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Karel Koštál

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DIFFERENTIAL EQUATIONS OF STOCHASTIC PROCESSES WHICH HAVE DERIVATIVE IN QUADRATIC MEAN

KAREL KOŠŤÁL, Praha

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INTRODUCTION AND NOTATION

It is usually assumed in physics that the changes of state of physical systems proceed at a finite velocity and hence that the functions describing the state of the system have finite derivative with respect to time. This is true particularly about the motion of material particles whose velocity and acceleration as well are always finite; therefore, the particle position vector has always finite first and second derivative with respect to time. If physical processes of a causal character are involved, such as the motion of material points in classical mechanics, the state of the system is described by ordinary functions the derivatives of which are defined in a usual manner. On the other hand random processes are described by random functions; accordingly if we wish to examine, in the study of random physical processes, the changes of state in arbitrarily short time intervals, we must require that the appropriate random functions should have a derivative, too. As it is well known the derivative of a random function can be defined with the aid of some criterion of convergence of random variables. As there exist several such criteria, the question arises which one is acceptable for the physical random processes. It is frequently assumed in physics that the function describing the state of a system has finite derivative with respect to time in each realization of the process (e.g. a material particle has a finite velocity in any motion). This is the reason why we can, when defining the derivative of a random function describing a physical process, use any of the standard criteria of convergence. So far as the physical aspects are concerned, the clearest mode of convergence is that of convergence almost surely. But from the theoretical point of view, the most convenient is the convergence in quadratic mean; this is why we use it, and that exclusively, in this paper.

The assumption that the random functions dealt with in this paper have derivative in quadratic mean of the first or possibly higher orders is of cardinal significance when evolving the differential equations in the discussion that follows.The remaining assumptions to be stated in the context are the usual assumptions customarily used in physics on the continuity or existence of continuous derivatives of the respective functions. The equations which we shall derive could evidently be arrived at also through the use of another criterion of the existence of the derivative of random functions, were we to supplement suitably other assumptions. This problem will not, however, be dealt with in this paper. The connections between different modes of convergence of random variables are discussed in detail in e.g. [2].

To complete our exposition, let us mention the fact that random functions non-differentiable with respect to time, e.g. functions with independent increments, are frequently used in the examination of some physical processes, such as the Brownian motion. Such stochastic models, even though very convenient in that they facilitate the mathematical analysis, are always but an approximation of a real physical process, as they can be justified physically only if the changes of the state of a physical system are examined on a sufficiently coarse time scale, i.e. in time intervals far longer than a very short but finite interval. For shorter time intervals they cannot be considered even approximately valid [11]. Should we consider such a stochastic model of e.g. the Brownian motion valid also for the changes of position or velocity of a particle in arbitrarily short time intervals, we would have to admit that the velocity or acceleration of the particle can assume infinitely large values [8].

This paper is in continuance of papers [3], [7] and [10] by DEDEBANT, MOYAL and WEHRLÉ, which for the first time deal with the differential equations of stochastic processes which have first and second derivative in quadratic mean and with their applications in physics in particular in hydrodynamics and in kinetic theory. It presents a systematic derivation of these equations and of other differential equations and systems of differential equations of higher orders. It first discusses processes which have only first derivative in quadratic mean (§ 1), further processes which have first and second derivative (§ 2), and finally processes with derivatives generaly to the r-th order inclusive (§ 3). The differential equations derived in § 3 have a general validity; equations of processes which have first and second derivative are their special cases. However, we derive in detail these equations, too, because they, especially, are suitable for physical applications [9]. In its concluding part, the paper deals with analogous differential equations satisfied by transition probability densities. Under a certain assumption, these equations take on a simple form and in that case they became to some extent analogies of Kolmogorov equations for Markov processes continuous in time.

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Notation. Let $X(t)$ and $Y(t)$ be random functions. We shall denote the conditional expectation of $Y(t)$ relative to $X(t)$ at instant t by $E[Y(t) \mid X(t)]$ and $E[Y(t) \mid X(t)]_{X(t)=x}$, i.e. the conditional expectation of $Y(t)$ for $X(t) = x$ at instant t, by $E[Y(t) \mid X(t) = x]$. If $X(t)$ is a vector random function with components $X_1(t), \ldots, \ldots, X_n(t)$, we write $X(t) = x$ for $X_1(t) = x_1, \ldots, X_n(t) = x_n$ where $x = (x_1, \ldots, x_n)$.

For short we shall often use other symbols for some conditional expectations, e.g. $\bar{v}_i(t \mid x, t)$ for $E[\dot{X}_i(t) \mid X(t) = x]$, $\overline{v_i v_j}(t \mid x, t)$ for $E[\dot{X}_i(t)\dot{X}_j(t) \mid X(t) = x]$, $\bar{a}_i(t \mid x, t)$ for $E[\ddot{X}_i(t) \mid X(t) = x]$ etc., where $\dot{X}_i(t)$ and $\dot{X}_j(t)$ denote the first derivatives of random functions $X_i(t)$ and $X_j(t)$ respectively, and $\ddot{X}_i(t)$ denotes the second derivative of $X_i(t)$. The abbreviation q.m. will be used for "quadratic mean".

In the whole paper, we consider a function a continuous one only if it is simultaneously finite.

1. DIFFERENTIAL EQUATION OF STOCHASTIC PROCESSES WHICH HAVE FIRST DERIVATIVE IN QUADRATIC MEAN

Let $X(t) = [X_1(t), \ldots, X_n(t)]$ be a vector random function of continuous parameter (time) $t \in T = (-\infty, +\infty)$, whose components $X_1(t), \ldots, X_n(t)$ are real second order random functions (see e.g. [1], [2]) having first derivatives in q.m. $\dot{X}_1(t), \ldots, \dot{X}_n(t)$ on T, i.e. they fulfil the following condition:

$$(1) \qquad E[X_i(t)]^2 < +\infty\,, \quad \lim_{\Delta t \to 0} E\left[\frac{X_i(t + \Delta t) - X_i(t)}{\Delta t} - \dot{X}_i(t)\right]^2 = 0 \quad (i = 1,2 \ldots, n)$$

whatever be $t \in T$.

We shall denote the distribution function of $X(t)$ at one instant t by $F(x, t)$, $x = (x_1, \ldots, x_n) \in R_n = (-\infty, +\infty)^n$, its distribution function at two instants t and t' by $F_2(x, t; x', t')$ and the conditional expectation $E[\dot{X}_i(t) \mid X(t) = x]$ by $\bar{v}_i(t \mid x, t)$.

We shall assume that the distribution function $F(x, t)$ has a density, i.e. that there exists a Lebesgue measurable and integrable function f, for which

$$(2) \qquad F(x_1, \ldots, x_n, t) = \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} f(\xi_1, \ldots, \xi_n, t)\, d\xi_1, \ldots, d\xi_n$$

for all $(x_1, \ldots, x_n) \in R_n$, $t \in T$.

Theorem 1. *Let besides condition* (1) *the following assumptions be fulfilled*:

(a_1) *There exists the density $f(x, t)$ corresponding to $F(x, t)$, whose derivative $\partial f/\partial t$ is continuous on $R_n \times T$.*

(a_2) *Derivatives $(\partial/\partial x_i)[\bar{v}_i(t \mid x, t) f(x, t)]$ $(i = 1, 2, \ldots, n)$ are continuous in $(x_1, \ldots, x_n)$ on R_n for every $t \in T$.*

Then equation

$$(3) \qquad \frac{\partial f(x, t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\bar{v}_i(t \mid x, t) f(x, t)] = 0$$

holds for all $x \in R_n$, $t \in T$. [1])

[1]) The continuity equation (3) for stochastic processes differentiable once in q.m. was first given in communication [3].

Proof. We shall use a method similar to that used by KOLMOGOROV [4] to derive the second equation for Markov processes continuous in time.[2]) Introduce an auxiliary function $\varphi(x) = \varphi(x_1, \ldots, x_n)$ which has the following properties:

(P_1) Function $\varphi(x)$ is non-negative and differs from zero only in the bounded interval $M = (x; a_1 < x_1 < b_1, \ldots, a_n < x_n < b_n)$, i.e.

$$\varphi(x) > 0 \quad \text{if} \quad a_1 < x_1 < b_1, \ldots, a_n < x_n < b_n\,, \tag{4}$$
$$\varphi(x) = 0 \quad \text{if} \quad x_i \leqq a_i \quad \text{or} \quad x_i \geqq b_i \quad (i = 1, 2, \ldots, n)\,.$$

(P_2) Function $\varphi(x)$ as well as its partial derivatives $\varphi'_i = \partial\varphi/\partial x_i$ and $\varphi''_{ij} = \partial^2\varphi/\partial x_i\, \partial x_j$ $(i, j = 1, 2, \ldots, n)$ are continuous on R_n.

Since, by its definition, the density function $f(x, t)$ is Lebesgue measurable in R_n and for every $t \in T$ there holds $0 \leqq \int_M f(x, t)\, \mathrm{d}x \leqq 1$, $\mathrm{d}x = \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n$, and since function $\varphi(x)$ is continuous and bounded on $M \subset R_n$, integral $\int_M \varphi(x) f(x, t)\, \mathrm{d}x$ exists and is finite. Thus, making use of assumption (a_1) and of properties (P_1) and (P_2) of function $\varphi(x)$, we obtain

$$\int_M \varphi(x) \frac{\partial f(x, t)}{\partial t}\, \mathrm{d}x = \frac{\partial}{\partial t} \int_M \varphi(x) f(x, t)\, \mathrm{d}x = \tag{5}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{E[\varphi(X(t + \Delta t))] - E[\varphi(X(t))]\} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\varphi(X(t + \Delta t)) - \varphi(X(t))] =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{R_{2n}} [\varphi(x') - \varphi(x)]\, \mathrm{d}F_2(x, t; x', t + \Delta t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{R_{2n}} [\sum_{i=1}^{n} \varphi'_i(x)\, (x'_i - x_i) +$$

$$+ \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi''_{ij}(\xi)\, (x'_i - x_i)\, (x'_j - x_j)]\, \mathrm{d}F_2(x, t; x', t + \Delta t)\,,$$

where $\mathrm{d}x = \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n$, $R_{2n} = R_n \times R_n$, $\xi = (\xi_1, \ldots, \xi_n) = [x_1 + \Theta(x'_1 - x_1), \ldots, x_n + \Theta(x'_n - x_n)]$, $0 < \Theta < 1$.

Since $|\varphi''_{ij}(\xi)| \leqq C < +\infty$ $(i, j = 1, 2, \ldots, n)$ and, as follows from condition (1), $E[(X_i(t + \Delta t) - X_i(t))/\Delta t]^2 \to E[\dot{X}_i(t)]^2 < +\infty$, $\Delta t \to 0$ $(i = 1, 2, \ldots, n)$, we have by an immediate application of Schwarz's inequality

$$\left| \frac{1}{\Delta t} \int_{R_{2n}} \tfrac{1}{2}\varphi''_{ij}(\xi)\, (x'_i - x_i)\, (x'_j - x_j)\, \mathrm{d}F_2(x, t; x', t + \Delta t) \right| \leqq \tag{6}$$

$$\leqq \frac{C}{2} |\Delta t| \,.\, E \left| \frac{X_i(t + \Delta t) - X_i(t)}{\Delta t} \,.\, \frac{X_j(t + \Delta t) - X_j(t)}{\Delta t} \right| \leqq$$

[2]) We should like to emphasize, that stochastic processes differentiable in q.m. as considered in this paper are not assumed to be Markovian. On the other hand, processes for which the aforementioned Kolmogorov equation holds (see also [5] and [6]) are Markovian and, moreover, though continuous in time, have not generally derivatives in q.m.

$$\leqq \frac{C}{2}\left|\Delta t \,.\, E^{1/2}\left[\frac{X_i(t+\Delta t)-X_i(t)}{\Delta t}\right]^2 . \, E^{1/2}\left[\frac{X_j(t+\Delta t)-X_j(t)}{\Delta t}\right]^2\right| \to 0\,, \quad \Delta t \to 0\,.$$

Furthermore, since $|\varphi_i'(x)| \leqq K < +\infty$ $(i = 1, 2, \ldots, n)$, Schwarz's inequality and condition (1) yield

$$\text{(7)} \qquad \left|E\left\{\varphi_i'(X(t))\left[\frac{X_i(t+\Delta t)-X_i(t)}{\Delta t} - \dot{X}_i(t)\right]\right\}\right| \leqq$$

$$\leqq K \,.\, E^{1/2}\left[\frac{X_i(t+\Delta t)-X_i(t)}{\Delta t} - \dot{X}_i(t)\right]^2 \to 0\,, \quad \Delta t \to 0\,,$$

and hence

$$\text{(8)} \qquad E\left[\varphi_i'(X(t))\,\frac{X_i(t+\Delta t)-X_i(t)}{\Delta t}\right] \to E[\varphi_i'(X(t))\,\dot{X}_i(t)]\,, \quad \Delta t \to 0\,.$$

Finally, using eq. (8), assumptions (a_1) and (a_2) and properties (P_1) and (P_2) of function $\varphi(x)$, we obtain

$$\text{(9)} \qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{R_{2n}} \varphi_i'(x)\,(x_i' - x_i)\,\mathrm{d}F_2(x, t; x', t+\Delta t) =$$

$$= E[\varphi_i'(X(t))\,\dot{X}_i(t)] = E\{\varphi_i'(X(t))\,E[\dot{X}_i(t) \mid X(t)]\} =$$

$$= \int_{R_n} \varphi_i'(x)\,\bar{v}_i(t \mid x, t)\,f(x, t)\,\mathrm{d}x = \int_{a_1}^{b_1}\ldots\int_{a_n}^{b_n}\varphi_i'(x)\,\bar{v}_i(t \mid x, t)\,f(x, t)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n =$$

$$= -\int_M \varphi(x)\,\frac{\partial}{\partial x_i}\left[\bar{v}_i(t \mid x, t)\,f(x, t)\right]\mathrm{d}x\,.$$

(the final result in this equation was obtained by the application of Fubini's theorem, and of assumption (a_2) and properties (P_1), (P_2) used when integrating by parts).

Substituting the results of equations (6) and (9) in equation (5) we obtain

$$\text{(10)} \qquad \int_M \varphi(x)\,\frac{\partial f(x, t)}{\partial t}\,\mathrm{d}x = -\int_M \varphi(x) \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\bar{v}_i(t \mid x, t)\,f(x, t)\right]\mathrm{d}x\,.$$

As $\varphi(x)$ is an arbitrary function except for properties (P_1), (P_2) stated above, derivatives $\partial f(x, t)/\partial t$ and $(\partial/\partial x_i)\left[\bar{v}_i(t \mid x, t)\,f(x, t)\right]$ $(i = 1, 2, \ldots, n)$ are continuous on R_n for all $t \in T$ (cf. assumptions (a_1) and (a_2)) and as for each $x \in R_n$ we can choose a bounded interval $M \subset R_n$ such that $x \in M$, equation (10) implies equation (3) for all $x \in R_n$, $t \in T$.

2. DIFFERENTIAL EQUATIONS OF STOCHASTIC PROCESSES WHICH HAVE FIRST AND SECOND DERIVATIVE IN QUADRATIC MEAN

Let the components of the vector random function $X(t)$ described in § 1 have also second derivatives in q.m. $\ddot{X}_1(t), \ldots, \ddot{X}_n(t)$ on T, i.e. let the following condition be fulfilled in addition to condition (1):

$$(11)\quad \lim_{\Delta t \to 0} E\left[\frac{\dot{X}_i(t+\Delta t) - \dot{X}_i(t)}{\Delta t} - \ddot{X}_i(t)\right]^2 = 0 \quad (i = 1, 2, \ldots, n) \quad \textit{whatever be } t \in T.$$

Denote by $Y(t)$ a vector random function $[Y_1(t), \ldots, Y_{2n}(t)]$ with components $Y_1(t) = X_1(t), \ldots, Y_n(t) = X_n(t)$, $Y_{n+1}(t) = \dot{X}_1(t), \ldots, Y_{2n}(t) = \dot{X}_n(t)$ and assume, that there exists a density of the probability distribution of $Y(t)$, i.e. a density of the joint probability distribution of $X_1(t), \ldots, X_n(t)$, $\dot{X}_1(t), \ldots, \dot{X}_n(t)$, at one instant t, which we denote by $f(y, t)$, $y = (y_1, \ldots, y_{2n}) \in R_{2n}$.

It follows from Theorem 1 that if assumptions (a_1) and (a_2) in which we replace $f(x, t)$, $x = (x_1, \ldots, x_n)$, $\bar{v}_i(t \mid x, t) = E[\dot{X}_i(t) \mid X(t) = x]$ and $i = 1, 2, \ldots, n$, respectively, by $f(y, t)$, $y = (y_1, \ldots, y_{2n})$, $E[\dot{Y}_j(t) \mid Y(t) = y]$ and $j = 1, 2, \ldots, 2n$, respectively, are fulfilled in addition to conditions (1), (11), the density $f(y, t)$ satisfies the following equation:

$$(12)\qquad \frac{\partial f(y, t)}{\partial t} + \sum_{j=1}^{2n} \frac{\partial}{\partial y_j} \{E[\dot{Y}_j(t) \mid Y(t) = y] f(y, t)\} = 0\,.$$

This equation can further take a simpler form. Proceeding analogously as in equation (9) and putting $y_1 = x_1, \ldots, y_n = x_n$, $y_{n+1} = v_1, \ldots, y_{2n} = v_n$, $(x_1, \ldots, x_n) = x \in R_n$, $(v_1, \ldots, v_n) = v \in R_n$, we namely obtain for $1 \leqq j \leqq n$

$$(13)\qquad E[\varphi'_j(Y(t))\, \dot{Y}_j(t)] = \int_{M_{2n}} \frac{\partial \varphi(y)}{\partial y_j}\, y_{j+n} f(y, t)\, \mathrm{d}y =$$

$$= -\int_{M_{2n}} \varphi(y)\, y_{j+n} \frac{\partial f(y, t)}{\partial y_j}\, \mathrm{d}y = -\int_{M_{2n}} \varphi(x, v)\, v_j \frac{\partial f(x, v, t)}{\partial x_j}\, \mathrm{d}x\, \mathrm{d}v\,,$$

where $\varphi(y)$ is a function with properties analogous to properties (P_1) and (P_2) of function $\varphi(x)$, $M_{2n} \subset R_n \times R_n$ is a bounded interval in which function $\varphi(y)$ differs from zero and $\mathrm{d}y = \mathrm{d}y_1\, \mathrm{d}y_2 \ldots \mathrm{d}y_{2n}$. Further procedure, similarly as when deriving (3), leads to (12) in the form of

$$(14)\quad \frac{\partial f(x, v, t)}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial f(x, v, t)}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial}{\partial v_i} [\bar{a}_i(t \mid x, v, t) f(x, v, t)] = 0\,,$$

where by $a_i(t \mid x, v, t)$ is denoted the expectation $E[\ddot{X}_i(t) \mid X(t) = x, \dot{X}(t) = v]$.

Hence, we may state the following

Theorem 1*. *Let besides conditions* (1) *and* (11) *the following assumptions be fulfilled*:

(a_1^*) *There exists the density* $f(x, v, t) = f(x_1, \ldots, x_n, v_1, \ldots, v_n, t)$ *of the joint probability distribution of* $X_1(t), \ldots, X_n(t), \dot{X}_1(t), \ldots, \dot{X}_n(t)$ *whose derivative* $\partial f / \partial t$ *is continuous on* $R_n \times R_n \times T$ *and derivatives* $\partial f / \partial x_i$ $(i = 1, 2, \ldots, n)$ *are continuous in* $(x_1, \ldots, x_n, v_1, \ldots, v_n)$ *on* $R_n \times R_n$ *for every* $t \in T$.

(a_2^*) *Derivatives* $(\partial / \partial v_i) [\bar{a}_i(t \mid x, v, t) f(x, v, t)]$ $(i = 1, 2, \ldots, n)$ *are continuous in* $(x_1, \ldots, x_n, v_1, \ldots, v_n)$ *on* $R_n \times R_n$ *for every* $t \in T$.

Then equation (14) *holds for all* $x \in R_n$, $v \in R_n$, $t \in T$.

From equation (14) we can derive in a straightforward manner other differential equations satisfied, like equation (3), by density $f(x, t)$. The general procedure is carried out in papers [7] and [8]. Equation (14) is first multiplied by a chosen function of variables $x_1, \ldots, x_n, v_1, \ldots, v_n, t$, then integrated with respect to $v_1, \ldots, v_n$ over the whole range R_n. Under convenient assumptions, one can then carry out integration by parts and exchange the sequence of integration and differentiation in some of the integrals. Following simple rearrangements, we obtain a differential equation in which the density $f(x, t)$ and, in addition to it, only relevant conditional expectations for given $X(t)$ appear.

Even though the procedure is simple, we must assume, when applying it, that equation (14), satisfied by density $f(x, v, t)$ rather than by density $f(x, t)$, holds and that density $f(x, v, t)$ fulfils a number of additional conditions.

However, it is possible to derive equations of the type mentioned, satisfied by $f(x, t)$, without making use of $f(x, v, t)$, namely by a method similar to that used when deriving (3). We shall derive in this manner several equations which are of significance particularly in physical applications [9]. Assumptions which we shall use in doing so are simple and, so far as the physical aspects are concerned, ordinary.

We shall use abbreviations $\bar{v}_i(t \mid x, t)$, $\overline{v_i v_j}(t \mid x, t)$ and $\bar{a}_i(t \mid x, t)$ for $E[\dot{X}_i(t) \mid X(t) = x]$, $E[\dot{X}_i(t) \dot{X}_j(t) \mid X(t) = x]$ and $E[\ddot{X}_i(t) \mid X(t) = x]$, respectively.

First, we shall prove the following

Theorem 2. *Let besides conditions* (1) *and* (11) *the following assumptions be fulfilled*:

(A_1) *There exists the density* $f(x, t) = f(x_1, \ldots, x_n, t)$ *for every* $t \in T$.

(A_2) *Derivatives* $(\partial / \partial t) [\bar{v}_i(t \mid x, t) f(x, t)]$ $(i = 1, 2, \ldots, n)$ *are continuous on* $R_n \times T$.

(A_3) *Derivatives* $(\partial / \partial x_j) [\overline{v_i v_j}(t \mid x, t) f(x, t)]$ $(i, j = 1, 2, \ldots, n)$ *are continuous in* $(x_1, \ldots, x_n)$ *on* R_n *for every* $t \in T$.

(A_4) *Functions* $\bar{a}_i(t \mid x, t) f(x, t)$ $(i = 1, 2, \ldots, n)$ *are continuous in* $(x_1, \ldots, x_n)$ *on* R_n *for every* $t \in T$.

Then the system of equations

$$(15)\quad \frac{\partial}{\partial t}[\bar{v}_i(t \mid x, t) f(x, t)] + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[\overline{v_i v_j}(t \mid x, t) f(x, t)] - \bar{a}_i(t \mid x, t) f(x, t) = 0$$

$$(i = 1, 2, \ldots, n)$$

holds for all $x \in R_n$ *and* $t \in T$.

Proof. We shall introduce again the auxiliary function $\varphi(x)$ with properties indicated in § 1. Making use of condition (1), of assumptions (A_1) and (A_2) and of the fact that $\varphi(x)$ is continuous and bounded on $M \subset R_n$, we obtain

$$(16)\quad \int_M \varphi(x) \frac{\partial}{\partial t}[\bar{v}_i(t \mid x, t) f(x, t)] \, dx = \frac{\partial}{\partial t} \int_M \varphi(x) \, \bar{v}_i(t \mid x, t) f(x, t) \, dx =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\varphi(X(t + \Delta t)) \dot{X}_i(t + \Delta t) - \varphi(X(t)) \dot{X}_i(t)] =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{[\varphi(X(t + \Delta t)) - \varphi(X(t))] \dot{X}_i(t)\} +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{\varphi(X(t + \Delta t)) [\dot{X}_i(t + \Delta t) - \dot{X}_i(t)]\} \quad (i = 1, 2, \ldots, n).$$

We shall first compute the second term in the resultant expression in (16). According to conditions (1) and (11) and assumption (A_1), we have

$$(17)\quad \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{\varphi(X(t + \Delta t)) [\dot{X}_i(t + \Delta t) - \dot{X}_i(t)]\} =$$

$$= \lim_{\Delta t \to 0} E\left\{\varphi(X(t)) \frac{\dot{X}_i(t + \Delta t) - \dot{X}_i(t)}{\Delta t}\right\} +$$

$$+ \lim_{\Delta t \to 0} E\left\{\frac{1}{\Delta t} \sum_{j=1}^{n} \varphi'_j(Q) [X_j(t + \Delta t) - X_j(t)] [\dot{X}_i(t + \Delta t) - \dot{X}_i(t)]\right\} =$$

$$= E[\varphi(X(t)) \ddot{X}_i(t)] = \int_M \varphi(x) \, \bar{a}_i(t \mid x, t) f(x, t) \, dx,$$

where $Q = \{X_1(t) + \vartheta[X_1(t + \Delta t) - X_1(t)], \ldots, X_n(t) + \vartheta[X_n(t + \Delta t) - X_n(t)]\}$, $0 < \vartheta < 1$, since $E\{\varphi(X(t)) [\dot{X}_i(t + \Delta t) - \dot{X}_i(t)]/\Delta t\} \to E[\varphi(X(t)) \ddot{X}_i(t)]$, $\Delta t \to 0$, as follows from (11) in view of the fact that $\varphi(x)$ is bounded in R_n [3]), and since

$$\left| E\left\{\frac{1}{\Delta t} \varphi'_j(Q) [X_j(t + \Delta t) - X_j(t)] [\dot{X}_i(t + \Delta t) - \dot{X}_i(t)]\right\} \right| \leqq$$

$$\leqq K \left| \Delta t \, . \, E^{1/2} \left[\frac{X_j(t + \Delta t) - X_j(t)}{\Delta t}\right]^2 . \, E^{1/2} \left[\frac{\dot{X}_i(t + \Delta t) - \dot{X}_i(t)}{\Delta t}\right]^2 \right| \to 0, \quad \Delta t \to 0,$$

[3]) This relation can be proved similarly as relation (8).

as follows by Schwarz's inequality from (1), (11) and from the fact, that $|\varphi_j'(Q)| \leqq \leqq K < +\infty$.

In the computation that follows, we shall introduce an auxiliary random function $U(t) = [U_1(t), \ldots, U_n(t)]$ defined by relations

$$(18) \qquad \begin{aligned} U_i(t) &= \dot{X}_i(t) \quad \text{for} \quad |\dot{X}_i(t)| \leqq A\,, \\ U_i(t) &= 0 \qquad \text{for} \quad |\dot{X}_i(t)| > A \quad (i = 1, 2, \ldots, n)\,, \end{aligned}$$

where A is a chosen number, $0 < A < +\infty$.

Now, we rearrange the first term in the resultant expression in (16) as follows:

$$(19) \qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{[\varphi(X(t + \Delta t)) - \varphi(X(t))]\, \dot{X}_i(t)\} =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{[\varphi(X(t + \Delta t)) - \varphi(X(t))]\, U_i(t)\} +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{[\varphi(X(t + \Delta t)) - \varphi(X(t))]\, [\dot{X}_i(t) - U(t)]\} =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{\sum_{j=1}^{n} \varphi_j'(X(t))\, [X_j(t + \Delta t) - X_j(t)]\, U_i(t)\} +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{\sum_{j=1}^{n} \sum_{k=1}^{n} \tfrac{1}{2}\varphi_{jk}''(S)\, [X_j(t + \Delta t) - X_j(t)]\, [X_k(t + \Delta t) - X_k(t)]\, U_i(t)\} +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{\sum_{j=1}^{n} \varphi_j'(Q)\, [X_j(t + \Delta t) - X_j(t)]\, [\dot{X}_i(t) - U_i(t)]\}\,,$$

where $S = \{X_1(t) + \Theta[X_1(t + \Delta t) - X_1(t)], \ldots, X_n(t) + \Theta[X_n(t + \Delta t) - X_n(t)]\}$, $0 < \Theta < 1$, $Q = \{X_1(t) + \vartheta[X_1(t + \Delta t) - X_1(t)], \ldots, X_n(t) + \vartheta[X_n(t + \Delta t) - - X_n(t)]\}$, $0 < \vartheta < 1$.

Since $|\varphi_j'(X(t))| \leqq K < +\infty$, $|\varphi_{jk}''(S)| \leqq C < +\infty$, $|U_i(t)| \leqq A < +\infty$ $(i, j, k = = 1, 2, \ldots, n)$ for all $t \in T$, it follows from Schwarz's inequality and from (1) first that

$$\frac{1}{\Delta t} E\{\varphi_j'(X(t))\, [X_j(t + \Delta t) - X_j(t)]\, U_i(t)\} \to E\{\varphi_j'(X(t))\, \dot{X}_j(t)\, U_i(t)\}\,, \quad \Delta t \to 0\,,$$

which can be proved similarly as (8), further that

$$\left| \frac{1}{\Delta t} E\{\tfrac{1}{2}\varphi_{jk}''(S)\, [X_j(t + \Delta t) - X_j(t)]\, [X_k(t + \Delta t) - X_k(t)]\, U_i(t)\} \right| \leqq$$

$$\leqq \tfrac{1}{2} A C \left| \Delta t \,.\, E^{1/2} \left[\frac{X_j(t + \Delta t) - X_j(t)}{\Delta t} \right]^2 .\, E^{1/2} \left[\frac{X_k(t + \Delta t) - X_k(t)}{\Delta t} \right]^2 \right| \to 0\,, \quad \Delta t \to 0\,,$$

eventually

$$\left|\frac{1}{\Delta t} E\{\varphi_j'(Q)\,[X_j(t+\Delta t) - X_j(t)]\,[\dot{X}_i(t) - U_i(t)]\}\right| \leqq$$

$$\leqq K \,.\, E^{1/2}\left[\frac{X_j(t+\Delta t) - X_j(t)}{\Delta t}\right]^2 .\, E^{1/2}[\dot{X}_i(t) - U_i(t)]^2 \to$$

$$\to K \,.\, E^{1/2}[\dot{X}_j(t)]^2 \,.\, E^{1/2}[\dot{X}_i(t) - U_i(t)]^2\,, \quad \Delta t \to 0\,.$$

According to condition (1) $E[\dot{X}_i(t)]^2 < +\infty$ $(i = 1, 2, \ldots, n)$. Therefore

$$K \,.\, E^{1/2}[\dot{X}_j(t)]^2 \,.\, E^{1/2}[\dot{X}_i(t) - U_i(t)]^2 \to 0\,, \quad A \to +\infty\,,$$

and hence

$$E[\varphi_j'(X(t))\,\dot{X}_j(t)\,U_i(t)] \to E[\varphi_j'(X(t))\,\dot{X}_j(t)\,\dot{X}_i(t)]\,, \quad A \to +\infty\,,$$

since $|E\{\varphi_j'(X(t))\,\dot{X}_j(t)\,[\dot{X}_i(t) - U_i(t)]\}| \leqq KE^{1/2}[\dot{X}_j(t)]^2\, E^{1/2}[\dot{X}_i(t) - U_i(t)]^2$.

Thus by letting $A \to +\infty$ and then by substituting the results just arrived at in (19), we obtain

$$\text{(20)}\quad \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\{[\varphi(X(t+\Delta t)) - \varphi(X(t))]\,\dot{X}_i(t)\} = E\{\sum_{j=1} \varphi_j'(X(t))\,\dot{X}_j(t)\,\dot{X}_i(t)\} =$$

$$= \int_M \{\sum_{j=1}^{n} \varphi_j'(x)\,\overline{v_i v_j}(t \mid x, t)\,f(x, t)\}\,\mathrm{d}x =$$

$$= -\int_M \varphi(x)\left\{\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\,[\overline{v_i v_j}(t \mid x, t)\,f(x, t)]\right\}\mathrm{d}x$$

according to condition (1), assumptions (A_1), (A_3) and properties (P_1) and (P_2).

Finally, substituting (17) and (20) in (16), we obtain

$$\text{(21)}\qquad \int_M \varphi(x)\,\frac{\partial}{\partial t}\,[\bar{v}_i(t \mid x, t)\,f(x, t)]\,\mathrm{d}x =$$

$$= -\int_M \varphi(x)\left\{\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\,[\overline{v_i v_j}(t \mid x, t)\,f(x, t)]\right\}\mathrm{d}x + \int_M \varphi(x)\,\bar{a}_i(t \mid x, t)\,f(x, t)\,\mathrm{d}x$$

$$(i = 1, 2, \ldots, n)\,.$$

Making use of assumptions (A_2), (A_3) and (A_4) and of the fact, that $\varphi(x)$ is an arbitrary function except for properties (P_1) and (P_2) and that for each $x \in R_n$ we can choose a bounded interval $M \subset R_n$ such that $x \in M$, we can prove immediately, that (21) implies (15) for all $x \in R_n$, $t \in T$.

Thus the proof of Theorem 2 is completed.

A further system of differential equations of the same type as (15) can be derived analogously:

$$\frac{\partial}{\partial t}\left[\overline{v_i v_j}(t \mid x, t) f(x, t)\right] + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left[\overline{v_i v_j v_k}(t \mid x, t) f(x, t)\right] - \tag{22}$$

$$- \overline{a_i v_j}(t \mid x, t) f(x, t) - \overline{a_j v_i}(t \mid x, t) f(x, t) = 0 \quad (i, j = 1, 2, \ldots, n)$$

where $\overline{v_i v_j}(t \mid x, t)$, $\overline{v_i v_j v_k}(t \mid x, t)$ and $\overline{a_i v_j}(t \mid x, t)$ denote $E[\dot{X}_i(t)\, \dot{X}_j(t) \mid X(t) = x]$, $E[\dot{X}_i(t)\, \dot{X}_j(t)\, \dot{X}_k(t) \mid X(t) = x]$ and $E[\ddot{X}_i(t)\, \dot{X}_j(t) \mid X(t) = x]$ $(i, j, k = 1, 2, \ldots, n)$ respectively.

Equations (15) and (22) are closely related inasmuch as all conditional moments appearing in (22) are one order higher with respect to $\dot{X}_j(t)$ than the conditional moments appearing in the corresponding terms of (15). It is thus obvious that the assumptions under which equations (22) hold differ from the assumptions $(A_1)-(A_4)$ only in that the conditional moments they concern are also one order higher with respect to $\dot{X}_j(t)$ than the conditional moments in the corresponding assumptions $(A_1)-(A_4)$. As to conditions (1) and (11), it is sufficient to complement them with the following subsidiary condition:

$$E[\dot{X}_i(t)]^4 < +\infty \quad (i = 1, 2, \ldots, n) \quad \textit{whatever be} \quad t \in T. \tag{23}$$

To demonstrate it, we shall prove the following

Theorem 3. *Let besides conditions* (1), (11) *and* (23) *the following assumptions be fulfilled*:

(A_1') *There exists the density* $f(x, t) = f(x_1, \ldots, x_n, t)$ *for every* $t \in T$.

(A_2') *Derivatives* $(\partial/\partial t)\left[\overline{v_i v_j}(t \mid x, t) f(x, t)\right]$ $(i, j = 1, 2, \ldots, n)$ *are continuous on* $R_n \times T$.

(A_3') *Derivatives* $(\partial/\partial x_k)\left[\overline{v_i v_j v_k}(t \mid x, t) f(x, t)\right]$ $(i, j, k = 1, 2, \ldots, n)$ *are continuous in* $(x_1, \ldots, x_n)$ *on* R_n *for every* $t \in T$.

(A_4') *Functions* $\overline{a_i v_j}(t \mid x, t) f(x, t)$ $(i, j = 1, 2, \ldots, n)$ *are continuous in* $(x_1, \ldots, x_n)$ *on* R_n *for every* $t \in T$.

Then the system of equations (22) *holds for all* $x \in R_n$ *and* $t \in T$.

Proof. Again we shall use the auxiliary function $\varphi(x)$ introduced in § 1. According to condition (1) and assumptions (A_1') and (A_2') we have

$$\int_M \varphi(x) \frac{\partial}{\partial t}\left[\overline{v_i v_j}(t \mid x, t) f(x, t)\right] \mathrm{d}x = \frac{\partial}{\partial t} \int_M \varphi(x)\, \overline{v_i v_j}(t \mid x, t) f(x, t)\, \mathrm{d}x = \tag{24}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{E[\varphi(X(t + \Delta t))\, \dot{X}_i(t + \Delta t)\, \dot{X}_j(t + \Delta t)] - E[\varphi(X(t))\, \dot{X}_i(t)\, \dot{X}_j(t)]\}\,.$$

Now, we shall introduce the function $U(t)$ defined by relations (18). According to conditions (1), (11) and properties (P_1), (P_2) of $\varphi(x)$, we may write

$$(25)\quad E[\varphi(X(t+\Delta t))\,\dot{X}_i(t+\Delta t)\,\dot{X}_j(t+\Delta t)] - E[\varphi(X(t))\,\dot{X}_i(t)\,\dot{X}_j(t)] =$$
$$= E\{[\varphi(X(t+\Delta t)) - \varphi(X(t))]\,U_i(t)\,U_j(t)\} +$$
$$+ E\{[\varphi(X(t+\Delta t)) - \varphi(X(t))]\,[\dot{X}_i(t)\,\dot{X}_j(t) - U_i(t)\,U_j(t)]\} +$$
$$+ E\{\varphi(X(t+\Delta t))\,[\dot{X}_i(t+\Delta t)\,\dot{X}_j(t+\Delta t) - \dot{X}_i(t)\,\dot{X}_j(t)]\} =$$
$$= E\{\sum_{k=1}^{n} \varphi'_k(X(t))\,[X_k(t+\Delta t) - X_k(t)]\,U_i(t)\,U_j(t)\} +$$
$$+ E\{\sum_{k=1}^{n}\sum_{l=1}^{n} \tfrac{1}{2}\varphi''_{kl}(S)\,[X_k(t+\Delta t) - X_k(t)]\,[X_l(t+\Delta t) - X_l(t)]\,U_i(t)\,U_j(t)\} +$$
$$+ E\{\sum_{k=1}^{n} \varphi'_k(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_i(t) - U_i(t)]\,\dot{X}_j(t)\} +$$
$$+ E\{\sum_{k=1}^{n} \varphi'_k(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_j(t) - U_j(t)]\,U_i(t)\} +$$
$$+ E\{\varphi(X(t+\Delta t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,\dot{X}_j(t)\} +$$
$$+ E\{\varphi(X(t+\Delta t))\,[\dot{X}_j(t+\Delta t) - \dot{X}_j(t)]\,\dot{X}_i(t)\} +$$
$$+ E\{\varphi(X(t+\Delta t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,[\dot{X}_j(t+\Delta t) - \dot{X}_j(t)]\}\,,$$

where $S = \{X_1(t) + \Theta[X_1(t+\Delta t) - X_1(t)], \ldots, X_n(t) + \Theta[X_n(t+\Delta t) - X_n(t)]\}$, $0 < \Theta < 1$, $Q = \{X_1(t) + \vartheta[X_1(t+\Delta t) - X_1(t)], \ldots, X_n(t) + \vartheta[X_n(t+\Delta t) - X_n(t)]\}$, $0 < \vartheta < 1$.

Substitute the resultant expression (25) in (24).

Since $|\varphi'_k(X(t))| \leqq K < +\infty$, $|\varphi''_{kl}(X(t))| \leqq C < +\infty$, $|U_i(t)| \leqq A < +\infty$ $(i, k, l = 1, 2, \ldots, n)$ for all $t \in T$, it follows from (1) that

$$(26)\quad \frac{1}{\Delta t}\,E\{\varphi'_k(X(t))\,[X_k(t+\Delta t) - X_k(t)]\,U_i(t)\,U_j(t)\} \to$$
$$\to E\{\varphi'_k(X(t))\,\dot{X}_k(t)\,U_i(t)\,U_j(t)\}\,, \quad \Delta t \to 0\,,$$

which can be proved similarly as (8), and that

$$(27)\quad \left|\frac{1}{\Delta t}\,E\{\tfrac{1}{2}\varphi''_{kl}(S)\,[X_k(t+\Delta t) - X_k(t)]\,[X_l(t+\Delta t) - X_l(t)]\,U_i(t)\,U_j(t)\}\right| \leqq$$
$$\leqq \tfrac{1}{2}CA^2\left|\Delta t\,.\,E^{1/2}\left[\frac{X_k(t+\Delta t) - X_k(t)}{\Delta t}\right]^2.\,E^{1/2}\left[\frac{X_l(t+\Delta t) - X_l(t)}{t\Delta}\right]^2\right| \to 0,\ \Delta t \to 0\,.$$

From (23) it follows by Schwarz's inequality that

$$(28)\quad E\{[\dot{X}_i(t) - U_i(t)]^2\,[\dot{X}_j(t)]^2\} \leqq E^{1/2}[\dot{X}_i(t) - U_i(t)]^4\,.\,E^{1/2}[\dot{X}_j(t)]^4 \to 0\,,$$
$$A \to +\infty \quad (i, j = 1, 2, \ldots, n)\,.$$

Therefore, according to (1),

$$(29) \qquad \lim_{\Delta t \to 0} \left| \frac{1}{\Delta t} E\{\varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_i(t) - U_i(t)\}]\,\dot{X}_j(t)\} \right| \leqq$$

$$\leqq K \,.\, \lim_{\Delta t \to 0} E^{1/2} \left\{ \frac{X_k(t+\Delta t) - X_k(t)}{\Delta t} \right\}^2 .\, E^{1/2}\{[\dot{X}_i(t) - U_i(t)]^2\,[\dot{X}_j(t)]^2\} =$$

$$= K \,.\, E^{1/2}\{\dot{X}_k(t)\}^2 \,.\, E^{1/2}\{[\dot{X}_i(t) - U_i(t)]^2\,[\dot{X}_j(t)]^2\} \to 0\,, \quad A \to +\infty\,,$$

$$\lim_{\Delta t \to 0} \left| \frac{1}{\Delta t} E\{\varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_j(t) - U_j(t)]\,U_i(t)\} \right| \leqq$$

$$\leqq K \,.\, E^{1/2}\{\dot{X}_k(t)\}^2 \,.\, E^{1/2}\{[\dot{X}_j(t) - U_j(t)]^2\,[U_i(t)]^2\} \to 0\,, \quad A \to +\infty\,.$$

From (29) it follows that

$$\left| E\{\varphi_k'(X(t))\,\dot{X}_k(t)\,[\dot{X}_i(t)\,\dot{X}_j(t) - U_i(t)\,U_j(t)]\} \right| \leqq$$

$$\leqq K \,.\, E\left|\dot{X}_k(t)\,[\dot{X}_i(t) - U_i(t)]\,\dot{X}_j(t)\right| + K \,.\, E\left|\dot{X}_k(t)\,[\dot{X}_j(t) - U_j(t)]\,U_i(t)\right| \to 0\,,$$

$$A \to +\infty\,.$$

Hence, by letting $A \to +\infty$, we obtain

$$(30) \qquad E\{\varphi_k'(X(t))\,\dot{X}_k(t)\,U_i(t)\,U_j(t)\} \to E\{\varphi_k'(X(t))\,\dot{X}_i(t)\,\dot{X}_j(t)\,\dot{X}_k(t)\} =$$

$$= \int_M \varphi_k'(x)\,\overline{v_i v_j v_k}(t \mid x, t)\,f(x,t)\,\mathrm{d}x = - \int_M \varphi(x)\,\frac{\partial}{\partial x_k}\left[\overline{v_i v_j v_k}(t \mid x, t)\,f(x,t)\right]\mathrm{d}x$$

according to condition (23), assumptions (A_1'), (A_3') and properties (P_1) and (P_2).
Since $|\varphi(X(t+\Delta t))| \leqq B < +\infty$, Schwarz's inequality and condition (11) yield

$$(31) \qquad \left| \frac{1}{\Delta t} E\{\varphi(X(t+\Delta t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,[\dot{X}_j(t+\Delta t) - \dot{X}_j(t)]\} \right| \leqq$$

$$\leqq B \left| \Delta t \,.\, E^{1/2} \left[\frac{\dot{X}_i(t+\Delta t) - \dot{X}_i(t)}{\Delta t} \right]^2 .\, E^{1/2} \left[\frac{\dot{X}_j(t+\Delta t) - \dot{X}_j(t)}{\Delta t} \right]^2 \right| \to 0\,, \quad \Delta t \to 0\,.$$

According to definition (18), conditions (1), (11) and properties (P_1) and (P_2) we may write

$$\frac{1}{\Delta t} E\{\varphi(X(t+\Delta t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,\dot{X}_j(t)\} =$$

$$= \frac{1}{\Delta t} E\{\varphi(X(t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,\dot{X}_j(t)\} +$$

$$+ \frac{1}{\Delta t} E\{\sum_{k=1}^{n} \varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,U_j(t)\} +$$

$$+ \frac{1}{\Delta t} E\{\sum_{k=1}^{n} \varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,[\dot{X}_j(t) - U_j(t)]\}\,.$$

Then, Schwarz's inequality and (1), (11), (28) yield

$$\left|\frac{1}{\Delta t} E\{\varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,U_j(t)\}\right| \leqq$$

$$\leqq KA\left|\Delta t\,.\,E^{1/2}\left[\frac{X_k(t+\Delta t) - X_k(t)}{\Delta t}\right]^2 .\,E^{1/2}\left[\frac{\dot{X}_i(t+\Delta t) - \dot{X}_i(t)}{\Delta t}\right]^2 \to 0\,,\quad \Delta t \to 0\,,\right.$$

$$\left|\frac{1}{\Delta t} E\{\varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,\dot{X}_i(t)\,[\dot{X}_j(t) - U_j(t)]\}\right| \leqq$$

$$\leqq K\,.\,E^{1/2}\left\{\frac{X_k(t+\Delta t) - X_k(t)}{\Delta t}\right\}^2 .\,E^{1/2}\{[\dot{X}_i(t)]^2\,[\dot{X}_j(t) - U_j(t)]^2\} \to 0\,,\quad A \to +\infty\,.$$

Hence also

$$\left|\frac{1}{\Delta t} E\{\varphi_k'(Q)\,[X_k(t+\Delta t) - X_k(t)]\,\dot{X}_i(t+\Delta t)\,[\dot{X}_j(t) - U_j(t)]\}\right| \to 0\,,\quad A \to +\infty\,.$$

Finally, as follows from (1) and (11),

$$\frac{1}{\Delta t} E\{\varphi(X(t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,\dot{X}_j(t)\} \to E\{\varphi(X(t))\,\ddot{X}_i(t)\,\dot{X}_j(t)\}\,,\quad \Delta t \to 0\,,$$

which can be proved similarly as (8).

Altogether, we have

$$\text{(32)}\qquad \lim_{\Delta t\to 0} \frac{1}{\Delta t} E\{\varphi(X(t+\Delta t))\,[\dot{X}_i(t+\Delta t) - \dot{X}_i(t)]\,\dot{X}_j(t)\} =$$

$$= E\{\varphi(X(t))\,\ddot{X}_i(t)\,\dot{X}_j(t)\} = \int_M \varphi(x)\,\overline{a_i v_j}(t \mid x, t)\,f(x, t)\,\mathrm{d}x\,,$$

according to conditions (1), (11) and assumption (A_1').

Substituting first (25) and then (26), (27), (29), (30), (31) and (32) in (24), we obtain

$$\text{(33)}\quad \int_M \varphi(x)\frac{\partial}{\partial t}[\overline{v_i v_j}(t \mid x, t)\,f(x, t)]\,\mathrm{d}x = -\int_M \varphi(x)\left\{\sum_{k=1}^{n}\frac{\partial}{\partial x_k}[\overline{v_i v_j v_k}(t \mid x, t)\,f(x, t)]\right\}\mathrm{d}x +$$

$$+ \int_M \varphi(x)\,[\overline{a_i v_j}(t \mid x, t) + \overline{a_j v_i}(t \mid x, t)]\,f(x, t)\,\mathrm{d}x\,.$$

Making use of assumptions (A_2'), (A_3') and (A_4') and of the fact, that $\varphi(x)$ is an arbitrary function except for properties (P_1) and (P_2), we can prove immediately, that (33) implies (22) for all $x \in R_n$, $t \in T$.

Thus the proof of Theorem 3 is completed.

Close connections between systems of differential equations (3), (15) and (22) can be revealed by comparing one with another. First of all, each successive system contains conditional moments which are one order higher with respect to $\dot{X}_i(t)$ $(i = 1, 2, \ldots, n)$ than the conditional moments appearing in the corresponding terms of the preceding system as already mentioned when comparing systems (15) and (22). Another connection is as follows: in the first equation, i.e. in (3), partial derivatives of products $\bar{v}_1(t \mid x, t) f(x, t), \ldots, \bar{v}_n(t \mid x, t) f(x, t)$ with respect to variables $x_1, \ldots, x_n$ appear and in the subsequent system of equations (15) partial derivatives of the same products with respect to t appear. Partial derivatives with respect to $x_1, \ldots, x_n$ found in (15) are those of products $\overline{v_i v_j}(t \mid x, t) f(x, t)$ which contain conditional moments one order higher than moments $\bar{v}_i(t \mid x, t)$. In the next system (22) partial derivatives of products $\overline{v_i v_j}(t \mid x, t) f(x, t)$ with respect to t appear etc. Of course, we can also derive other systems of differential equations of this type which, together with the systems mentioned, form a family in which each two succeeding systems relate in the manner stated above. There thus follows in this family, after system (22), a system containing derivatives $(\partial/\partial t) [\overline{v_i v_j v_k}(t \mid x, t) f(x, t)]$, further a system containing derivatives $(\partial/\partial t) [\overline{v_i v_j v_k v_l}(t \mid x, t) f(x, t)]$ $(i, j, k, l = 1, 2, \ldots, n)$ etc. It is possible to derive these systems by a method similar to that used when deriving (3), (15) and (22) and that under the conditions (1) and (11) suitably complemented by some subsidiary condition similar to (23) and under the assumptions which are appropriately modified assumptions $(A_1)-(A_4)$, or possibly $(A_1')-(A_4')$.

The systems of partial differential equations of this type can generally be written in the form

$$\begin{aligned}(34)\qquad & \frac{\partial}{\partial t} \{E[(\dot{X}_1(t))^{k_1} \ldots (\dot{X}_n(t))^{k_n} | X(t) = x] f(x, t)\} + \\ & + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \{E[(\dot{X}_1(t))^{k_1} \ldots (\dot{X}_j(t))^{k_j+1} \ldots (\dot{X}_n(t))^{k_n} | X(t) = x] f(x, t)\} - \\ & - \sum_{j=1}^{n} k_j E[\ddot{X}_j(t) (\dot{X}_1(t))^{k_1} \ldots (\dot{X}_j(t))^{k_j-1} \ldots (\dot{X}_n(t))^{k_n} | X(t) = x] f(x, t) = 0 \\ & \qquad (k_1, \ldots, k_n = 0, 1, 2, \ldots),\end{aligned}$$

where we put $(\dot{X}_i(t))^{k_i} = 1$, when $k_i = 0$ $(i = 1, 2, \ldots, n)$.

When $k_1 = k_2 = \ldots = k_n = 0$, (34) turns into (3); when $k_i = 1$ and $k_j = 0$ for $j \neq i$, (34) turns into (15) etc.

To derive equations (34), we can start from the integral $\int_M \varphi(x) (\partial/\partial t) \,.$ $. \{E[(\dot{X}_1(t))^{k_1} \ldots (\dot{X}_n(t))^{k_n} \mid X(t) = x] f(x, t)\} \, dx$ and proceed analogously as when deriving (15) or (22). However, this derivation is much more extensive than that of (15) or (22). Therefore, we shall not give it.

As a corollary of Theorems 1 and 2 we obtain

Theorem 4. *Let conditions* (1), (11), *assumptions* (a_1), (a_2) *of Theorem* 1 *and assumptions* (A_2), (A_3) *and* (A_4) *of Theorem* 2 *be fulfilled. Further, let the following assumptions be fulfilled*:

(A_5) *Derivatives* $(\partial^2/\partial x_i\,\partial t)\,[\bar{v}_i(t \mid x, t) f(x, t)]$ $(i = 1, 2, \ldots, n)$ *are continuous in* (x_i, t) *on* $R_1 \times T$ *for all* $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R_{n-1}$.

(A_6) *There exist derivatives* $(\partial^2/\partial x_j\,\partial x_i)\,[\overline{v_i v_j}(t \mid x, t) f(x, t)]$ *and* $(\partial/\partial x_i)\,.$ $.\,[\bar{a}_i(t \mid x, t) f(x, t)]$ $(i, j = 1, 2, \ldots, n)$ *for all* $x \in R_n$, $t \in T$.

Then equation

$$\frac{\partial^2 f(x, t)}{\partial t^2} - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j\,\partial x_i} [\overline{v_i v_j}(t \mid x, t) f(x, t)] + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\bar{a}_i(t \mid x, t) f(x, t)] = 0 \tag{35}$$

holds for all $x \in R_n$, $t \in T$.

Proof. From equation (3) (cf. Theorem 1) and assumptions (a_2), (A_2) and (A_5) it follows that

$$\frac{\partial^2 f(x, t)}{\partial t^2} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \frac{\partial}{\partial t} [\bar{v}_i(t \mid x, t) f(x, t)] \right\} = 0$$

for all $x \in R_n$, $t \in T$. Substituting for $(\partial/\partial t)\,[v_i(t \mid x, t) f(x, t)]$ from (15) (cf. Theorem 2) and making use of assumption (A_6) we obtain (35).

3. DIFFERENTIAL EQUATIONS OF STOCHASTIC PROCESSES WHICH HAVE DERIVATIVES IN QUADRATIC MEAN UP TO THE r-th ORDER

Let the components of the vector random function $X(t)$ described in § 1 have derivatives in q.m. up to the r-th order inclusive on T, i.e. let the following condition be fulfilled:

$$E[X_i(t)]^2 < +\infty\,, \quad \lim_{\Delta t \to 0} E\left[\frac{X_i^{(l)}(t + \Delta t) - X_i^{(l)}(t)}{\Delta t} - X_i^{(l+1)}(t)\right]^2 = 0 \tag{36}$$

$$(i = 1, 2, \ldots, n;\; l = 0, 1, 2, \ldots, r - 1) \text{ whatever be } t \in T\,.$$

(here $X_i^{(l)}(t)$ denotes the derivative of the l-th order of random function $X_i(t)$).

Denote by $Z(t)$ a vector random function $[Z_1(t), \ldots, Z_N(t)]$, $N = rn$, with components $Z_1(t) = X_1(t), \ldots, Z_n(t) = X_n(t)$, $Z_{n+1}(t) = \dot{X}_1(t), \ldots, Z_{2n}(t) = \dot{X}_n(t)$, $Z_{2n+1}(t) = \ddot{X}_1(t), \ldots, Z_{3n}(t) = \ddot{X}_n(t), \ldots, Z_{(r-1)n+1}(t) = X_1^{(r-1)}(t), \ldots, Z_N(t) = X_n^{(r-1)}(t)$, i.e.

$$Z_j(t) = X_j(t) \qquad \text{for} \quad j = 1, 2, \ldots, n\,,$$

$$Z_j(t) = \dot{Z}_{j-n}(t) \qquad \text{for} \quad j = n + 1, n + 2, \ldots, N\,.$$

Assume that there exists a density of the probability distribution of $Z(t)$, i.e. a density of the joint probability distribution of functions $X_1(t), \ldots, X_n(t)$ and their derivatives in q.m. up to the order $r-1$ inclusive, at one instant t, which we denote by $f_r(z, t)$, $z = (z_1, \ldots, z_N) \in R_N = (-\infty, +\infty)^N$, $N = rn$.

As a generalization of Theorem 1*, we have the following

Theorem 1.** *Let besides condition* (36) *the following assumptions be fulfilled*:

(a_1^{**}) *There exists the density* $f_r(z, t) = f_r(z_1, \ldots, z_N, t)$ *of the joint probability distribution of* $Z_1(t), \ldots, Z_N(t)$ *whose derivative* $\partial f_r / \partial t$ *is continuous on* $R_N \times T$ *and derivatives* $\partial f_r / \partial z_j$ $(j = 1, 2, \ldots, N-n)$ *are continuous in* $(z_1, \ldots, z_N)$ *on* R_N *for every* $t \in T$.

(a_2^{**}) *Derivatives* $(\partial/\partial z_j)\{E[\dot{Z}_j(t) \mid Z(t) = z] f_r(z, t)\}$ $(j = N-n+1, \ldots, N)$ *are continuous in* $(z_1, \ldots, z_N)$ *on* R_N *for every* $t \in T$.

Then equation

$$(37) \qquad \frac{\partial f_r(z,t)}{\partial t} + \sum_{j=1}^{N-n} z_{j+n} \frac{\partial f_r(z,t)}{\partial z_j} + \sum_{j=N-n+1}^{N} \frac{\partial}{\partial z_j} \{E[\dot{Z}_j(t) \mid Z(t) = z] f_r(z,t)\} = 0$$

holds for all $z \in R_N$, $t \in T$.

The proof of Theorem 1** is analogous to that of Theorem 1*.

Equation (37) turns into (14), or into (3), when $N = 2n$, or when $N = n$.

It is possible to derive other differential equations from equation (37) in a similar manner as from equation (14) (cf. § 2). Equation (37) is first multiplied by a chosen real function g of variables $z_1, \ldots, z_N, t$ and then integrated with respect to $z_{mn+1}, \ldots, \ldots, z_N$ $(1 \leqq m \leqq r-1)$ over the range R_{N-mn}. Under well-known assumptions, one can then carry out integration by parts and exchange the sequence of integration and differentiation in some of the integrals. Following simple rearrangements, we obtain

$$(38) \qquad \frac{\partial}{\partial t}[(\overline{g})_m f_m] + \sum_{j=1}^{mn} \frac{\partial}{\partial z_j}[(\overline{g\dot{Z}_j})_m f_m] - \left[\overline{\left(\frac{\partial g}{\partial t}\right)}_m + \sum_{j=1}^{N} \overline{\left(\frac{\partial g}{\partial z_j} \dot{Z}_j\right)}_m\right] f_m = 0\,,$$

where $f_m = f_m(z_1, \ldots, z_{mn}, t)$ denotes the density of the joint probability distribution of $Z_1(t), \ldots, Z_{mn}(t)$ and $(\overline{g})_m$, $(\overline{g\dot{Z}_j})_m$, $(\overline{\partial g/\partial t})_m$, $(\overline{(\partial g/\partial z_j)\,\dot{Z}_j})_m$ denote the conditional expectations of the corresponding functions for $Z_1(t) = z_1, \ldots, Z_{mn}(t) = z_{mn}$ at instant t (for short, we do not write arguments of the functions in (38)). There is

$$f_m = \int_{R_{N-mn}} f_r \, dz_{mn+1} \ldots dz_N\,,$$

$$(\overline{g})_m f_m = \int_{R_{N-mn}} g f_r \, dz_{mn+1} \ldots dz_N\,,$$

$$(\overline{g\dot{Z}_j})_m f_m = \int_{R_{N-mn}} g z_{j+n} f_r \, dz_{mn+1} \ldots dz_N \quad (1 \leqq j \leqq mn)$$

etc.

It can easily be proved, that if $m \geqq 2$, we may substitute $(\overline{g\dot{Z}_j})_m = (\bar{g})_m z_{j+n}$ and $((\overline{\partial g/\partial z_j) \dot{Z}_j})_m = (\overline{\partial g/\partial z_j})_m z_{j+n}$ for $j \leqq (m-1)n$ in (38).

It is possible to derive equation (38) by a method similar to that used when deriving (15) and (22), too, namely by computing the integral $\int_{M_{mn}} \psi(\partial/\partial t) [(\bar{g})_m f_m] \,\mathrm{d}z_1 \ldots \mathrm{d}z_{mn}$ $(m = 1, 2, \ldots, r-1)$ where $\psi = \psi(z_1, \ldots, z_{mn})$ is a function with properties analogous to those of function $\varphi(x_1, \ldots, x_n)$ introduced in § 1 and $M_{mn} \subset R_{mn}$ is a bounded interval in which ψ differs from zero. This method is particularly well suited, when function g has such a form that it follows immediately from condition (36), or possibly from (36) and some simple subsidiary condition similar to (23), that function $G(t) = g(Z_1(t), \ldots, Z_N(t), t)$ is differentiable in q.m. By this method we can prove in a straightforward manner the following

Theorem 5. *Let $H(t)$ be a real second order random function having first derivative in q.m. $\dot{H}(t)$ on T. Let besides condition* (36) *the following assumptions be fulfilled (write $(\bar{H})_m$, $(\bar{\dot{H}})_m$ and $(\overline{H\dot{Z}_j})_m$ for $E[H(t) \mid Z_1(t) = z_1, \ldots, Z_{mn}(t) = z_{mn}]$, $E[\dot{H}(t) \mid Z_1(t) = z_1, \ldots, Z_{mn}(t) = z_{mn}]$ and $E[H(t)\dot{Z}_j(t) \mid Z_1(t) = z_1, \ldots, Z_{mn}(t) = z_{mn}]$, respectively):*

($\mathscr{A}_1$) *There exists the density $f_m = f_m(z_1, \ldots, z_{mn}, t)$ of the joint probability distribution of $Z_1(t), \ldots, Z_{mn}(t)$, $1 \leqq m \leqq r$, r being the positive integer appearing in* (36), *for every $t \in T$.*

($\mathscr{A}_2$) *Derivative $(\partial/\partial t)[(\bar{H})_m f_m]$ is continuous in $(z_1, \ldots, z_{mn}, t)$ on $R_{mn} \times T$.*

($\mathscr{A}_3$) *Derivatives $(\partial/\partial z_j)[(\overline{H\dot{Z}_j})_m f_m]$ $(j = 1, 2, \ldots, mn)$ are continuous in $(z_1, \ldots, z_{mn})$ on R_{mn} for every $t \in T$.*

($\mathscr{A}_4$) *Function $(\bar{\dot{H}})_m f_m$ is continuous in $(z_1, \ldots, z_{mn})$ on R_{mn} for every $t \in T$.*

Then equation

$$\frac{\partial}{\partial t}[(\bar{H})_m f_m] + \sum_{j=1}^{mn} \frac{\partial}{\partial z_j}[(\overline{H\dot{Z}_j})_m f_m] - (\bar{\dot{H}})_m f_m = 0 \tag{39}$$

holds for all $(z_1, \ldots, z_{mn}) \in R_{mn}$, $t \in T$.

If $m \geqq 2$, we can substitute $(\partial/\partial z_j)[(\overline{H\dot{Z}_j})_m f_m] = z_{j+n}(\partial/\partial z_j)[(\bar{H})_m f_m]$ for $j \leqq (m-1)n$ in (39).

The proof of Theorem 5 is analogous to that of Theorem 2. To prove it, it is sufficient to replace function $\dot{X}_i(t)$ and its conditional expectation $\bar{v}_i(t \mid x, t)$ by $H(t)$ and $(\bar{H})_m$ respectively, and also $[X_1(t), \ldots, X_n(t)]$, $(x_1, \ldots, x_n)$, $f(x, t)$, $\varphi(x)$ and $j, k = 1, 2, \ldots, n$ by $[Z_1(t), \ldots, Z_{mn}(t)]$, $(z_1, \ldots, z_{mn})$, $f_m(z_1, \ldots, z_{mn}, t)$, $\psi(z_1, \ldots, z_{mn})$ and $j, k = 1, 2, \ldots, mn$ respectively, in the procedure used when deriving (15) (cf. equations (16)–(21)).

Remark 1. If $H(t) = g(Z_1(t), \ldots, Z_N(t), t)$ and, at the same time, $\dot{H} = \partial g/\partial t + \sum_{j=1}^{N} (\partial g/\partial Z_j)\dot{Z}_j$, $\dot{H}$ being the derivative in q.m. of H, (39) turns into (38).

Remark 2. Since $[Z_1(t), \ldots, Z_{mn}(t)] = [X_1(t), \ldots, X_n(t)] = X(t)$, $(z_1, \ldots, z_{mn}) = (x_1, \ldots, x_n), f_m(z_1, \ldots, z_{mn}, t) = f(x, t)$ for $m = 1$, (39) turns into (15), when $m = 1$, $H(t) = \dot{X}_i(t)$ or into (22), when $m = 1$, $H(t) = \dot{X}_i(t)\, \dot{X}_j(t)$.

Furthermore, putting $m = 1$, $H(t) = \ddot{X}_i(t)$ in (39), we obtain

$$(40) \quad \frac{\partial}{\partial t}[\bar{a}_i(t \mid x, t) f(x, t)] + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} [\overline{a_i v_j}(t \mid x, t) f(x, t)] - \bar{b}_i(t \mid x, t) f(x, t) = 0\,,$$

where $\bar{a}_i(t \mid x, t) = E[\ddot{X}_i(t) \mid X(t) = x]$, $\overline{a_i v_j}(t \mid x, t) = E[\ddot{X}_i(t)\, \dot{X}_j(t) \mid X(t) = x]$, in accordance with the notation introduced in § 2, and $\bar{b}_i(t \mid x, t) = E[\dddot{X}_i(t) \mid X(t) = x]$.

As a corollary of Theorems 1, 2, 3 and 5 we obtain

Theorem 6. *Let condition* (36) *be fulfilled for* $r = 3$. *Let condition* (23), *assumptions* (a_1), (a_2) *of Theorem* 1, *assumptions* (A_2), (A_3), (A_4) *of Theorem* 2, *assumptions* (A_2'), (A_3'), (A_4') *of Theorem* 3 *and assumption* (A_5) *of Theorem* 4 *be fulfilled. Assumptions* ($\mathscr{A}_2$), ($\mathscr{A}_3$) *and* ($\mathscr{A}_4$) *of Theorem* 5 *let be fulfilled for* $m = 1$, $H(t) = \ddot{X}_i(t)$ $(i = 1, 2, \ldots, n)$. *In addition to it, let the following assumptions be fulfilled*:

(A_7) *Derivatives* $(\partial^2/\partial x_j\, \partial t)\, [\overline{v_i v_j}(t \mid x, t) f(x, t)]$ $(i, j = 1, 2, \ldots, n)$ *are continuous in* (x_j, t) *on* $R_1 \times T$ *for all* $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in R_{n-1}$.

(A_8) *Derivatives* $(\partial^2/\partial x_i\, \partial t)\, \{(\partial/\partial t)\, [\bar{v}_i(t \mid x, t) f(x, t)]\}$ $(i = 1, 2, \ldots, n)$ *are continuous in* (x_i, t) *on* $R_1 \times T$ *for all* $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R_{n-1}$.

(A_9) *There exist derivatives* $(\partial^2/\partial x_k\, \partial x_j)\, [\overline{v_i v_j v_k}(t \mid x, t) f(x, t)]$, $(\partial^3/\partial x_k\, \partial x_j\, \partial x_i)\, .\, [\overline{v_i v_j v_k}(t \mid x, t) f(x, t)]$, $(\partial/\partial x_j)\, [\overline{a_j v_i}(t \mid x, t) f(x, t)]$ *and* $(\partial/\partial x_i)\, [\bar{b}_i(t \mid x, t) f(x, t)]$ $(i, j, k = 1, 2, \ldots, n)$ *for all* $x \in R_n$, $t \in T$.

(A_{10}) *Derivatives* $(\partial^2/\partial x_j\, \partial x_i)\, [\overline{a_i v_j}(t \mid x, t) f(x, t)]$ $(i, j = 1, 2, \ldots, n)$ *are continuous in* (x_i, x_j) *on* R_2 *for all* $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in R_{n-2}$, $t \in T$.

Then equation

$$(41) \quad \frac{\partial^3 f(x, t)}{\partial t^3} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^3}{\partial x_k\, \partial x_j\, \partial x_i} [\overline{v_i v_j v_k}(t \mid x, t) f(x, t)] - 3 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j\, \partial x_i} [\overline{a_i v_j}(t \mid x, t) f(x, t)] + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\bar{b}_i(t \mid x, t) f(x, t)] = 0$$

holds for all $x \in R_n$, $t \in T$.

Proof. From equation (15) (cf. Theorem 2), assumptions (A_3), (A_2'), (A_7) and assumption ($\mathscr{A}_2$) for $m = 1$, $H(t) = \ddot{X}_i(t)$ $(i = 1, 2, \ldots, n)$ it follows that

$$(42) \quad \frac{\partial^2}{\partial t^2} [\bar{v}_i(t \mid x, t) f(x, t)] = - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ \frac{\partial}{\partial t} [\overline{v_i v_j}(t \mid x, t) f(x, t)] \right\} + \frac{\partial}{\partial t} [\bar{a}_i(t \mid x, t) f(x, t)] \quad (i = 1, 2, \ldots, n)$$

for all $x \in R_n$, $t \in T$.

Making use of equation (3) (cf. Theorem 1), assumptions (a_2), (A_2), (A_5), (A_8) and of the fact, that derivatives $(\partial^2/\partial t^2)\left[\bar{v}_i(t \mid x, t) f(x, t)\right]$ exist for all $x \in R_n$, $t \in T$ according to assumptions (A_7), ($\mathscr{A}_2$) and equation (42), we obtain (cf. the proof of Theorem 4)

$$\frac{\partial^3 f(x, t)}{\partial t^3} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{\frac{\partial^2}{\partial t^2}\left[\bar{v}_i(t \mid x, t) f(x, t)\right]\right\} = 0 \tag{43}$$

for all $x \in R_n$, $t \in T$.

Substituting first for $(\partial/\partial t)\left[\overline{v_i v_j}(t \mid x, t) f(x, t)\right]$ from (22) and for $(\partial/\partial t)$ . . $\left[\bar{a}_i(t \mid x, t) f(x, t)\right]$ from (40) into (42) and then for $(\partial^2/\partial t^2)\left[\bar{v}_i(t \mid x, t) f(x, t)\right]$ from (42) into (43) and making use of assumptions (A_9), (A_{10}) we obtain (41).

Remark 3. It is possible to derive equation (41), too, by differentiating (35) with respect to t and by substituting for the corresponding terms from (22) and (40).

Equations (3), (35) and (41) form a family in which each successive equation contains a partial derivative of the density $f(x, t)$ with respect to t, which is one order higher than that found in the preceding equation. The other terms appearing in these equations are, altogether, partial derivatives with respect to variables $x_1, \ldots, x_n$, namely partial derivatives of products of $f(x, t)$ with conditional product moments of derivatives of random functions $X_1(t), \ldots, X_n(t)$, the highest order of derivatives of $X_1(t), \ldots, X_n(t)$ in these moments in each equation being equal to the order of the partial derivative of $f(x, t)$ with respect to t. Of course, we can also derive other partial differential equations of this type which, together with the equations mentioned, form a family of partial differential equations, in which each two succeeding equations relate in the manner stated above. There thus follows in this family, after equation (41), an equation containing derivative $\partial^4 f(x, t)/\partial t^4$, further an equation containing $\partial^5 f(x, t)/\partial t^5$ etc. It is possible to derive these equations by a procedure similar to that used when deriving (35) and (41), namely by differentiating the preceding equation with respect to t and by substituting for the corresponding terms from relevant equations of the type (34) or (39).

4. DIFFERENTIAL EQUATIONS SATISFIED BY TRANSITION PROBABILITY DENSITIES

In differential equations stated in §§ 1–3, there appear exclusively densities of non-conditional probability distributions. However, analogous equations, satisfied by the corresponding densities of conditional probability distributions (transition probability densities), can be derived by similar methods for stochastic processes dealt with in this paper. Each of those equations differs from its counterpart stated in §§ 1–3 only by that there appears in it the corresponding transition probability density instead of non-conditional probability density and that all the moments it contains are conditional moments relative to the same conditions as those in its counterpart supplemented by the conditions to which the transition probability density relates.

Thus e.g. the counterpart of equation (3) is the following differential equation:

$$\frac{\partial}{\partial t}\,\varrho(x, t \mid x_0, t_0) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\bar{v}_i(t \mid x_0, t_0; x, t)\,\varrho(x, t \mid x_0, t_0)\right] = 0\,. \tag{44}$$

Here, $\varrho(x, t \mid x_0, t_0)$ denotes the density of the conditional probability distribution of random function $X(t) = [X_1(t), \ldots, X_n(t)]$, described in § 1, for $X(t_0) = x_0$, $t_0 \in T$, $t \in T - \{t_0\}$, $x_0 = (x_{01}, \ldots, x_{0n}) \in R_n$, $x = (x_1, \ldots, x_n) \in R_n$ and $\bar{v}_i(t \mid x_0, t_0; x, t)$ denotes $E[\dot{X}_i(t) \mid X(t_0) = x_0, X(t) = x]$.

It is obvious that equation (44) can be derived by the same method as equation (3) under assumptions analogous to those of Theorem 1. In this manner, we can prove e.g. the following theorem which is entirely analogous to Theorem 1:

Theorem 7. *Let $x_0 = (x_{01}, \ldots, x_{0n}) \in R_n$, $t_0 \in T$. Let the condition*

$$E\{[X_i(t)]^2 \mid X(t_0) = x_0\} < +\infty\,, \tag{45}$$

$$\lim_{\Delta t \to 0} E\left\{\left[\frac{X_i(t + \Delta t) - X_i(t)}{\Delta t} - \dot{X}_i(t)\right]^2 \middle| X(t_0) = x_0\right\} = 0 \quad (i = 1, 2, \ldots, n)$$

be fulfilled for every $t \in T$. Further, let the following assumptions be fulfilled:

(a_1') *There exists the density $\varrho(x, t \mid x_0, t_0)$ of the conditional probability distribution of $X(t) = [X_1(t), \ldots, X_n(t)]$ for $X(t_0) = x_0$, whose derivative $\partial\varrho/\partial t$ is continuous in $(x_1, \ldots, x_n, t)$ on $R_n \times (T - \{t_0\})$.*

(a_2') *Derivatives $(\partial/\partial x_i)\,[\bar{v}_i(t \mid x_0, t_0; x, t)\,\varrho(x, t \mid x_0, t_0)]$ $(i = 1, 2, \ldots, n)$ are continuous in $(x_1, \ldots, x_n)$ on R_n for every $t \in T - \{t_0\}$.*

Then equation (44) *holds for all $x \in R_n$, $t \in T - \{t_0\}$.*

To prove Theorem 7, we start from the integral $\int_M \varphi(x)\,(\partial/\partial t)\,\varrho(x, t \mid x_0, t_0)\,\mathrm{d}x$, where $\varphi(x)$ is the auxiliary function introduced in § 1, and proceed similarly as when proving Theorem 1 (cf. equations 5–10).

Analogously, we can prove, as a modification of Theorem 7 in which condition (1) instead of (45) is used, the following

Theorem 7*. *Let $t_0 \in T$. Let besides condition* (1) *the following assumptions be fulfilled*:

(a_1'') *There exists the density $f_2(x_0, t_0; x, t)$ of the probability distribution of $X(t) = [X_1(t), \ldots, X_n(t)]$ at two instants t_0, t for every $t \in T - \{t_0\}$, whose derivative $\partial f_2/\partial t$ is continuous in $(x_{01}, \ldots, x_{0n}, x_1, \ldots, x_n, t)$ on $R_n \times R_n \times (T - \{t_0\})$.*

(a_2'') *Derivatives $(\partial/\partial x_i)\,[\bar{v}_i(t \mid x_0, t_0; x, t)\,f_2(x_0, t_0; x, t)]$ $(i = 1, 2, \ldots, n)$ are continuous in $(x_{01}, \ldots, x_{0n}, x_1, \ldots, x_n)$ on $R_n \times R_n$ for every $t \in T - \{t_0\}$.*

Then equation (44) *holds for all* $x \in R_n$, $t \in T - \{t_0\}$ *and for each* x_0 *for which* $\varrho(x, t \mid x_0, t_0)$ *is defined.*

Proof. We start from the integral $\int_{M_0 \times M} \varphi_0(x_0)\, \varphi(x)\, (\partial/\partial t) f_2(x_0, t_0; x, t)\, dx_0\, dx$ where $\varphi_0(x_0)$ is an auxiliary function with properties analogous to those of function $\varphi(x)$, $M_0 \subset R_n$ is a bounded interval in which $\varphi_0(x_0)$ differs from zero, $dx_0 = dx_{01}\, dx_{02} \ldots dx_{0n}$ and $dx = dx_1\, dx_2 \ldots dx_n$. Proceeding analogously as when deriving equation (3) we find easily, that

$$\frac{\partial}{\partial t} f_2(x_0, t_0; x, t) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\bar{v}_i(t \mid x_0, t_0; x, t) f_2(x_0, t_0; x, t)\right] = 0 \tag{46}$$

holds for all $x_0 \in R_n$, $x \in R_n$, $t \in T - \{t_0\}$.

Considering the fact that $\varrho(x, t \mid x_0, t_0)$ is defined by the relation

$$\varrho(x, t \mid x_0, t_0) = \frac{f_2(x_0, t_0; x, t)}{\int_{R_n} f_2(x_0, t_0; x, t)\, dx} \tag{47}$$

for each x_0 for which $\int_{R_n} f_2(x_0, t_0; x, t)\, dx$ does not vanish, we obtain (44) for each x_0 for which $\varrho(x, t \mid x_0, t_0)$ is defined by dividing (46) by $\int_{R_n} f_2(x_0, t_0; x, t)\, dx$.

Thus the proof of Theorem 7* is completed.

As a corollary of Theorems 1 and 7* we obtain

Theorem 8. *Let* $t_0 \in T$. *Let condition* (1), *assumptions* (a_1''), (a_2'') *of Theorem* 7*, *and assumptions* (a_1), (a_2) *of Theorem* 1, *in which* $f(x, t) = \int_{R_n} f_2(x_0, t_0, x, t)\, dx_0$, *be fulfilled. Further, let the following assumption be fulfilled*:

(a_3'') *There exist derivatives* $\partial f_2/\partial x_i$ *for every* $x_0 \in R_n$, $x \in R_n$, $t \in T - \{t_0\}$ *and derivatives* $\partial f/\partial x_i$ *for every* $x \in R_n$, $t \in T$ $(i = 1, 2, \ldots, n)$.

Then equation

$$\frac{\partial}{\partial t} \varrho(x_0, t_0 \mid x, t) + \sum_{i=1}^{n} \bar{v}_i(t \mid x, t) \frac{\partial}{\partial x_i} \varrho(x_0, t_0 \mid x, t) + \tag{48}$$

$$+ \frac{1}{f(x, t)} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \{\varrho(x_0, t_0 \mid x, t) f(x, t) \left[\bar{v}_i(t \mid x_0, t_0; x, t) - \bar{v}_i(t \mid x, t)\right]\} = 0\,,$$

where $\varrho(x_0, t_0 \mid x, t)$ *denotes the density of the conditional probability distribution of* $X(t_0) = [X_1(t_0), \ldots, X_n(t_0)]$ *for* $X(t) = x$, *holds for all* $x_0 \in R_n$, $t \in T - \{t_0\}$ *and for each* x *for which* $\varrho(x_0, t_0 \mid x, t)$ *is defined.*

Proof. Putting $\int_{R_n} f_2(x_0, t_0; x, t)\, dx_0 = f(x, t)$, we have

$$\varrho(x_0, t_0 \mid x, t) = \frac{f_2(x_0, t_0; x, t)}{f(x, t)} \tag{49}$$

for each x for which $f(x, t) \neq 0$, according to definition (47).

From (49) and assumptions (a_1), (a_1'') it follows that

$$(50)\quad \frac{\partial}{\partial t}\varrho(x_0, t_0 \mid x, t) - \frac{1}{f(x, t)} \cdot \frac{\partial}{\partial t} f_2(x_0, t_0; x, t) + \frac{f_2(x_0, t_0; x, t)}{f^2(x, t)} \cdot \frac{\partial}{\partial t} f(x, t) = 0$$

for $f(x, t) \neq 0$.

Substituting first for $\partial f_2/\partial t$ from (46) (cf. the proof of Theorem 7*) and for $\partial f/\partial t$ from (3) (cf. Theorem 1) and then for f_2 from (49) into (50) and making use of assumption (a_3'') we obtain (48) for all $x_0 \in R_n$, $t \in T - \{t_0\}$ and for each $x \in R_n$ for which $f(x, t) \neq 0$, i.e. for each x for which $\varrho(x_0, t_0 \mid x, t)$ is defined by (49).

Thus the proof of Theorem 8 is completed.

Remark 4. To derive equation (48), we can use also equation (44) instead of (46).

Remark 5. Assume, that $X(t + \Delta t)$ is conditionaly independent of $X(t_0)$ given $X(t)$ for $t_0 < t < t + \Delta t$ or possibly for $t + \Delta t < t < t_0$. Then, if condition (1), assumption (a_1'') and assumption (a_2'') in which we replace $\bar{v}_i(t \mid x_0, t_0; x, t)$ by $\bar{v}_i(t \mid x, t)$ are fulfilled, we can find by the procedure used when proving Theorem 7*, that (44) turns into

$$(51)\qquad \frac{\partial}{\partial t}\varrho(x, t \mid x_0, t_0) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\bar{v}_i(t \mid x, t)\, \varrho(x, t \mid x_0, t_0)\right] = 0$$

and hence, if also assumptions (a_1), (a_2) and (a_3'') are fulfilled, (48) turns into

$$(52)\qquad \frac{\partial}{\partial t}\varrho(x_0, t_0 \mid x, t) + \sum_{i=1}^{n} \bar{v}_i(t \mid x, t) \frac{\partial}{\partial x_i}\varrho(x_0, t_0 \mid x, t) = 0\,,$$

which can be proved quite analogously as Theorem 8. Equations (51) and (52) are to some extent analogies of the forward equation and the backward equation, respectively, for continuous Markov processes derived by Kolmogorov [4], [12].

Similarly as equation (44) is an analogy of equation (3), the remaining equations stated in §§ 1 – 3 have their analogies, too, satisfied by the corresponding transition probability densities. We shall present these equations here no further; as already mentioned, we obtain them immediately from the equations stated in §§ 1 – 3 if we supplement in each of them both the probability density and all the conditional moments by an appropriate condition. These equations can be, similarly as (44) and (48), simplified if the assumption stated in Remark 5 or possibly an analogous assumption is fulfilled. The problems just mentioned will be dealt with in detail in our next paper.

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Author's address: Praha 6 - Dejvice, Technická ul. 1902, ČSSR (Strojní fakulta ČVUT).

Резюме

ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ ДЛЯ СТОХАСТИЧЕСКИХ ПРОЦЕССОВ, ОБЛАДАЮШИХ ПРОИЗВОДНОЙ В СРЕДНЕМ КВАДРАТИЧЕСКОМ

КАРЕЛ КОШТЯЛ (Karel Košťál), Прага

В работе систематически выводятся дифференциальные уравнения для векторных стохастических процессов, обладающих производной в среднем квадратическом. Сначала изучаются процессы обладающие первой и второй производной. Для них выводятся подробно дифференциальные уравнения, находящие приложения особенно в физике. Затем выводятся дифференциальные уравнения для процессов, имеющих производные в общем до порядка r. В заключительной части работы исследуются аналогичные уравнения, которым удовлетворяют соответствующие плотности вероятностей перехода.