remarkable sequence in S and, by (σ_3^*), it is $\mathfrak{S}^*$-convergent, i.e. there is $b \in S$ such that $(\{x_n\}, b) \in \mathfrak{S}^*$. This is a contradiction since $a \neq b$.

II. To prove the converse suppose that the space $(S, \mathfrak{S}, \sigma)$ is a sequential envelope of $(L, \mathfrak{L}, \lambda)$ but that it does not satisfy condition (σ_3^*). Let $(M, \mathfrak{M}, \mu)$ be the sequential envelope of $(S, \mathfrak{S}, \sigma)$. Then we have $M \neq S$ by Theorem 9. It is easy to prove that the space $(M, \mathfrak{M}, \mu)$ satisfies conditions (σ_0) through (σ_2) with respect to $(L, \mathfrak{L}, \lambda)$. This is a contradiction by (σ_3).

Corollary 8. *If $(L, \mathfrak{L}, \lambda)$ is a sequentially regular space, then $\sigma(\sigma(L)) = \sigma(L)$.*

The following example of a sequentially regular space $(L, \mathfrak{L}, \lambda)$ for which $\sigma(L) \neq L$ is constructed in [7].

Let $L = \bigcup_{\alpha \leqq \omega_1} \bigcup_{k=1}^{\infty} (\alpha, k)$; $(\{z\}, z) \in \mathfrak{L}$ for each $z \in L$, $(\{(\alpha_n, 1)\}, (\alpha, 1)) \in \mathfrak{L}$ whenever $\lim \alpha_n = \alpha$, $(\{(\alpha_n, k)\}, (\omega_1, k)) \in \mathfrak{L}$ whenever $k > 1$ and $\alpha_m \neq \alpha_n$ for $m \neq n$, and $(\{\alpha, k_j)\}, (\alpha, 1)) \in \mathfrak{L}$ for each $\alpha \leqq \omega_1$ and for each subsequence $\{k_j\}$ of $\{n\}$. Let $L' = L - (\omega_1, 1)$. Then $\sigma(L') = L$.

Since the space $(L, \mathfrak{L}, \lambda)$ is $\{0, 1\}$ sequentially regular, it follows (see [7]) that it can be realized by a convergence system of sets. The folowing is an example of such a system and it will be used to show the existence of a convergence ring **P** for which $\sigma(\mathbf{P}) \neq \mathbf{P}$.

Example 6. Let $R = X \cup Y$ where X and Y are disjoint sets of power $\aleph_1$ and $2^{\aleph_0}$ respectively. Let $\mathfrak{R}$ be the usual convergence of sequences of sets in **R** and let ϱ be the corresponding convergence topology.

For each $\alpha \leqq \omega_1$ let X_α be a countably infinite subset of X such that $X_\beta \cap X_\gamma = \emptyset$ whenever $\beta \neq \gamma$. For each $\alpha \leqq \omega_1$ arrange the points of X_α into a one-to-one sequence $\{x_{\alpha n}\}$ so that $X_\alpha = \bigcup_{n=1}^{\infty} x_{\alpha n}$.

Let $S = \{(\alpha, k) : \alpha < \omega_1,\ k = 2, 3, \ldots\}$. Let $T = \{\{\xi_n, k_n\}\}$ be the set of all sequences of points of S such that both sequences $\{\xi_n\}$ and $\{k_n\}$ are one-to-one. Let $\mathscr{U} = \{V : V = \bigcup_{n=1}^{\infty} (\xi_n, k_n),\ \{\xi_n, k_n\} \in T\}$. Since $P(\mathscr{U}) = 2^{\aleph_0}$ there is a one-to-one mapping ψ of the family $\mathscr{U}$ onto the set Y.

Let $\beta < \omega_1$ be an ordinal. To each $\alpha \leqq \beta$ we assign a positive integer $n(\alpha, \beta)$ in the following way:

If the ordinal β is finite, then $n(\alpha, \beta) = \alpha + 1$ for each $\alpha \leqq \beta$.

If the ordinal β is not finite, then arrange the points of the set $W(\beta + 1) = \{\xi : \xi \leqq \beta\}$ into a fixed sequence $\{\xi_n^\beta\}$; for each $\alpha \leqq \beta$ let $n(\alpha,\beta)$ be an integer for which $\xi^\beta_{n(\alpha,\beta)} = \alpha$.

Define the convergence space $(\mathbf{M}, \mathfrak{M}, \mu)$.

$\mathbf{M} = \{A_{\alpha k} : \alpha \leqq \omega_1, k \in N\}$ where $A_{\alpha k}$ are the following subsets of R:

for $\alpha < \omega_1$, $k > 1$: $A_{\alpha k} = Z_k \cup B_{\alpha k} \cup \{y : y \in Y, (\alpha, k) \in \psi^{-1}(y)\}$,

for $\alpha = \omega_1$, $k > 1$: $A_{\omega_1 k} = Z_k$,

for $\alpha \leqq \omega_1$, $k = 1$: $A_{\alpha_1} = \bigcup_{\beta \geqq \alpha} X_\beta$,

where $Z_k = \bigcup_{i=1}^{k} x_{\omega_1 i}$ and $B_{\alpha k} = \bigcup_{\alpha \leqq \beta < \omega_1} \bigcup_{m=1}^{k-n(\alpha,\beta)} x_{\beta m}$ $\left(\bigcup_{m=1}^{i} x_{\beta m} = \emptyset \text{ for } i < 1 \text{ and } \beta < \omega_1\right)$.

It is clear that $A_{\alpha k} \neq A_{\beta j}$ for $(\alpha, k) \neq (\beta, j)$.

Let $\mathfrak{M} = \mathfrak{R}_{\mathbf{M}}$ (partial convergence [7]) and let $\mu = \varrho \mid \mathbf{M}$. Denote $\mathbf{M}' = \mathbf{M} - (A_{\omega_1 1})$.

Let φ be a mapping on $(L, \mathfrak{L}, \lambda)$ onto $(\mathbf{M}, \mathfrak{M}, \mu)$ such that $\varphi((\alpha, k)) = A_{\alpha k}$. It can be proved that the mapping φ is a homeomorphism. (Note that $\{B_{\alpha k}\}$ is an increasing sequence for each $\alpha < \omega_1$. It is also easy to see that, for any $k > 1$, $\bigcap_{\alpha \in S} B_{\alpha k} = \emptyset$ whenever the set S is infinite.) Consequently $\sigma(\mathbf{M}') = \mathbf{M}$.

Denote $\mathbf{P} = \mathbf{R}(\mathbf{M}')$ the ring generated by $\mathbf{M}'$ and $(\mathbf{P}, \mathfrak{P}, \pi)$ the corresponding convergence subspace of $(\mathbf{R}, \mathfrak{R}, \varrho)$. We will show that $\sigma(\mathbf{P}) \neq \mathbf{P}$. By Theorem 9 it is sufficient to show that the space $(\mathbf{P}, \mathfrak{P}, \pi)$ is not $\mathscr{L}$-complete, i.e. that there is a totally $\mathfrak{P}$-divergent remarkable sequence in $\mathbf{P}$. We will prove that $\{A_{\omega_1 k}\}$ is such a sequence.

Let $f \in \mathfrak{F}(\mathbf{P})$. Denote $g = f \mid \mathbf{M}' \in \mathfrak{F}(\mathbf{M}')$. Since $\{A_{\omega_1 k}\}$ is a remarkable sequence in $\mathbf{M}'$, it follows that the sequence $\{g(A_{\omega_1 k})\}$ and consequently also the sequence $\{f(A_{\omega_1 k})\}$ are convergent. Hence $\{A_{\omega_1 k}\}$ is a remarkable sequence in $\mathbf{P}$.

Since $\mathfrak{R} - \lim A_{\omega_1 k} = A_{\omega_1 1}$, it remains to prove that $A_{\omega_1 1} \notin \mathbf{P}$. Let $C \in \mathbf{P}$ and suppose that $A_{\omega_1 1} \subset C$. We will prove that $C - A_{\omega_1 1} \neq \emptyset$. Since $C \in \mathbf{P} = \mathbf{R}(\mathbf{M}')$ we have $C = \mathop{\triangle}_{i=1}^{r} \bigcap_{j=1}^{s_i} C_{ij}$[6]) where $C_{ij} \in \mathbf{M}'$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s_i$. Denote $C_i = \bigcap_{j=1}^{s_i} C_{ij}$, $i = 1, 2, \ldots, r$, so that $C = \mathop{\triangle}_{i=1}^{r} C_i$. We can assume, without loss of generality, that there is $p \leqq r$ and $\alpha(i, j) < \omega_1$ such that $C_{ij} = A_{\alpha(i,j)1}$, $i \leqq p, j = 1, 2, \ldots, s_i$, while $C_i \subset A_{\alpha_i k_i}$, $i > p$, where $k_i > 1$. It follows that $C_i = A_{\alpha_i 1}$, $i \leqq p$, where $\alpha_i =$

[6]) $\mathop{\triangle}_{i=1}^{r} C_i$ denotes the symmetric difference of the sets $C_1, C_2, \ldots, C_r$, i.e. the set of all points belonging precisely to an odd number of the sets C_i.

$= \max_{1 \leq j \leq s_i} (\alpha(i, j))$. Since the set $\bigwedge_{i=p+1}^{r} C_i$ can contain only a finite number of points $x_{\omega_1 n}$ and $A_{\omega_1 1} \subset C$, it follows that $p \geqq 1$. Consequently $A_{\beta 1} \subset \bigwedge_{j=1}^{p} C_i$ where $\beta = \max_{1 \leqq i \leqq p} \alpha_i$. On the other hand $x_{\beta m} \in \bigcup_{i=p+1}^{r} C_i$ for a finite number of m only. Hence there is m_0 such that $x_{\beta m_0} \notin \bigcup_{i=p+1}^{r} C_i$ and therefore $x_{\beta m_0} \in C$. It follows that $C \neq A_{\omega_1 1}$ and hence $A_{\omega_1 1} \notin \mathbf{P}$.

The space $(\mathbf{M}, \mathfrak{M}, \mu)$ is also an example of a convergence space for which $\mu \neq \mu^{\omega_1} \neq \tilde{\mu}$ (cf. Note 3 p. 236). If we denote $\mathbf{B} = \{A_{\alpha k} : \alpha < \omega_0, k > 1\}$, then $A_{\omega_0 1} \in \mu^{\omega_1}\mathbf{B} - \mu\mathbf{B}$ and therefore $\mu^{\omega_1} \neq \mu$. It is also easy to see that the point $A_{\omega_1 1}$ and the closed set $\{A_{\alpha 1} : \alpha < \omega_1\}$ cannot be separated by open neighbourhoods and consequently the space $(\mathbf{M}, \mu^{\omega_1})$ is not regular. Hence $\tilde{\mu} \neq \mu^{\omega_1}$.

It is pointed out in [4] that the definition of a sequential envelope $\sigma(L)$ is similar to that of the Stone-Čech compactification $\beta(L)$ but that the spaces $\sigma(L)$ and $\beta(L)$ can be completely different. Now we are going to examine their relation more closely.

Theorem 11. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space, let $\tilde{\lambda}$ be the completely regular modification of λ and let (P, u) be the Stone-Čech compactification of $(L, \tilde{\lambda})$. Let $(P, \mathfrak{P}, \pi)$ be the convergence space associated with (P, u). Then $\sigma(L) = \pi^{\omega_1}L$.*

Proof. Let φ be a special homeomorphism on $(L, \mathfrak{L}, \lambda)$ into the convergence cube space $(C, \mathfrak{C}, \gamma)$ of the dimension $P(\mathfrak{F}(L))$, i.e. $\varphi(x) = (f_\alpha(x)), x \in L, \mathfrak{F}(L) = \{f_\alpha : \alpha \in I\}$. Let v be the usual product topology for C. Then by Theorem 3 it follows that φ is a homeomorphism on $(L, \tilde{\lambda})$ onto the subspace $(\varphi(L), v|\varphi(L))$ of (C, v). Since $v\,\varphi(L) = \beta(\varphi(L))$, it follows that there is a homeomorphism h' on (P, u) onto $(v\,\varphi(L), v|v\,\varphi(L))$ such that $h'|_L = \varphi$. Let $A = \pi^{\omega_1}L$, $B = \gamma^{\omega_1}\,\varphi(L)$ and $h = h'|_A$. Then h is a homeomorphism on $(A, \mathfrak{P}_A, \pi \mid A)$ onto $(B, \mathfrak{C}_B, \gamma \mid B)$ such that $h(x) = \varphi(x)$ for $x \in L$. The assertion of the theorem follows by Theorem 13 in [7].

Corollary 9. *The sequential envelope of a sequentially regular space is the smallest sequentially closed subset in the Stone-Čech compactification of the corresponding completely regular space which contains the given space.*

Theorem 12. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space. If the space $(L, \tilde{\lambda})$ is normal, then $\sigma(L) = L$.*

Proof. Let (P, u) be the Stone-Čech compactification of $(L, \tilde{\lambda})$ and let $(P, \mathfrak{P}, \pi)$ be the convergence space associated with (P, u). According to Theorem 11 we have $\sigma(L) = \pi^{\omega_1}L$. Suppose that $\sigma(L) \neq L$. Then there is a one-to-one sequence $\{x_n\}$ of points $x_n \in L$, $n \in N$, and a point $x \in \pi L - L$ such that $(\{x_n\}, x) \in \mathfrak{P}$. According to $(\mathscr{L}_2)$ we have $(\{x_{n_i}\}, x) \in \mathfrak{P}$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. However, since the space $(L, \tilde{\lambda})$ is normal, we have $\pi A \cap \pi B = \emptyset$ whenever $\tilde{\lambda}A \cap \tilde{\lambda}B = \emptyset$. This is a contradiction and the theorem follows.

On the other hand, if $\sigma(L) = L$, then it does not follow that the space $(L, \tilde{\lambda})$ is normal. Let us mention the following example.

Example 7. Let L be the subset of all points (x, y) of the Euclidean plane such that $y \geqq 0$. The topology λ for L is defined as follows: Let $(x_0, y_0) \in L$. If $y_0 > 0$, then the family of sets $U_n(x_0, y_0) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < 1/n^2\}$, where $n \in N$ and is such that $1/n < y_0$, is a complete collection of neighbourhoods of (x_0, y_0). If $y_0 = 0$, then the family of sets $U_n(x_0, 0) = \{(x, y) : (x - x_0)^2 + (y - 1/n)^2 < < 1/n^2\} \cup (x_0, 0)$, where $n \in N$, is a complete collection of neighbourhoods of $(x_0, 0)$. The space (L, λ) is a well-known example of a completely regular non-normal space. It is easy to see that space (L, λ) is a sequentially regular convergence space. Since $\tilde{\lambda} = \lambda$, the space $(L, \tilde{\lambda})$ is not normal but on the other hand $\sigma(L) = L$ by Theorem 9.

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Резюме

О СЕКВЕНЦИАЛЬНО РЕГУЛЯРНЫХ ПРОСТРАНСТВАХ СХОДИМОСТИ

ВАЦЛАВ КОУТНИК (Václav Koutník), Прага

В статье рассматриваются пространства сходимости, т.е. пространства, в которых операция замыкания определена посредством сходимости последовательностей.

Обозначим через $\mathfrak{F}(L)$ множество всех непрерывных действительных функций

таких, что $0 \leqq f(x) \leqq 1$ для всех $x \in L$. Пространство сходимости L называется секвенциально регулярным [3], если к каждой точке $x \in L$ и к каждой последовательности точек $\{x_n\}$, причем ниодна выбранная из нее подпоследовательность не сходится к точке x, существует функция $f \in \mathfrak{F}(L)$ такая, что последовательность $\{f(x_n)\}$ не сходится к $f(x)$.

В статье дается необходимое и достаточное условие для того, чтобы секвенциальная топология была секвенциально регулярной.

Пространство сходимости имеет свойство (P), если для $x \neq y$ существует $f \in \mathfrak{F}(L)$ такая, что $f(x) \neq f(y)$. В пространстве сходимости, которое имеет свойство (P), назовем секвенциальную топологию $\hat{\lambda}$ секвенциально регулярной модификацией секвенциальной топологии λ, если $\hat{\lambda}$ является слабейшей из всех секвенциально регулярных топологий более сильных чем λ.

Вполне регулярную топологию $\tilde{\lambda}$ назовем вполне регулярной модификацией секвенциальной топологии λ, если $\tilde{\lambda}$ является слабейшей из всех вполне регулярных топологий более сильных чем λ. Для того, чтобы существовала вполне регулярная модификация секвенциальной топологии необходимо и достаточно, чтобы было выполнено условие (P). Если λ – секвенциальная топология, то $\tilde{\tilde{\lambda}} = \tilde{\lambda}$.

Всякому отделимому пространству соответствует пространство сходимости, в котором последовательность $\{x_n\}$ сходится к точке x, если всякая окрестность точки x содержит почти все точки x_n. Секвенциально регулярная модификация $\hat{\lambda}$ секвенциональной топологии λ совпадает с секвенциальной топологией, соответствующей вполне регулярной модификации $\tilde{\lambda}$. Класс секвенциально регулярных пространств совпадает с классом пространств сходимости, соответствующих вполне регулярным пространствам.

Исследуется класс ***P*** вполне регулярных пространств, топология которых является вполне регулярной модификацией некоторой секвенциальной топологии. Дается необходимое и достаточное условие для того, чтобы вполне регулярное пространство принадлежало классу ***P***.

Показывается, что секвенциальную оболочку $\sigma(L)$ секвенциально регулярного пространства L можно получить постепенным расширением данного пространства.

Секвенциально регулярное пространство назовем $\mathscr{L}$-полным, если сходится каждая последовательность $\{x_n\}$ такая, что последовательность $\{f(x_n)\}$ сходится для каждой $f \in \mathfrak{F}(L)$. Показывается, что $\mathscr{L}$-полнота секвенциально регулярного пространства L является необходимым и достаточным условием для равенства $\sigma(L) = L$.

Исследуется соотношение между бикомпактным расширением вполне регулярного пространства. Секвенциальная оболочка секвенциально регулярного пространства является наименьшим секвенциально замкнутым подмножеством бикомпактного расширения соответствующего вполне регулярного пространства.