Ivo Vrkoč

Extension of the averaging method to stochastic equations

Czechoslovak Mathematical Journal, Vol. 16 (1966), No. 4, 518–544

Persistent URL: http://dml.cz/dmlcz/100748

Чехословацкий математический журнал, т. 16 (91) 1966, Прага

EXTENSION OF THE AVERAGING METHOD TO STOCHASTIC EQUATIONS

IVO VRKOČ, Praha

(Received August 18, 1965)

In the book of BOGOLIUBOV and MITROPOL'SKIJ [1] there were laid the foundations of averaging method for systems of ordinary differential equations. I shall show conditions under which it is possible to apply this method to Ito's stochastic equations and for such equations with adhesive barriers. There will also be given conditions for stability in average and asymptotic stability in average, since stability is very useful in this theory. Finally we shall deal with systems of integral equations which are not in the form (2).

Let the triplet $(\Omega, \mathscr{F}, P)$ be given. The random values are the $\mathscr{F}$-measurable mappings from Ω into an n-dimensional Euclidean space. The norm $|x|$ of a point in the Euclidean space is defined in the usual way as $|x| = \sqrt{(\sum x_i^2)}$. Let $z(t, \omega)$ be an arbitrary stochastic process; the expression $\operatorname*{l.i.m.}_{t \to t_0} z(t, \omega) = 0$ has the usual meaning that $\lim_{t \to t_0} E|z(t, \omega)|^2 = 0$ and for brevity we denote

$$\|z(t, \omega)\| = \sqrt{E|z(t, \omega)|^2}\,, \quad \|z(t, \omega)\|_L = \sqrt{E \sup_{t \in \langle 0, L \rangle} |z(t, \omega)|^2}\,.$$

First we shall formulate several assumptions:

i) Let $a(t, x)$ be a vector and $B(t, x)$ a square matrix, both continuous in t, x and Lipschitz continuous in x with a constant K. The components of a and B are denoted by a_i and b_{ij}. The norm of the matrix B is $|B| = \sqrt{(\sum_{i,j} b_{ij}^2)}$.

ii) $|a(t, 0)| \leqq K$, $|B(t, 0)| \leqq K$.

iii) Let $w_\varepsilon(t)$ be vector stochastic processes with independent increments which are defined for all $t \geqq 0$, and with

$$E(w_\varepsilon(t_2) - w_\varepsilon(t_1)) = 0\,, \quad E|w_\varepsilon(t_2) - w_\varepsilon(t_1)|^2 = F_\varepsilon(t_2) - F_\varepsilon(t_1) \tag{1}$$

where $F_\varepsilon(t)$ are continuous functions.

iv) Let there exist a function $\varphi(\varepsilon) > 0$ such that $\operatorname*{l.i.m.}_{\varepsilon \to 0} (\bar{w}_\varepsilon(t_2) - \bar{w}_\varepsilon(t_1)) = w_0(t_2) - w_0(t_1)$ uniformly on every compact set of the t_1, t_2, where $\bar{w}_\varepsilon(t) = \sqrt{(\varphi(\varepsilon))}\, w_\varepsilon(t/\varepsilon)$

v) $w_\varepsilon^*(t) = \bar{w}_\varepsilon(t) - w_0(t)$ is a process with independent increments.

vi) Let $\mathscr{F}(t)$ and $\overline{\mathscr{F}}_\varepsilon(t)$ be the smallest σ-fields corresponding to the random values $w_0(t_2) - w_0(t_1)$ and $\bar{w}_\varepsilon(t_2) - \bar{w}_\varepsilon(t_1)$, respectively, for $0 \leqq t_1 < t_2 \leqq t$; let $\mathscr{F}_\varepsilon^*(t)$ contain the smallest σ-field which corresponds to $w_\varepsilon^*(t_2) - w_\varepsilon^*(t_1)$, $0 \leqq t_1 < t_2 \leqq t$, and let the smallest σ-field corresponding to $w_\varepsilon^*(t_4) - w_\varepsilon^*(t_3)$, $t \leqq t_3 < t_4$ be independent of $\mathscr{F}_\varepsilon^*(t)$ for all t. These σ-fields are to fulfil

$$\overline{\mathscr{F}}_\varepsilon(t) \subset \mathscr{F}(t)\,, \quad \overline{\mathscr{F}}_\varepsilon(t) \subset \mathscr{F}_\varepsilon^*(t)\,.$$

vii) Let there exist a vector $\bar{a}(x)$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T a(t, x)\,\mathrm{d}t = \bar{a}(x) \quad \text{uniformly with respect to } x\,.$$

viii) Let there exist a square matrix $\bar{B}(x)$ such that

$$\lim_{T\to\infty} \int_{\alpha T}^{\alpha T + \beta T} \left|B(t, x) - \bar{B}(x)\right|^2 \mathrm{d}F\left(\frac{t}{T}\right) = 0$$

uniformly with respect to x for all α, β, $0 \leqq \alpha \leqq L$, $0 < \beta < L$, where L is a given positive number.

ix) Let $x_0(\omega)$ be a random value independent of all $w_\varepsilon(t_2) - w_\varepsilon(t_1)$ and with $E\left|x_0(\omega)\right|^2 < \infty$.

In the last section of this paper it is sketched the proof that the solution $x(t)$ of the equation

$$x(t) = x_0 + \varepsilon \int_0^t a(\tau, x(\tau))\,\mathrm{d}\tau + \sqrt{\varphi(\varepsilon)} \int_0^t B(\tau, x(\tau))\,\mathrm{d}w_\varepsilon(\tau) \tag{2}$$

exists (under the assumptions i) ii) iii) and ix)), $x(t)$ is a Markov process and $x(t) \in \mathscr{M}$ (for the definition of $\mathscr{M}$ see the last section).

Now we have all prepared to formulate the fundamental result:

Theorem 1.[1]) *Let the assumptions* i) *to* ix) *be fulfilled, and let* $y(t)$ *be a solution of the equation*

$$y(t) = x_0 + \int_0^t \bar{a}(y(\tau))\,\mathrm{d}\tau + \int_0^t \bar{B}(y(\tau))\,\mathrm{d}w_0(\tau)\,; \tag{3}$$

---

[1]) A result similar but actually distinct was published by Gichman I. I. in „Зимняя школа по теории вероятностей и математической статистике", Киев 1964. This book was not known to me at the time when this paper was completed.

then to every $\eta > 0$ *there exists an* $\varepsilon_0 > 0$ *such that*

$$E(\sup_{\langle 0, L/\varepsilon\rangle} |x(t) - y(\varepsilon t)|^2) < \eta \quad \textit{for} \quad 0 < \varepsilon \leqq \varepsilon_0 .$$

Proof. On transforming $\varepsilon t = \xi$, $x(t) = \bar{x}(\xi)$, $\bar{w}_\varepsilon(t) = \sqrt{(\varphi(\varepsilon))}\, w_\varepsilon(t/\varepsilon)$, equation (2) changes to

$$(1,1) \qquad \bar{x}(\xi) = x_0 + \int_0^\xi a(\tau/\varepsilon, \bar{x}(\tau))\, d\tau + \int_0^\xi B(\tau/\varepsilon, \bar{x}(\tau))\, d\bar{w}_\varepsilon(\tau) .$$

We shall estimate the expression $\|x(\xi) - y(\xi)\|_\gamma$. By the Hölder inequality

$$(2,1) \qquad \|\bar{x}(\xi) - y(\xi)\|_\gamma \leqq \left\| \int_0^\xi (a(\tau/\varepsilon, \bar{x}(\tau)) - \bar{a}(y(\tau))\, d\tau \right\|_\gamma +$$
$$+ \left\| \int_0^\xi (B(\tau/\varepsilon, \bar{x}(\tau)) - \bar{B}(y(\tau))\, dw_0(\tau) \right\|_\gamma + \left\| \int_0^\xi B(\tau/\varepsilon, \bar{x}(\tau))\, dw_\varepsilon^*(\tau) \right\|_\gamma .$$

First estimate the first term on the right-hand side of (2,1). Choose a positive number v and a sequence of points $\tau_i : 0 = \tau_0 < \tau_1 < \ldots < \tau_m = \gamma$, $\max_i (\tau_{i+1} - \tau_i) = v$. Set $f(t, \omega, \varepsilon) = a(t/\varepsilon, \bar{x}(t)) - \bar{a}(y(t))$. We may define the random value $\tau(\omega)$ by

$$\sup_{\langle 0, \gamma\rangle} \left| \int_0^t f(\tau, \omega, \varepsilon)\, d\tau \right| = \left| \int_0^{\tau(\omega)} f(\tau, \omega, \varepsilon)\, d\tau \right| .$$

Let $\tau^+(\omega)$ be defined as the maximal τ_i for which $\tau_i \leqq \tau(\omega)$. There is a function $v(\eta)$ with $v(\eta) > 0$ and

$$(3,1) \qquad \sqrt{\left(\int \left(\left| \int_0^{\tau(\omega)} f(\tau, \omega, \varepsilon)\, d\tau \right|^2 - \left| \int_0^{\tau^+(\omega)} f(\tau, \omega, \varepsilon)\, d\tau \right|^2 \right) dP \right)} < \eta$$
$$\text{for} \quad v < v(\eta) .$$

The function $v(\eta)$ is independent of ε and of the sequence τ_i. This assertion follows easily from the inequalities

$$\sqrt{\left(\int \left(\left| \int_0^{\tau(\omega)} f(\tau, \omega, \varepsilon)\, d\tau \right|^2 - \left| \int_0^{\tau^+(\omega)} f(\tau, \omega, \varepsilon)\, d\tau \right|^2 dP \right)} \leqq$$
$$\leqq \sqrt{E} \left[\left| \int_{\tau^+(\omega)}^{\tau(\omega)} f(\xi, \omega, \varepsilon)\, d\xi \right| \cdot \left(\left| \int_0^{\tau^+(\omega)} |f(\xi, \omega, \varepsilon)\, d\xi \right| + \left| \int_0^{\tau(\omega)} f(\xi, \omega, \varepsilon)\, d\xi \right| \right) \right] \leqq$$
$$\leqq \sqrt{\left(2E \left[\int_{\tau^+(\omega)}^{\tau^+(\omega)+v} |f(\xi, \omega, \varepsilon)|\, d\xi \int_0^L |f(\xi, \omega, \varepsilon)|\, d\xi \right] \right)} \leqq$$
$$\leqq \sqrt{(2)}\, \sqrt[4]{\left(E \left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+v} |f(\xi, \omega, \varepsilon)|\, d\xi)^2 \right)^2 \right.}\, \sqrt[4]{\left(E \left(\int_0^L |f(\xi, \omega, \varepsilon)|\, d\xi^2 \right) \right.} .$$

$$\sqrt[4]{E\left(\int_0^L |f(\xi,\omega,\varepsilon)\,\mathrm{d}\xi\right)^2} \leqq \sqrt{\int_0^L \sqrt{(E|f(\xi,\omega,\varepsilon)|^2)}\,\mathrm{d}\xi} \leqq$$

$$\leqq \sqrt{\left[\int_0^L \sqrt{(E|a(\tau/\varepsilon, |\bar{x}(\tau))|^2)}\,\mathrm{d}\tau + \int_0^L \sqrt{(E|\bar{a}(y(\tau))|^2)}\,\mathrm{d}\tau\right]} \leqq$$

$$\leqq \sqrt{\left[2KL + K\int_0^L \sqrt{E|\bar{x}(\tau)|^2}\,\mathrm{d}\tau + K\int_0^L \sqrt{E|y(\tau)|^2}\,\mathrm{d}\tau\right]} \leqq \sqrt{[KL(2 + \|\bar{x}\|_L + \|y\|_L)]}\,.$$

$$\sqrt[4]{E\left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} |f(\xi,\omega,\varepsilon)|\,|\mathrm{d}\xi\right)^2} \leqq$$

$$\leqq \sqrt{\left[\sqrt{E}\left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} |a(\tau/\varepsilon, \bar{x}(\tau))|\,\mathrm{d}\tau\right)^2 + \sqrt{E}\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} |\bar{a}(y(\tau))|\,\mathrm{d}\tau)^2\right]} \leqq$$

$$\leqq \sqrt{\left[2K\nu + K\sqrt{E}\left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} |\bar{x}(\tau)|\,\mathrm{d}\tau\right)^2 + K\sqrt{E}\left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} |y(\tau)|\,\mathrm{d}\tau\right)^2\right]} \leqq$$

$$\leqq \sqrt{\left[2K\nu + K\sqrt{E}\left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} \sup_{\langle 0,L\rangle} |\bar{x}(\xi)|\,\mathrm{d}\tau\right)^2 + K\sqrt{E}\left(\int_{\tau^+(\omega)}^{\tau^+(\omega)+\nu} \sup_{\langle 0,L\rangle} |y(\xi)|\,\mathrm{d}\tau\right)^2\right]} \leqq$$

$$\leqq \sqrt{[2K\nu + K\nu\sqrt{E}\sup|\bar{x}(\xi)|^2 + K\nu\sqrt{E}\sup|y(\xi)|^2]} \leqq$$

$$\leqq \sqrt{(K\nu(2 + \|\bar{x}\|_L + \|y\|_L))}\,.$$

According to (8,7) for the norm $\|\bar{x}\|_L \leqq C$, C is independent of ε and τ_i; thus we may choose $\nu(\eta) = \eta^2(1 + C)^{-2}(8K^2L)^{-1}$.

Hence one need only consider the expression

$$\sqrt{E\left|\int_0^{\tau^+(\omega)} (a(\tau/\varepsilon, x(\tau)) - \bar{a}(y(\tau)))\,\mathrm{d}\tau\right|^2} = \sqrt{E\left|\int_0^{\gamma} (a(\tau/\varepsilon, x(\tau)) - \bar{a}(y(\tau)))\,\chi(\tau)\,\mathrm{d}\tau\right|^2} =$$

where $\chi(t, \omega)$ is the characteristic function of the interval $\langle 0, \tau^+(\omega)\rangle$. From the Hölder inequality,

$$= \left\|\int_0^{\gamma} (a(\tau/\varepsilon, \bar{x}(\tau)) - \bar{a}(y(\tau)))\,\chi(\tau)\,\mathrm{d}\tau\right\| \leqq \tag{5,1}$$

$$\leqq \sum_i \left\|\int_{\tau_i}^{\tau_{i+1}} (a(\tau/\varepsilon, \bar{x}(\tau)) - \bar{a}(y(\tau)))\,\chi(\tau)\,\mathrm{d}\tau\right\| \leqq \sum_i \left\|\int_{\tau_i}^{\tau_{i+1}} (a(\tau/\varepsilon, \bar{x}(\tau)) -\right.$$

$$\left. - a(\tau/\varepsilon, \bar{x}(\tau_i)))\,\chi(\tau)\,\mathrm{d}\tau\right\| + \sum_i \left\|\int_{\tau_i}^{\tau_{i+1}} (a(\tau/\varepsilon, \bar{x}(\tau_i)) - \bar{a}(\bar{x}(\tau_i)))\,\chi(\tau)\,\mathrm{d}\tau\right\| +$$

$$+ \sum_i \left\|\int_{\tau_i}^{\tau_{i+1}} (\bar{a}(\bar{x}(\tau)) - \bar{a}(\bar{x}(\tau_i)))\,\chi(\tau)\,\mathrm{d}\tau\right\| + \sum_i \left\|\int_{\tau_i}^{\tau_{i+1}} (\bar{a}(\bar{x}(\tau)) - \bar{a}(y(\tau)))\,\chi(\tau)\,\mathrm{d}\tau\right\|\,.$$

As $a(t, x)$ is Lipschitz continuous with constant K, so is $\bar{a}(x)$, and thus one may continue

$$(6{,}1) \qquad \leqq \sum_i \left\| \int_{\tau_i}^{\tau_{i+1}} (a(\tau/\varepsilon, \bar{x}(\tau_i)) - \bar{a}(\bar{x}(\tau_i)))\, \mathrm{d}\tau \right\| +$$

$$+ 2K \sum_i \int_{\tau_i}^{\tau_{i+1}} \|\bar{x}(\tau) - \bar{x}(\tau_i)\|\, \mathrm{d}\tau + K \int_0^{\gamma} \|\bar{x}(\tau) - y(\tau)\|\, \mathrm{d}\tau\,.$$

The first term at (6,1) converges to 0 as $\varepsilon \to 0$ for fixed τ_i since it may be written as

$$\sum_i |\tau_{i+1} - \tau_i| \frac{\varepsilon}{|\tau_{i+1} - \tau_i|} \left\| \int_{\tau_i/\varepsilon}^{\tau_{i+1}/\varepsilon} (a(\tau, \bar{x}(\tau_i)) - \bar{a}(\bar{x}(\tau_i)))\, \mathrm{d}\tau \right\|$$

and one may apply Lemma 1. From assumption iv) it follows that the functions $\bar{F}_\varepsilon(\tau_2) - \bar{F}_\varepsilon(\tau_1)$ converge to $F_0(\tau_2) - F_0(\tau_1)$ uniformly with respect to $\tau_1, \tau_2 \in \langle 0, L\rangle$. According to (8,7) there are numbers $\varepsilon_0 > 0$ and $C > 0$ such that $\|\bar{x}(t, \omega)\|_L \leqq C$ for $0 < \varepsilon \leqq \varepsilon_0$. From (7,7) it follows that the second term in (6,1) can be estimated as

$$(7{,}1) \qquad \leqq 2K^2(1 + C)\left(v + 2\sqrt{n}\sqrt{\left(\max_{\langle 0, L-v\rangle} (F_0(\lambda + v) - F_0(\lambda)) + \right.}\right.$$

$$\left.\left. + \max_{\langle 0, L\rangle} |\bar{F}_\varepsilon(\lambda) - F_0(\lambda)|\right)\right).$$

Thus, we obtain that the second term in (6,1) converges to zero if v and ε converge to zero.

From (3,1) to (7,1) we conclude

$$(8{,}1) \qquad \left\| \int_0^{\xi} (a(\tau/\varepsilon, \bar{x}(\tau)) - \bar{a}(y(\tau)))\, \mathrm{d}\tau \right\|_{\gamma} \leqq \varphi_1(\varepsilon, v) + K \int_0^{\gamma} \|\bar{x}(\tau) - y(\tau)\|_\tau\, \mathrm{d}\tau$$

where $\lim_{v \to 0} \overline{\lim_{\varepsilon \to 0}}\, \varphi_1(\varepsilon, v) = 0$.

Next estimate the third term in (2,1). Since $|\int_0^{\xi} (B(\tau/\varepsilon, \bar{x}(\tau)) - \bar{B}(y(\tau)))\, \mathrm{d}w_0(\tau)|$ is a semi-martingale, we may use Theorem 3,4 from Chap. VII of [2],

$$(9{,}1) \qquad \left\| \int_0^{\xi} (B(\tau/\varepsilon, \bar{x}(\tau)) - \bar{B}(y(\tau)))\, \mathrm{d}w_0(\tau) \right\|_{\gamma} \leqq$$

$$\leqq 2 \left\| \int_0^{\gamma} (B(\tau/\varepsilon, \bar{x}(\tau)) - \bar{B}(y(\tau)))\, \mathrm{d}w_0(\tau) \right\|,$$

and by (4,7)

$$(10{,}1) \qquad \leqq 2 \sqrt{\left(n \int_0^{\gamma} \|B(\tau/\varepsilon, \bar{x}(\tau)) - \bar{B}(y(\tau))\|^2\, \mathrm{d}F_0(\tau)\right)}\,.$$

Now proceed similarly as in the case of (5,1):

$$(11,1) \qquad \leqq 4\sqrt{\left(n\sum_i\int_{\tau_i}^{\tau_{i+1}}\|B(\tau/\varepsilon,\bar{x}(\tau_i)) - \bar{B}(\bar{x}(\tau_i))\|^2\,\mathrm{d}F_0(\tau)\right)} +$$

$$+ 8K\sqrt{\left(n\sum_i\int_{\tau_i}^{\tau_{i+1}}\|\bar{x}(\tau) - \bar{x}(\tau_i)\|^2\,\mathrm{d}F_0(\tau)\right)} + 4K\sqrt{\left(n\int_0^{\gamma}\|\bar{x}(\tau) - y(\tau)\|^2\,\mathrm{d}F_0(\tau)\right)}.$$

The first term in (11,1) can be put in the form

$$4\sqrt{\left(n\sum_i\int_{\tau_i/\varepsilon}^{\tau_{i+1/\varepsilon}}E|B(\tau,\bar{x}(\tau_i)) - \bar{B}(\bar{x}(\tau_i))|^2\,\mathrm{d}F_0(\tau\varepsilon)\right)},$$

and according to viii) it converges to zero for $\varepsilon \to 0$. As in the previous case we prove that the second term of (11,1) converges to 0 for $v \to 0$, $\varepsilon \to 0$. From (7,1) to (11,1) it follows that

$$(12,1) \qquad \left\|\int_0^{\xi}(B(\tau/\varepsilon,\bar{x}(\tau)) - \bar{B}(y(\tau)))\,\mathrm{d}w_0(\tau)\right\|_{\gamma} \leqq \varphi_2(\varepsilon, v) +$$

$$+ 4K\sqrt{\left(n\int_0^{\gamma}\|\bar{x}(\tau) - y(\tau)\|^2\,\mathrm{d}F_0(\tau)\right)}$$

where $\lim_{v\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varphi_2(\varepsilon, v) = 0$.

It remains to estimate the last term of (2,1). There is

$$(13,1) \qquad \left\|\int_0^{\xi}B(\tau/\varepsilon,\bar{x}(\tau))\,\mathrm{d}w_\varepsilon^*(\tau)\right\|_{\gamma} \leqq 2\sqrt{\left(n\int_0^{\gamma}\|B(\tau/\varepsilon,\bar{x}(\tau))\|^2\,\mathrm{d}F_\varepsilon^*(\tau)\right)} \leqq$$

$$\leqq 4K\sqrt{\left(n\int_0^{\gamma}(1 + \|\bar{x}(\tau)\|^2)\,\mathrm{d}F_\varepsilon^*(\tau)\right)} \leqq 4K(1 + C)\sqrt{n}\,\sqrt{(F_\varepsilon^*(\gamma) - F_\varepsilon^*(0))} \leqq$$

$$\leqq 4K(1 + C)\sqrt{n}\,\|\sqrt{(\varphi(\varepsilon))}\,w_\varepsilon(\gamma/\varepsilon) - \sqrt{(\varphi(\varepsilon))}\,w_\varepsilon(0) - w_0(\gamma) + w_0(0)\| \to 0$$

for $\varepsilon \to 0$ in accordance with the uniform convergence $\bar{w}_\varepsilon(t)$ to $w_0(t)$.

By (8,1), (12,1) and (13,1),

$$(14,1) \qquad \|\bar{x}(\xi) - y(\xi)\|_{\gamma} \leqq \varphi(\varepsilon, v) + K\int_0^{\gamma}\|\bar{x}(\tau) - y(\tau)\|_{\tau}\,\mathrm{d}\tau +$$

$$+ 4K\sqrt{\left(\int_0^{\gamma}\|\bar{x}(\tau) - y(\tau)\|_{\tau}^2\,\mathrm{d}F_0(\tau)\right)}$$

where $\lim_{v\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varphi(\varepsilon, v) = 0$. From Lemma 2 it follows that $\lim_{\varepsilon\to 0}\|\bar{x}(\xi) - y(\xi)\|_{\gamma} = 0$, and this is the assertion of Theorem 1.

Remark 1. The assumptions $\overline{\mathscr{F}}_{\varepsilon}(t) \subset \mathscr{F}(t)$, $\overline{\mathscr{F}}_{\varepsilon}(t) \subset \mathscr{F}^*_{\varepsilon}(t)$ may be replaced by $\mathscr{F}(t) \subset \overline{\mathscr{F}}_{\varepsilon}(t)$, $\mathscr{F}(t) \subset \mathscr{F}^*_{\varepsilon}(t)$ since the expression in (2,1) may be estimated in another way. In the case $w^*_{\varepsilon}(t) \equiv 0$ the assumption $\overline{\mathscr{F}}_{\varepsilon}(t) \subset \mathscr{F}^*_{\varepsilon}(t)$ need not be considered as $w^*_{\varepsilon}(t_4) - w^*_{\varepsilon}(t_3)$ is already independent of $\mathscr{F}^*_{\varepsilon}(t)$.

Remark 2. If $\mathrm{d}F(t) = t^{\lambda}\,\mathrm{d}t$, the assumption viii) may be formulated in a simplier manner, as

$$\lim_{T\to\infty} \frac{1}{T^{\lambda+1}} \int_0^T |B(t, x) - \bar{B}(x)|^2\, t^{\lambda}\,\mathrm{d}t = 0$$

uniformly with respect to x.

Proof. If the assertion did not hold, there would exist a number $\bar{\varepsilon} > 0$ and sequences x_i and $T_i \to \infty$ such that

$$\frac{1}{(\beta T_i)^{\lambda+1}} \int_{\alpha T_i}^{\alpha T_i + \beta T_i} |B(t, x_i) - \bar{B}(x_i)|^2\, t^{\lambda}\,\mathrm{d}t \geqq \bar{\varepsilon} > 0$$

for large i's. Put $\overline{T}_i = \alpha T_i + \beta T_i$. Then

$$\frac{1}{\overline{T}_i^{\lambda+1}} \int_0^{\overline{T}_i} |B(t, x_i) - \bar{B}(x_i)|^2\, t^{\lambda}\,\mathrm{d}t \geqq \left(\frac{\alpha}{\alpha+\beta}\right)^{\lambda+1} \frac{1}{(\alpha T_i)^{\lambda+1}} \int_0^{\alpha T_i} |B(t, x_i) -$$

$$- \bar{B}(x_i)|^2\, t^{\lambda}\,\mathrm{d}t + \left(\frac{\beta}{\alpha+\beta}\right)^{\lambda+1} \frac{1}{(\beta T_i)^{\lambda+1}} \int_{\alpha T_i}^{\alpha T_i+\beta T_i} |B(t, x_i) - \bar{B}(x_i)|^2\, t^{\lambda}\,\mathrm{d}t \geqq$$

$$\geqq \left(\frac{\beta}{\alpha+\beta}\right)^{\lambda+1} \bar{\varepsilon} > 0\,,$$

in contradiction with the assumption of Remark 2.

Remark 3. If $w(t)$ is a one-dimensional Wiener process (cf. the definition in [3]), then the condition for $B(t, x)$ is $\lim_{T\to\infty} (1/T) \int_0^T |B(t, x) - \bar{B}(x)|^2\,\mathrm{d}t = 0$. This condition is obviously more restricting than condition vii). However, the equation $x(t) = \sqrt{(\varepsilon)} \int_0^t \sin\tau\,\mathrm{d}w_{\varepsilon}(\tau)$ where $w_{\varepsilon}(t) = (1/\sqrt{(\varepsilon)})\, w(\varepsilon t)$, $w(t)$ is a Wiener process shows that condition vii) for $B(t, x)$ would be not sufficient[2]). To complete the proof of the theorem it remains to prove the Lemmas 1 and 2 .

Lemma 1. *Let* $a(t, x)$ *fulfil assumptions* i) ii) *and* $\lim_{T\to\infty} (1/T) \int_0^T a(t, x)\,\mathrm{d}t = \bar{a}(x)$ *uniformly for all* x; *then* $\lim_{T\to\infty} (1/T) \int_{\alpha T}^{\alpha T+T} a(t, x)\,\mathrm{d}t = \bar{a}(x)$ *uniformly for all* x *and arbitrary* $\alpha \geqq 0$.

Proof. If the assertion of the Lemma were not valid, there would exist a number

[2]) Such an example also appears in the paper mentioned above.

$\bar{\varepsilon} > 0$, a index j and sequences x_i and $T_i \to \infty$ such that $|(1/T_i) \int_{\alpha T_i}^{\alpha T_i + T_i} a_j(t, x_i)\, dt - \bar{a}_j(x_i)| \geqq \bar{\varepsilon}$. Without loss of generality we may assume $\alpha > 0$ and

$$(15{,}1) \qquad \frac{1}{T_i} \int_{\alpha T_i}^{\alpha T_i + T_i} a_j(t, x_i)\, dt \geqq \bar{a}_j(x_i) + \bar{\varepsilon}\,.$$

In the opposite case the proof is analogous. Put $T_i^* = \alpha T_i + T_i$; obviously $T_i^* \to \infty$. It follows that

$$\frac{1}{T_i^*} \int_0^{T_i^*} a_j(t, x_i)\, dt = \frac{\alpha T_i}{T_i^*} \frac{1}{\alpha T_i} \int_0^{\alpha T_i} a_j(t, x_i)\, dt + \frac{T_i}{T_i^*} \frac{1}{T_i} \int_{\alpha T_i}^{\alpha T_i + T_i} a_j(t, x_i)\, dt\,.$$

By (15,1) we obtain

$$\frac{1}{T_i^*} \int_0^{T_i^*} a_j(t, x_i)\, dt \geqq \frac{\alpha}{\alpha + 1} \frac{1}{\alpha T_i} \int_0^{\alpha T_i} (a_j(t, x_i) - \bar{a}(x_i))\, dt + \bar{a}_j(x_i) + \frac{1}{\alpha + 1} \bar{\varepsilon}$$

Since $\alpha T_i \to \infty$, the second term converges to zero and we find

$$\frac{1}{T_i^*} \int_0^{T_i^*} a_j(t, x_i)\, dt \geqq \bar{a}_j(x_i) + \frac{1}{\alpha + 2} \bar{\varepsilon}$$

for large i. The last inequality is in contradiction with the assumptions of Lemma 1.

Lemma 2. *Let $u(t)$ be a nonnegative integrable function on the interval $\langle 0, L\rangle$ which fulfils the inequality*

$$(16{,}1) \qquad u(t) \leqq a + K \int_0^t u(\tau)\, d\tau + K \sqrt{\left(\int_0^t u^2(\tau)\, dF(\tau)\right)},$$

where $F(t)$ is a continuous non-decreasing function on $\langle 0, L\rangle$. The constant a is nonnegative, the constant K is positive. Then $\lim_{a \to 0} u(t) = 0$ uniformly with respect to t in the interval $\langle 0, L\rangle$.

Proof. From (16,1) it follows that

$$u^2(t) \leqq 3a^2 + 3K^2 t \int_0^t u^2(\tau)\, d\tau + 3K^2 \int_0^t u^2(\tau)\, dF(\tau)\,.$$

We take the number $L > 0$ and put $c = 3K^2(L + 1)$, $F^*(t) = F(t) + t$. Then $u^2(t) \leqq 3a^2 + c \int_0^t u^2(\tau)\, dF^*(\tau)$ for $0 \leqq t \leqq L$. Using this inequality we obtain

$$u^2(t) \leqq 3a^2 \exp\{c(F^*(t) - F^*(0))\} = 3a^2 \exp\{c(F(t) - F(0)) + ct\}$$

for $0 \leqq t \leqq L$. The last inequality implies the assertion of the Lemma.

Before treating the case of the adhesive barrier we have to introduce several definitions.

Definition 1. Let a region G be given. The frontier G' of G is called smooth if it is possible to cover the frontier G' by a finite system of neighbourhoods U_j in which there are given functions ψ_j and indices $i(j)$ such that ψ_j have Hölder continuous second derivatives and that

$$G' \cap U_j = \{x : x_{i(j)} = \psi_j(x_1, \ldots, x_{i(j)-1}, x_{i(j)+1}, \ldots, x_n)\} \cap U_j .$$

Definition 2. Let a square matrix $B(x)$ of $b_{ij}(x)$ be defined on a set H. The matrix $B(x)$ is called canonic on H if there is a constant $m > 0$ such that

$$\sum_{i,j,k=1}^{n} b_{ik}(x)\, b_{jk}(x)\, \lambda_i \lambda_j \geqq m \sum_{i=1}^{n} \lambda_i^2$$

for all real λ_i, $x \in H$.

Definition 3. Let a process $x^*(t, \omega)$ be defined on the whole space and let G be a region from this space. Denote by $\tau(\omega)$ the Markov time of the first exit from G provided that it is a random value. Put $x(t, \omega) = x^*(\tau(\omega), \omega)$ for $t \geqq \tau(\omega)$, $x(t, \omega) = x^*(t, \omega)$ for $t < \tau(\omega)$. The process $x(t, \omega)$ is said to have the adhesive barrier G' which corresponds to the process x^*.

Remark 4. Let $x^*(t)$, $y^*(t)$ be solutions of (2), (3), where $w(t)$ is a Wiener process, and let a region G have smooth frontier. The processes $x(t)$, $y(t)$ are defined as the processes with adhesive barrier G' which correspond to $x^*(t)$, $y^*(t)$. The processes $x(t)$, $y(t)$ are solutions of the differential equations

$$x(t) = x_0 + \varepsilon \int_0^t a(\tau, x(\tau))\, \chi(x(\tau))\, d\tau + \sqrt{(\varphi(\varepsilon))} \int_0^t B(\tau, x(\tau))\, \chi(x(\tau))\, dw_\varepsilon(\tau)$$

$$y(t) = x_0 + \int_0^t \bar{a}(y(\tau))\, \chi(y(\tau))\, d\tau + \int_0^t \bar{B}(y(\tau))\, \chi(y(\tau))\, dw_0(\tau)$$

where $\chi(x)$ is the characteristic function of the region G (the proof of this statement is in [3]). We cannot use Theorem 1 in this case, since the functions $a_i(t, x)\, \chi(x)$, $b_{ij}(t, x)\, \chi(x)$ are not continuous in x; nevertheless, there is a similar assertion. The following theorem deals with this case.

Theorem 2. *Let a region G be given with compact boundary and smooth frontier. Let $a(t, x)$, $B(t, x)$ fulfil the assumptions of Theorem 1 throughout the boundary $\bar{G}$, and let the matrix $\bar{B}(x)$ be canonic in $\bar{G}$. Let $w_0(t)$ be a Wiener process, and $\mathscr{F}_\varepsilon(t) = \mathscr{F}(t)$ for all $t \geqq 0$. If $x^*(t)$, $y^*(t)$ are the solutions of (2), (3) and $x(t)$, $y(t)$ the processes with adhesive barrier G' which correspond to $x^*(t)$, $y^*(t)$, then statement of Theorem 1 holds for the processes $x(t)$, $y(t)$.*

In order to prove this theorem in detail we would have to utilize several more fundamental properties of Markov processes, and the proof would be very complicated, so it will only be sketched. The functions $a_i(t, x)$, $b_{ij}(t, x)$ may be extended to the whole space in such a manner they are bounded and that the assumptions are fulfilled on the whole space. Put $\varepsilon t = \xi$, $\bar{x}(\xi) = x(t)$, $\bar{x}^*(\xi) = x^*(t)$; this is the same transformation as at the beginning of the proof of Theorem 1. $\bar{x}^*(\xi)$ is then a solution of (1,1). Let τ_1 and τ_2 be the Markov time of the first exit from G of the process $\bar{x}^*$ or y^* respectively. We may decompose the set Ω into three parts,

$$A_1 = \{\omega : \tau_1 = \tau_2 \text{ or } \min \tau_i \geqq L\}\,, \quad A_2 = \{\omega : \tau_1 < \tau_2\}\,, \quad A_3 = \{\omega : \tau_2 < \tau_1\}\,.$$

Since $\bar{x}(t, \omega) = \bar{x}^*(t, \omega)$, $y(t, \omega) = y^*(t, \omega)$ for $t \leqq \tau_i$, $\omega \in A_1$

$$(1,2) \qquad \int_{A_1} \sup_{\langle 0,L\rangle} |\bar{x}(t) - y(t)|^2 \,\mathrm{d}P = \int_{A_1} \sup_{\langle 0,\min(\tau_i,L)\rangle} |\bar{x}^*(t) - y^*(t)|^2 \,\mathrm{d}P \leqq$$

$$\leqq \int_{\Omega} \sup_{\langle 0,L\rangle} |\bar{x}^*(t) - y^*(t)|^2 \,\mathrm{d}P\,.$$

By Theorem 1, the last expression in (1,2) converges to 0 for $\varepsilon \to 0$.

Denote by $S_\delta(H)$ the δ-neighbourhood of the set H. The following Lemma plays an important part in the proof.

Lemma 3. *There is a function $\varphi(\delta)$ with $\varphi(\delta) > 0$, $\varphi(\delta) \to 0$, $\varepsilon \to 0$ such that $E(\tau(z)) < \varphi(\delta)$ for every solution of* (3) *with initial condition z_0, $z_0 \in \bar{G}$, $P(z_0 \in S_\delta(G')) > 1 - \delta$ (instead of x_0) where $\tau(z)$ is the Markov time of the first exit from the region G.*

This lemma is a modification of Theorem 13,7, [3]. From $\mathscr{F}_\varepsilon(t) = \mathscr{F}(t)$ it follows that τ_1 is a Markov time of y^*, and thus $\tau_3 = \min(\tau_1, \tau_2)$ is a Markov time of y^*. By Theorem 1 we obtain $\int_{A_2} \sup_{\langle 0,\tau_1\rangle} |\bar{x}^*(t) - y^*(t)|^2 \,\mathrm{d}P < \varphi_1(\varepsilon)$, where $\varphi_1(\varepsilon)$ si a function with $\varphi_1(\varepsilon) > 0$, $\varphi_1(\varepsilon) \to 0$ for $\varepsilon \to 0$. There is a function $\varphi_2(\varepsilon)$ with $\varphi_2(\varepsilon) > 0$, $\varphi_2(\varepsilon) \to 0$ for $\varepsilon \to 0$, such that

$$(3,2) \qquad P(y^*(\tau_3) \in S_{\varphi_2}(G')) > 1 - \varphi_2\,.$$

Put $z(t) = y^*(t + \tau_3)$. As the process y^* has the strong Markov property, the process z is equivalent to some solution of (3). The Markov time of the first exit from G of z is $\tau_2 - \tau_3$, and its initial distribution is $z(0) = y^*(\tau_3)$; by (3,2) we obtain that $P(z(0) \in S_{\varphi_2}(G')) > 1 - \varphi_2$. From Lemma 3, $E(\tau_2 - \tau_3) < \varphi_3(\varepsilon)$, where $\varphi_3(\varepsilon) > 0$, $\varphi_3(\varepsilon) \to 0$ for $\varepsilon \to 0$, and, according to the definition of τ_3,

$$\int_{A_2} (\tau_2 - \tau_1) \,\mathrm{d}P < \varphi_3(\varepsilon)\,.$$

From (7,7), (8,7) and the strong Markov property of y, we obtain

$$(4,2) \quad \int_{A_2} \sup_{\langle 0,L\rangle} |\bar{x}(t) - y(t)|^2 \,\mathrm{d}P < \varphi_4(\varepsilon) \quad \text{where} \quad \varphi_4 > 0,\ \varphi_4 \to 0 \quad \text{for} \quad \varepsilon \to 0\,.$$

It remains to show that there is a function $\varphi_5(\varepsilon)$ with $\varphi_5(\varepsilon) > 0$, $\varphi_5(\varepsilon) \to 0$ for $\varepsilon \to 0$, such that

$$(5,2) \qquad \int_{A_3} \sup_{\langle 0,L\rangle} |\bar{x}(t) - y(t)|^2 \, dP < \varphi_5(\varepsilon) .$$

The proof of this inequality is based on the following

Lemma 4. *Let $z(t, \omega)$ be a solution of (3) with initial condition $z(0, \omega) \in G'$. Then there is a sequence of regions $\ldots G_n \supset \bar{G}_{n+1} \supset G_{n+1} \ldots$ such that $\cap G_n = G$, G_n have smooth frontiers, $\bar{G}_n$ are compact, and also a sequence of numbers $\delta_n > 0$, $\lim_{n\to\infty} \delta_n = 0$ that* 1) *$P(|z(\hat{\tau}_n) - z(0)| < \delta_n) \to 1$ for $n \to \infty$, where $\hat{\tau}_n$ is the Markov time of the first exit of z from G_n, and* 2) *$E(\hat{\tau}_n) \to 0$ for $n \to \infty$.*

This lemma follows from Theorem 13.16 of [3]. Now put $z(t) = y^*(t + \tau_2)$. Since the process y^* has the strong Markov property, the process z is equivalent to a solution of (3) with initial condition $z(0) = y^*(\tau_2) \in G'$. The Markov time of the first exit of z from G_n is $\tau_2^{(n)} - \tau_2$, and

$$P(|z(\tau_2^{(n)} - \tau_2) - z(0)| < \eta) = P(|y^*(\tau_2^{(n)}) - y^*(\tau_2)| < \eta) .$$

By Lemma 4, to every $\eta > 0$ there is an n such that

$$(6,2) \qquad \begin{gathered} P(|z(\tau_2^{(n)} - \tau_2) - z(0)| < \eta) > 1 - \eta , \\ E(\tau_2^{(n)} - \tau_2) < \eta . \end{gathered}$$

To this n take $\alpha = \min \varrho(x, y)$, $x \in G'$, $y \in G_n'$ (ϱ is the Euclidean distance of the points x, y. By Theorem 1 there exists an $\varepsilon_0(\eta) > 0$ such that $E(\sup_{\langle 0,L\rangle} |\bar{x}^*(t) - y^*(t)|^2) < \alpha^2\eta/4$ whenever $0 < \varepsilon \leqq \varepsilon_0(\eta)$ and $P(\sup_{\langle 0,\tau_2^{(n)}\rangle} |\bar{x}^*(t) - y^*(t)| \geqq \alpha/2) \leqq \eta$. Since the distance between G' and G_n' is less than α, we obtain that

$$(7,2) \qquad \tau_1 \leqq \tau_2^{(n)} \quad \text{with the exception of } \omega \text{ in } A_3' ,$$

$A_3' \subset A_3$, $P(A_3') \leqq \varphi_5(\eta)$, $\varphi_5(\eta) > 0$, $\varphi_5(\eta) \to 0$ for $\eta \to 0$. From (6,2), (7,2) it follows that there is a constant $C > 1$ such that $\int \sup_{\langle 0,L\rangle} |\bar{x}(t) - y(t)|^2 \, dP \leqq C\eta$ for $0 < \varepsilon \leqq \varepsilon_0(\eta)$ where it is integrated throught A_3; then this assertion is equivalent to (5,2). The assertion of the theorem then follows from (1,2), (4,2) and (5,2).

Remark 5. In Theorem 2 it is sufficient to assume that the frontier has continuous curvature. The proof is then based on results from [6] and on immersion theorems.

From Theorem 1 it follows that $E|\bar{x}(t) - y(t)|^2 < \eta$ for $t \in \langle 0, L\rangle$, $0 < \varepsilon \leqq \varepsilon_0$. In the case that the solution $y(t)$ is asymptotically stable in average and if other assumptions are fulfilled then this inequality holds for every t.

Next we shall consider a stochastic integral equation

$$z(t) = z_0 + \int_{t_0}^{t} a(\tau, z(\tau))\,\mathrm{d}\tau + \int_{t_0}^{t} B(\tau, z(\tau))\,\mathrm{d}w(\tau) \tag{1,3}$$

where a, B fulfil the assumptions i) ii), the process $w(t)$ fulfils iii) and the initial random value z_0 fulfils ix).

First there will be presented the definitions of solutions stable in average and of solutions asymptotically stable in average.

Definition 4. A solution $z(t)$ of (1,3) is stable in average if there is a function $\delta(\varepsilon) > 0$, such that there is $\sup\limits_{\langle t_0 \infty)} E|z(t) - \tilde{z}(t)|^2 < \varepsilon$ for every $t_0 \geqq 0$ and for every solution $\tilde{z}(t)$ of $\tilde{z}(t) = \tilde{z}(t_0) + \int_{t_0}^t a(\tau, \tilde{z}(\tau))\,\mathrm{d}\tau + \int_{t_0}^t B(\tau, \tilde{z}(\tau))\,\mathrm{d}w(\tau)$ with initial condition $E|z(t_0) - \tilde{z}(t_0)|^2 < \delta(\varepsilon)$.

Definition 5. The solution $z(t)$ of (1,3) is asymptotically stable in average if it is stable in average and if there are a number $A > 0$ and a function $T(\delta, \eta)$ defined for $\delta < A$, $\eta < A$ such that $E|z(t) - \tilde{z}(t)|^2 < \eta$ for $t \geqq t_0 + T(\delta, \eta)$ if $E|z(t_0) - \tilde{z}(t_0)|^2 < \delta$.

Theorem 3. *Let the assumptions of Theorem* 1 *be fulfilled,*

$$\lim_{T\to\infty} \frac{1}{T} \int_{\alpha T}^{\alpha T + T} a\,(t,x)\,\mathrm{d}t = \bar{a}(x)\,, \quad \lim_{T\to\infty} \frac{1}{T} \int_{\alpha T}^{\alpha T + \beta T} |B(t, x) - \bar{B}(x)|^2\,\mathrm{d}t = 0$$

uniformly with respect to x and α for every β; $w(t)$ is a Wiener process (viz [3]). *If $y(t)$, $y(t) = y_0$ for $t \geqq t_0$, is a solution of* (3), *which is constant and asymptotically stable in average then to every $\eta > 0$ there are $\varepsilon_0 > 0$ and $\delta > 0$ such that*

$$\sup_{\langle t_0, \infty)} E|x(t) - y_0|^2 < \eta \quad \textit{for} \quad 0 < \varepsilon \leqq \varepsilon_0$$

where $x(t)$ is any solution of (2) *with initial condition $E|x(t_0) - y_0|^2 < \delta$.*

Proof. Without loss of generality take $y_0 = 0$. In accordance with Definition 4 we find to given $\eta > 0$ a number δ, $0 < \delta < A$, such that $E|\tilde{y}(t_0)|^2 < \delta$ implies

$$E|\tilde{y}(t)|^2 < \eta/4 \quad \text{for all} \quad t \geqq t_0\,. \tag{2,3}$$

According to Theorem 1 and the assumptions of Theorem 3, to $T(\delta, \delta/4)$ there is a number $\varepsilon_0 > 0$ (independent of t_0), such that

$$\sup_{\langle t_0, t_0 + T(\delta, \delta/4)\rangle} E|x(t) - \tilde{y}(t)|^2 < \delta/4 \quad \text{for} \quad 0 < \varepsilon \leqq \varepsilon_0\,. \tag{3,3}$$

The numbers δ and ε_0 have the required properties. Let $\bar{x}(t)$ be the solution of (1,1) and $E|\bar{x}(t_0)|^2 < \delta$ (cf. $y_0 = 0$). Let $\tilde{y}(t)$ be the solution of (3) with initial condition

$\tilde{y}(t_0) = \bar{x}(t_0)$. With respect to the stability in average of $y(t)$, one obtains (2,3). By (2,3) and (3,3),

$$E|\bar{x}(t)|^2 < \eta \quad \text{for} \quad t \in \langle t_0, t_0 + T(\delta, \delta/4)\rangle \,. \tag{4,3}$$

By definition of $T(\delta, \delta/4)$, cf. Definition 5, there is $E|\tilde{y}(t_0 + T(\delta, \delta/4))|^2 < \delta/4$; and according to the choice of ε_0 we obtain $E|\bar{x}(t_0 + T(\delta, \delta/4)) - \tilde{y}(t_0 + T(\delta, \delta/4))|^2 < \delta/4$. Hence

$$E|\bar{x}(t_0 + T(\delta, \delta/4))|^2 < \delta \,. \tag{5,3}$$

Put $t_k = t_0 + kT(\delta, \delta/4)$. Since $E|\bar{x}(t_1)|^2 < \delta$ i. e. (5,3) holds, we obtain that (4,3) holds again in the interval $\langle t_1, t_2\rangle$ and that (5,3) holds on substituting $t_1 + T(\delta, \delta/4)$ for $t_0 + T(\delta, \delta/4)$. The proof is then concluded by induction.

Sufficient conditions for stability in average and for asymptotic stability in average are given in the next section.

Let a function $F(t)$ be absolutely continuous; then by (2,7) the function $F_{ij}(t)$ are also absolutely continuous. Denote by $f(t)$ the derivative of $F(t)$ and by $f_{ij}(t)$ the derivatives of $F_{ij}(t)$.

Theorem 4. *Let the equation* (1,3) *have the solution* $z(t) \equiv 0$, *let* $a(t, z)$, $B(t, z)$, $w(t)$ *fulfil the conditions* i) iii) ix), *and let* $F(t)$ *be absolutely continuous. If there is a quadratic form* $V(t, z) = \sum_{i,j} c_{ij}(t)\, z_i z_j$ *which fulfils the condition that the* $c_{ij}(t)$ *have continuous second derivatives and that there are constants* $d_1 > 0$, $d_2 > 0$ *with*

$$d_1|z|^2 \leqq V(t, z) \leqq d_2|z|^2 \tag{1,4}$$

$$W(t, z) = \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial z_i} a_i + \sum_{i,j,k,l} c_{ij}(t)\, b_{ik} b_{jl} f_{kl}(t) \leqq 0 \tag{2,4}$$

for almost all t, *then the solution* $z(t) \equiv 0$ *is stable in average.*

Proof. First we shall prove the equation

$$E(V(t_2, z(t_2))) = E(V(t_1, z(t_1))) + E\int_{t_1}^{t_2} W(\tau, z(\tau))\, d\tau \quad \text{for} \quad 0 \leqq t_1 \leqq t_2 \,. \tag{3,4}$$

Obviously

$$E(V(t_2, z(t_2)) = E\left[V(t_1, z(t_2)) + \frac{\partial V}{\partial t}(t_1, z(t_2))\,(t_2 - t_1)\right] + \tag{4,4}$$
$$+ E\left[\frac{1}{2}\frac{\partial^2 V}{\partial t^2}(t^*, z(t_2))\,(t_2 - t_1)^2\right]$$

where $t_1 < t^* < t_2$. Choose a number $T > 0$ and assume $0 \leqq t_1 < t_2 \leqq T$. It will be said that $\Theta(t_1, t_2)$ has the property $\bar{o}(t_2 - t_1)$ if there is a function $\varphi(t)$ such that

$|\Theta(t_1, t_2)| \leqq \varphi(t_2 - t_1)$ for $0 \leqq t_1 < t_2 \leqq T$ and with $\lim_{t\to 0} \varphi(t)/t = 0$ or if $|\Theta(t_1, t_2))| \leqq \psi(t_2 - t_1)(F(t_2) - F(t_1))$ with $\lim_{t\to 0} \psi(t) = 0$. Since the $c''_{ij}(t)$ are bounded on $\langle 0, T\rangle$ and $E|z(t)|^2 \leqq \|z(t)\|_T^2 \leqq C^2$ (cf. (8,7)) for all $0 \leqq t \leqq T$, the last expression has property $\bar{o}(t_2 - t_1)$. Since

$$\left|\frac{\partial V}{\partial t}(t_1, z(t_2)) - \frac{\partial V}{\partial t}(t_1, z(t_1))\right|(t_2 - t_1) \leqq$$
$$\leqq K^*[|z(t_1)| \,.\, |z(t_2) - z(t_1)| + |z(t_2)| \,.\, |z(t_2) - z(t_1)|](t_2 - t_1),$$

we obtain

$$E\left[\left|\frac{\partial V}{\partial t}(t_1, z(t_2)) - \frac{\partial V}{\partial t}(t_1, z(t_1))\right|\right](t_2 - t_1) = \bar{o}(t_2 - t_1),$$

by the Hölder inequality and by (7,7). The equation (4,4) may be modified to

$$(5,4) \qquad = E\Big[V(t_1, z(t_1)) + \sum_{i,j} c_{ij}(t_1)\,(z_j(t_1)\,(z_i(t_2) - z_i(t_1)) +$$
$$+ z_i(t_1)\,(z_j(t_2) - z_j(t_1)) + \sum_{i,j} c_{ij}(t_1)\,(z_i(t_2) - z_i(t_1)) \,.\, (z_j(t_2) - z_j(t_1)) +$$
$$+ \frac{\partial V}{\partial t}(t_1, z(t_1))\,(t_2 - t_1)\Big] + \bar{o}(t_2 - t_1).$$

If we substitude the right-hand side of (1,3) instead of $z(t_2) - z(t_1)$, and consider the relation $E\left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,dw(\tau)\,|\mathscr{F}(t_1)\right) = 0$ we may proceed.

$$(6,4) \qquad = E\Big\{V(t_1, z(t_1)) + \sum_{i,j} c_{ij}(t_1)\Big[z_j(t_1)\int_{t_1}^{t_2} a_i(\tau, z(\tau))\,d\tau +$$
$$+ z_i(t_1)\int_{t_1}^{t_2} a_j(\tau, z(\tau))\,d\tau\Big] + \sum_{i,j} c_{ij}(t_1)\Big[\int_{t_1}^{t_2} a_i(\tau, z(\tau))\,d\tau \int_{t_1}^{t_2} a_j(\tau, z(\tau))\,d\tau +$$
$$+ \int_{t_1}^{t_2} a_i(\tau, z(\tau))\,d\tau\left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,dw(\tau)\right)_j + \int_{t_1}^{t_2} a_j(\tau, z(\tau))\,d\tau \times$$
$$\times \left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,dw(\tau)\right)_i + \left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,dw(\tau)\right)_i \times \left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,dw(\tau)\right)_j\Big] +$$
$$+ \frac{\partial V}{\partial t}(t_1, z(t_1))\,(t_2 - t_1)\Big\} + \bar{o}(t_2 - t_1)\,^3).$$

---

[3]) ($)_i$ denotes the i-th component of the vector within the parentheses.

We will estimate the terms from (6,4). By the Hölder inequality,

$$(7,4) \qquad E\left[\int_{t_1}^{t_2} a_i(\tau, z(\tau))\,d\tau \int_{t_1}^{t_2} a_j(\tau, z(\tau))\,d\tau\right] \leqq$$

$$\leqq \sqrt{E}\left|\int_{t_1}^{t_2} a_i(\tau, z(\tau))\,d\tau\right|^2 \times \sqrt{E}\left|\int_{t_1}^{t_2} a_j(\tau, z(\tau))\,d\tau\right|^2 \leqq$$

$$\leqq \int_{t_1}^{t_2} \sqrt{E|a_i(\tau, z(\tau))|^2}\,d\tau \times \int_{t_1}^{t_2} \sqrt{E|a_j(\tau, z(\tau))|^2}\,d\tau = \bar{o}(t_2 - t_1)\,.$$

The last equality holds since the a_i fulfil ii) and $E|z(t)|^2 \leqq c^2$ for $0 \leqq t \leqq T$. Similarly,

$$(8,4) \qquad E\left[\int_{t_1}^{t_2} a_i(\tau, z(\tau))\,d\tau \left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,dw(\tau)\right)_j\right] \leqq$$

$$\leqq \sqrt{(n)} \int_{t_1}^{t_2} \sqrt{E|a_i(\tau, z(\tau))|^2}\,d\tau \sqrt{\int_{t_1}^{t_2} E|B(\tau, z(\tau))|^2 f(\tau)\,d\tau} = \bar{o}(t_2 - t_1)\,.$$

Obviously

$$(9,4) \qquad \int_{t_1}^{t_z} E(z_j(t_1)a_i(\tau, z(\tau))\,d\tau = (t_2 - t_1)\,E(z_j(t_1)a_i(t_1, z(t_1)) +$$

$$+ \int_{t_1}^{t_2} E(z_j(t_1)(a_i(\tau, z(\tau)) - a_i(\tau, z(t_1))))\,d\tau + \int_{t_1}^{t_2} E(z_j(t_1)\,(a_i(\tau, z(t_1)) - a_i(t_1, z(t_1))))\,d\tau\,.$$

The third expression has the property $\bar{o}(t_2 - t_1)$, according to (7,7), (8,7) and to the Lipschitz continuity of a_i. To show that the last expression has the property $\bar{o}(t_2 - t_1)$, take $M_\varepsilon > 0$ such that $K \sqrt{\int_{G_\varepsilon} (1 + \sup_{0 \leqq t \leqq T} |z(t)|^2)\,dP} < \varepsilon$ where $G_\varepsilon = \{\omega : \sup_{0 \leqq t \leqq T} |z(t, \omega)| > M_\varepsilon\}$. The functions a_i are uniformly continuous in x throughout the closed region $|z| \leqq M_\varepsilon$, $0 \leqq t \leqq T$, so that one may take $\delta > 0$ with the property that $|\tau - t| < \delta$ implies $|a(t, z) - a(\tau, z)| < \varepsilon$. It follows that

$$\sqrt{E(|a_i(\tau, z(t_1)) - a_i(t_1, z(t_1))|^2)} \leqq \sqrt{\int_{\Omega - G_\varepsilon} |a_i(\tau, z(t_1)) - a_i(t_1, z(t_1))|^2\,dP} +$$

$$+ 4K \sqrt{\int_{G_\varepsilon} (1 + |z(t)|^2)\,dP} < 3\varepsilon \quad \text{for} \quad |\tau - t_1| < \delta\,.$$

That is, the last expression also has property $\bar{o}(t_2 - t_1)$, and

$$(10,4) \qquad \int_{t_1}^{t_2} E(z_j(t_1)a_i(\tau, z(\tau)))\,d\tau = (t_2 - t_1)\,E(z_j(t_1)a_i(t_1, z(t_1))) + \bar{o}(t_2 - t_1)\,.$$

Similarly one proves (cf. (5,7)) that

$$(11,4)\qquad E\left[\left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,\mathrm{d}w(\tau)\right)_i \times \left(\int_{t_1}^{t_2} B(\tau, z(\tau))\,\mathrm{d}w(\tau)\right)_j\right] =$$
$$= \sum_{k,l} E(b_{ik}(t_1, z(t_1))\, b_{jl}(t_1, z(t_1))\,(F_{kl}(t_2) - F_{kl}(t_1)) + \bar{o}(t_2 - t_1)\,.$$

On substituting (7,4) to (11,4) into (6,4) we obtain

$$E(V(t_2, z(t_2)) = E\left[V(t_1, z(t_1)) + \frac{\partial V}{\partial t}(t_1, z(t_1))(t_2 - t_1) + \right.$$
$$+ \sum_i \frac{\partial V}{\partial z_i}(t_1, z(t_1))\, a_i(t_1, z(t_1))(t_2 - t_1) + \sum_{i,j,k,l} c_{ij}(t_1)\, b_{ik}(t_1, z(t_1)) \times$$
$$\left. \times\, b_{jl}(t_1, z(t_1))\,(F_{kl}(t_2) - F_{kl}(t_1))\right] + \bar{o}(t_2 - t_1)\,.$$

Let $t_1 = \tau_0^{(k)} < \tau_1^{(k)} < \ldots < \tau_k^{(k)} = t_2$ be a sequence of subdivisions $\lim_{k\to\infty} \max_i \tau_{i+1}^{(k)} - \tau_i^{(k)} = 0$; if we use the last equation for $\langle \tau_i^{(k)}, \tau_{i+1}^{(k)} \rangle$ and take $k \to \infty$ we obtain (3,4). By (2,4).

$$(12,4)\qquad E(V(t_2, z(t_2)) \leqq E(V(t_1, z(t_1)) \quad \text{for} \quad t_1 \leqq t_2\,.$$

From (12,4) and according to (1,4) it follows that

$$d_1 E|z(t)|^2 \leqq EV(t, z(t)) \leqq E(V(t_0, z(t_0)) \leqq d_2 E|z(t_0)|^2\,.$$

Put $\delta = d_1 \varepsilon / d_2$; then $E|z(t_0)|^2 < \delta$ implies $E|z(t)|^2 < \varepsilon$ for all $t \geqq t_0$.

The next theorems deals with the asymptotic stability in average.

Theorem 5. *Let the assumptions from Theorem* 4 *be fulfilled, with* (2,4) *replaced by*

$$(1,5)\qquad W(t, z) \leqq -d_3|z|^2 \quad \textit{for almost all } t$$

where d_3 is a positive constant; then $z(t) \equiv 0$ is asymptotically stable in average.

Proof. By (3,4) and (1,5),

$$E(V(t_2, z(t_2)) \leqq E(V(t_1, z(t_1)) - d_3 \int_{t_1}^{t_2} E|z(t)|^2\,\mathrm{d}t$$

and with respect to (1,4),

$$d_1 E|z(t)|^2 \leqq d_2 E|z(t_0)|^2 - d_3 \int_{t_0}^{t} E|z(\tau)|^2\,\mathrm{d}\tau\,.$$

By Theorem 4, $z(t) \equiv 0$ is stable in average. The number A from Definition 5 may be chosen as $A = \delta(1)$ (the meaning of $\delta(\varepsilon)$ is the same as in Definition 4.) Assume given δ η, $0 < \delta < A$, $0 < \eta < A$. To every $\eta > 0$ there is a $\delta_1 > 0$ such that $E|z(t_0)|^2 < \delta_1$ implies

$$E|z(t)|^2 < \eta \quad \text{for} \quad t \geqq t_0 . \tag{2,5}$$

It is not possible to have $E|z(t)|^2 \geqq \delta_1$ for all $t \geqq t_0$ Indeed, then $d_1 E|z(t)|^2 \leqq \leqq d_2 E|z(t_0)|^2 - d_3(t - t_0)\,\delta_1$, and hence $E|z(t)|^2 < 0$ for $t = t_0 + d_2 E|z(t_0)|^2/(d_3\delta_1)$. This contradiction shows that there is a $T(\delta, \eta)$ such that $E|z(t_0 + T(\delta, \eta))|^2 < \delta_1$. However by (2,5) $E|z(t)|^2 < \eta$ for all $t \geqq t_0 + T(\delta, \eta)$.

The applications of this method to stochastic integral equations not in the form (2) are very important. In the case of ordinary differential equations we may apply this method to the equation $\ddot{x} + \omega^2 x = \varepsilon f(t, x)\, x$. Using the transformation $x = r \sin(\omega t + \varphi)$, $\dot{x} = r\omega \cos(\omega t + \varphi)$ one obtains the system

$$\dot{r} = \frac{\varepsilon}{2\omega}\, r f(t, r \sin(\omega t + \varphi)) \sin(2\omega t + 2\varphi)$$

$$\dot{\varphi} = -\frac{\varepsilon}{\omega}\, f(t, r \sin(\omega t + \varphi)) \sin^2(\omega t + \varphi) .$$

This form is adequate for using averaging method. To arrive at stochastic integral equations we rewrite the original differential equation as $\mathrm{d}x = y\,\mathrm{d}t$, $\mathrm{d}y = -\omega^2 x\,\mathrm{d}t + \varepsilon f(t, x(t))\, x(t)\,\mathrm{d}t$. The corresponding form for stochastic differential equations is $\mathrm{d}x = y\,\mathrm{d}t$, $\mathrm{d}y = -\omega^2 x\,\mathrm{d}t + \varepsilon a(t, x(t))\, x(t)\,\mathrm{d}t + \sqrt{(\varphi(\varepsilon))}\, B(t, x(t))\, x(t)\,\mathrm{d}w_\varepsilon(t)$. The integral form of the last system is $x(t) = x_0 + \int_{t_0}^t y(\tau)\,\mathrm{d}\tau$, $y(t) = y_0 - \omega^2 \int_{t_0}^t x(\tau)\,\mathrm{d}\tau + \varepsilon \int_{t_0}^t a(\tau, x(\tau))\, x(\tau)\,\mathrm{d}\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^t B(\tau, x(\tau))\, x(\tau)\,\mathrm{d}w_\varepsilon(\tau)$. There are several difficulties which are connected with utilisation of non-linear transformations: thus it would be necessary to investigate expressions similar to those in [5]. I shall introduce another transformation, which may also be applied for stochastic differential equations. We shall limit ourselves to the second order differential equation

$$\ddot{x} + \mu\dot{x} + \omega^2 x = \varepsilon f(t, x) , \tag{1,6}$$

but reasoning may also be applied to systems of equations. The system of stochastic integral equations corresponding of (1,6) is now

$$x(t) = x_0 + \int_{t_0}^t y(\tau)\,\mathrm{d}\tau \; y(t) = \tag{2,6}$$

$$= y_0 - \omega^2 \int_{t_0}^t x(\tau)\,\mathrm{d}\tau - \mu \int_{t_0}^t y(\tau)\,\mathrm{d}\tau + \varepsilon \int_{t_0}^t a(\tau, x(\tau))\,\mathrm{d}\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^t B(\tau, x(\tau))\,\mathrm{d}w(\tau) .$$

Assume $w(t)$ is a Wiener process.

Theorem 5. *Let $x(t)$, $y(t)$ be a solution of* (2.6) $Q_1(t)$, $Q_2(t)$ *solutions of* (1,6) *for* $\varepsilon = 0$ *and with initial condition* $Q_1(0) = Q_2'(0) = 1$, $Q_2(0) = Q_1'(0) = 0$. *Then there are the solutions* $u(t)$, $v(t)$ *of* (13,6) *such that*

$$x(t) = Q_1(t - t_0)\, u(t) + Q_2(t - t_0)\, v(t)$$

$$y(t) = -\omega^2 Q_2(t - t_0)\, u(t) + (Q_1(t - t_0) - \mu\, Q_2(t - t_0))\, v(t)\,.$$

Remark 6. The system (13,6) is already in the proper form for utilising the preceding theorems.

Proof. First we shall prove that (2,6) is equivalent to

$$(3,6)\quad x(t) = x_0\, Q_1(t - t_0) + y_0\, Q_2(t - t_0) + \varepsilon \int_{t_0}^{t} Q_2(t - \tau)\, a(\tau, x(\tau))\, d\tau + $$
$$+ \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2(t - \tau)\, B(\tau, x(\tau))\, dw(\tau)\,,$$

$$y(t) = -\omega^2 x_0\, Q_2(t - t_0) + (Q_1(t - t_0) - \mu\, Q_2(t - t_0))\, y_0 +$$
$$+ \varepsilon \int_{t_0}^{t} [Q_1(t - \tau) - \mu\, Q_2(t - \tau)]\, a(\tau, x(\tau))\, d\tau +$$
$$+ \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} [Q_1(t - \tau) - \mu\, Q_2(t - \tau)]\, B(\tau, x(\tau))\, dw(\tau)\,.$$

Using simple transformations we obtain from (2,6)

$$(4,6)\qquad x(t) = x_0 + y_0(t - t_0) - \omega^2 \int_{t_0}^{t}\int_{t_0}^{\xi} x(\tau)\, d\tau\, d\xi - \mu \int_{t_0}^{t}\int_{t_0}^{\xi} y(\tau)\, d\tau\, d\xi +$$
$$+ \varepsilon \int_{t_0}^{t}\int_{t_0}^{\xi} a(\tau, x(\tau))\, d\tau\, d\xi + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t}\int_{t_0}^{\xi} B(\tau, x(\tau))\, dw(\tau)\, d\xi$$

$$(5,6)\qquad y(t) = -\omega^2(t - t_0)\, x_0 + (1 - \mu(t - t_0))\, y_0 + \mu\omega^2 \int_{t_0}^{t}\int_{t_0}^{\xi} x(\tau)\, d\tau\, d\xi +$$
$$+ (\mu^2 - \omega^2) \int_{t_0}^{t}\int_{t_0}^{\xi} y(\tau)\, d\tau\, d\xi + \varepsilon \int_{t_0}^{t} a(\tau, x(\tau))\, d\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} B(\tau, x(\tau))\, dw(\tau) -$$
$$- \varepsilon\mu \int_{t_0}^{t}\int_{t_0}^{\xi} a(\tau, x(\tau))\, d\tau\, d\xi - \mu \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t}\int_{t_0}^{\xi} B(\tau, x(\tau))\, dw(\tau)\, d\xi\,.$$

Choose $t_i : t_0 < t_1 < \ldots < t_i < \ldots < t_k = t$, $t_{i+1} - t_i = \Delta t > 0$. Let $x(t)$, $y(t)$ be a solution of (2,6), and $x_i^*(t)$, $y_i^*(t)$ be defined on every interval $\langle t_i, t_{i+1}\rangle$ by

$$(6,6)\qquad x_i^*(t) = x(t_i)\, Q_1(t - t_i) + y(t_i)\, Q_2(t - t_i) +$$

$$+ \varepsilon \int_{t_i}^{t} Q_2(t - \tau)\, a(\tau, x(\tau))\, d\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_i}^{t} Q_2(t - \tau)\, B(\tau, x(\tau))\, dw(\tau)$$

$$y^*(t_i) = -\omega^2 x(t_i)\, Q_2(t - t_i) + y(t_i)\,(Q_1(t - t_i) - \mu\, Q_2(t - t_i)) +$$

$$+ \varepsilon \int_{t_i}^{t} (Q_1(t - \tau) - \mu\, Q_2(t - \tau))\, a(\tau, x(\tau))\, d\tau +$$

$$+ \sqrt{(\varphi(\varepsilon))} \int_{t_i}^{t} (Q_1(t - \tau) - \mu\, Q_2(t - \tau))\, B(\tau, x(\tau))\, dw(\tau)\,.$$

We shall prove that

$$(7,6)\qquad E\sum_i |x^*(t_{i+1}) - x(t_{i+1})| \to 0\,, \quad E\sum_i |y^*(t_{i+1}) - y(t_i)| \to 0 \quad \text{for} \quad \Delta t \to 0\,.$$

According to definition of $x^*(t)$, $y^*(t)$, $x(t)$, $y(t)$ and (4,6), we have

$$(8.6)\quad \sum E|x_{i+1}^* - x_{i+1}| \leqq \sum E|x_i| \,.\, |Q_1(\Delta t) - 1| + \sum E|y_i| \,.\, |Q_2(\Delta t) - \Delta t| +$$

$$+ \omega^2 E \sum \left| \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} x(\tau)\, d\tau\, d\xi \right| + |\mu| \sum E \left| \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} y(\tau)\, d\tau\, d\xi + \right.$$

$$\left. + \varepsilon \sum E \left| \int_{t_i}^{t_{i+1}} Q_2(t_{i+1} - \tau)\, a(\tau, x(\tau))\ d\tau \right| + \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} |a(\tau, x(\tau))|\, d\tau\, d\xi + \right.$$

$$+ \sqrt{(\varphi(\varepsilon))} \sum_i E \left| \int_{t_i}^{t_{i+1}} Q_2(t_{i+1} - \tau)\, B(\tau, x(\tau))\, dw(\tau) - \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} B(\tau, x(\tau))\, dw(\tau)\, d\xi \right|$$

$$(9,6)\qquad \sum_i E|y_{i+1}^* - y_{i+1}| \leqq \omega^2 \sum E|x_i| \,.\, |Q_2(\Delta t) - \Delta t| +$$

$$+ \sum E|y_i| \,.\, |Q_1(\Delta t) - \mu\, Q_2(\Delta t) - 1 + \mu\, \Delta t| + |\omega^2 - \mu^2| \sum E \left| \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} y(\tau)\, d\tau\, d\xi \right| +$$

$$+ |\mu|\, \omega^2 \sum E \left| \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} x(\tau)\, d\tau\, d\xi \right| + \varepsilon \sum E \left| \int_{t_i}^{t_{i+1}} (Q_1(t_{i+1} - \tau) - \right.$$

$$\left. - 1 - \mu\, Q_2(t_{i+1} - \tau))\, a(\tau, x(\tau))\, d\tau \right| + \sqrt{(\varphi(\varepsilon))} \sum E \left| \int_{t_i}^{t_{i+1}} (Q_1(t_{i+1} - \tau) - \right.$$

$$\left. - 1 - \mu\, Q_2(t_{i+1} - \tau))\, B(\tau, x(\tau))\, dw(\tau) \right| + \varepsilon |\mu| \sum E \left| \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} a(\tau, x(\tau))\, d\tau\, d\xi \right. +$$

$$\left. + |\mu| \sqrt{(\varphi(\varepsilon))} \sum E \left| \int_{t_i}^{t_{i+1}} \int_{t_i}^{\xi} B(\tau, x(\tau))\, dw(\tau)\, d\xi \right|, \right.$$

where $x_i = x(t_i)$, $y_i = y(t_i)$, $x_i^* = x_{i-1}^*(t_i)$, $y_i^* = y_{i-1}^*(t_i)$. Since $x(t)$, $y(t)$ is a solution of (2,6), there is $\|x(t)\| \leqq C$, $\|y(t)\| \leqq C$ and all terms on the right-hand sides of (8,6), and (9,6) converge to 0 with $\Delta t \to 0$. Thus (7,6) is proved. Let $\tilde{x}(t)$, $\tilde{y}(t)$ be the left-hand side of (3,6) if into the right-hand side there is substituted the solution $x(t)$, $y(t)$ of (2,6) as well as (4,6). Since $Q_1(t)$, $Q_2(t)$ are solutions of (1,6) for $\varepsilon = 0$, it follows from (3,6) that $\tilde{x}(t)$, $\tilde{y}(t)$ fulfil

$$(10,6) \qquad \tilde{x}(t_{i+1}) = Q_1(\Delta t)\, \tilde{x}(t_i) + Q_2(\Delta t)\, \tilde{y}(t_i) +$$

$$+ \varepsilon \int_{ti}^{t_{i+1}} Q_2(t_{i+1} - \tau)\, a(\tau, x(\tau))\, \mathrm{d}\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_i}^{t_{i+1}} Q_2(t_{i+1} - \tau)\, B(\tau, x(\tau))\, \mathrm{d}w(\tau)$$

$$\tilde{y}(t_{i+1}) = -\omega^2\, x(t_i)\, Q_2(\Delta t) + (Q_1(\Delta t) - \mu\, Q_2(\Delta t))\, \tilde{y}(t_i) +$$

$$+ \varepsilon \int_{t_i}^{t_{i+1}} [Q_1(t_{i+1} - \tau) - \mu\, Q_2(t_{i+1} - \tau)]\, a(\tau, x(\tau))\, \mathrm{d}\tau +$$

$$+ \sqrt{(\varphi(\varepsilon))} \int_{ti}^{t_{i+1}} [Q_1(t_{i+1} - \tau) - \mu\, Q_2(t_{i+1} - \tau)]\, B(\tau, x(\tau))\, \mathrm{d}w(\tau)\,.$$

From (6,6), (10,6) we obtain

$$(11,6) \qquad |x(t_{i+1}) - \tilde{x}(t_{i+1})| + |y(t_{i+1}) - \tilde{y}(t_{i+1})| \leqq |x(t_{i+1}) - x_i^*(t_{i+1})| +$$

$$+ |y(t_{i+1}) - y_i^*(t_{1+1})| + (|Q_1(\Delta t)| + \omega^2 |Q_2(\Delta t)|)\, |x(t_i) - \tilde{x}(t_i)| +$$

$$+ (|Q_1(\Delta t)| + |Q_2(\Delta t)| + |\mu| \cdot |Q_2(\Delta t)|)\, |y(t_i) - \tilde{y}(t_i)|\,.$$

The expressions $|Q_1(\Delta t)| + \omega^2 |Q_2(\Delta t)|$, $|Q_1(\Delta t)| + |\mu| \cdot |Q_2(\Delta t)|$ are of the type $1 + \alpha_i\, \Delta t$ where $\alpha_i \to \alpha_i^{(0)}$ for $\Delta t \to 0$, $i = 1, 2$. Set $\beta = 2 \max\, (\alpha_1^{(0)}, \alpha_2^{(0)}, 0)$. By (11,6),

$$|x(t_{i+1}) - \tilde{x}(t_{i+1})| + |y(t_{i+1}) - \tilde{y}(t_{i+1})| \leqq$$
$$\leqq (1 + \beta\, \Delta t)^s \left(\sum |x(t_j) - x^*(t_j)| + |y(t_j) - y^*(t_j)|\right)$$

where $s = 1/\Delta t$. Obviously $(1 + \beta\, \Delta t)^s \to e^\beta$, and according to (7,6),

$$E[|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)|] = 0\,.$$

As $Q_1(t)$, $Q_2(t)$ are solutions of (1,6) for $\varepsilon = 0$,

$$Q_2(t - \tau) = Q_1(t - t_0)\, Q_2(t_0 - \tau) + Q_2(t - t_0)\, Q_2'(t_0 - \tau)$$

$$Q_1(t - \tau) - \mu\, Q_2(t - \tau) =$$

$$= -\omega^2\, Q_2(t - t_0)\, Q_2(t_0 - \tau) + (Q_1(t - t_0) - \mu\, Q_2(t - t_0))\, Q_2'(t_0 - \tau)\,.$$

The equations (3,6) may be rewritten as

$$(12,6)\qquad x(t) = Q_1(t - t_0)\left(x_0 + \varepsilon \int_{t_0}^{t} Q_2(t_0 - \tau)\, a(\tau, x(\tau))\, d\tau + \right.$$

$$\left. + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2(t_0 - \tau)\, B(\tau, x(\tau))\, dw(\tau)\right) +$$

$$+ Q_2(t - t_0)\left(y_0 + \varepsilon \int_{t_0}^{t} Q_2'(t_0 - \tau)\, a(\tau, x(\tau))\, d\tau + \right.$$

$$\left. + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2'(t_0 - \tau)\, B(\tau, x(\tau))\, dw(\tau)\right)$$

$$y(t) = -\omega^2 Q_2(t - t_0)\left(x_0 + \varepsilon \int_{t_0}^{t} Q_2(t_0 - \tau)\, a(\tau, x(\tau))\, d\tau + \right.$$

$$\left. + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2(t_0 - \tau)\, B(\tau, x(\tau))\, dw(\tau)\right) + (Q_1(t - t_0) - \mu\, Q_2(t - t_0)) \,.$$

$$. \left(y_0 + \varepsilon \int_{t_0}^{t} Q_2'(t_0 - \tau)\, a(\tau, x(\tau))\, d\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2'(t_0 - \tau)\, B(\tau, x(\tau))\, dw(\tau)\right).$$

Let u, v be solutions of the equations

$$(13,6)\quad u(t) = x_0 + \varepsilon \int_{t_0}^{t} Q_2(t_0 - \tau)\, a(\tau, u(\tau)\, Q_1(\tau - t_0) + v(\tau)\, Q_2(\tau - t_0))\, d\tau +$$

$$+ \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2(t_0 - \tau)\, B(\tau, u(\tau)\, Q_1(\tau - t_0) + v(\tau)\, Q_2(\tau - t_0))\, dw(\tau)$$

$$v(t) = y_0 + \varepsilon \int_{t_0}^{t} Q_2'(t_0 - \tau)\, a(\tau, u(\tau)\, Q_1(\tau - t_0) + v(\tau)\, Q_2(\tau - t_0))\, d\tau +$$

$$+ \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2'(t_0 - \tau)\, B(\tau, u(\tau)\, Q_1(\tau - t_0) + v(\tau)\, Q_2(\tau - t_0))\, dw(\tau)\,.$$

By (2,6), (3,6) and (13,6), the solution $x(t)$, $y(t)$ is given by

$$x(t) = Q_1(t - t_0)\, u(t) + Q_2(t - t_0)\, v(t)$$

$$y(t) = -\omega^2 Q_2(t - t_0)\, u(t) + (Q_1(t - t_0) - \mu\, Q_2(t - t_0))\, v(t)$$

and the theorem is proved.

Last section is devoted to some properties of the stochastic integral $\int_0^t f(t, \omega)\, d\omega(t)$ and of solutions of stochastic equations.

Let a process $w(t)$ fulfil assumption iii), and let $\mathscr{F}(t)$ denote the smallest σ-field which corresponds to $w(t_2) - w(t_1)$ for $0 \leqq t_1 < t_2 \leqq t$. There exist functions $F_{ij}(t)$ with

$$F_{ij}(t_2) - F_{ij}(t_1) = E((w_i(t_2) - w_i(t_1))(w_j(t_2) - w_j(t_1))) .$$

These functions fulfil the following conditions:

$$(1,7) \qquad F_{ii}(t) \text{ are non-decreasing continuous functions and } \sum_i (F_{ii}(t_2) - F_{ii}(t_1)) = F(t_2) - F(t_1)$$

$$(2,7) \qquad F_{ij}(t_2) - F_{ij}(t_1) = \tfrac{1}{2}[E|w_i(t_2) + w_j(t_2) - w_i(t_1) - w_j(t_1)|^2 - - (F_{ii}(t_2) - F_{ii}(t_1)) - (F_{jj}(t_2) - F_{jj}(t_1))]$$

and $\operatorname*{var}_{\langle t_1, t_2\rangle} F_{ij}(t) \leqq F(t_2) - F(t_1)$.

Let $f(t, \omega)$ be a quadratic matrix. Denote by $f_{ij}(t, \omega)$ the components of $f(t, \omega)$. Assume that the $f_{ij}(t, \omega)$ are measurable in both variables, $f_{ij}(t, \omega)$ are $\mathscr{F}(t)$ - - measurable for every t and $\int_{t_1}^{t_2} E f_{ij}^2(t, \omega)\, \mathrm{d}F(t) < \infty$ for all i, j. Defining the stochastic integral in the usual way, we obtain

$$(3,7) \qquad E\left(\int_{t_1}^{t_2} f(\tau, \omega)\, \mathrm{d}w(\tau)\right) = 0$$

$$(4,7) \qquad \left|E\left(\int_{t_1}^{t_2} f(\tau, \omega)\, \mathrm{d}w(\tau), \int_{t_1}^{t_2} g(\tau, \omega)\, \mathrm{d}w(\tau)\right)\right| \leqq \psi(f, t_1, t_2)\, \psi(g, t_1, t_2)$$

where $(,)$ is the scalar product and $\psi(f, t_1, t_2) = \sqrt{n}\sqrt{\int_{t_1}^{t_2} \|f\|^2\, \mathrm{d}F}$. Then, easily,

$$(5,7) \qquad E\left[\left(\int_{t_1}^{t_2} f(\tau, \omega)\, \mathrm{d}w(\tau)\right)_i \cdot \left(\int_{t_1}^{t_2} f(\tau, \omega)\, \mathrm{d}w(\tau)\right)_j\right] = = \int_{t_1}^{t_2} \sum_{k,l} E(f_{il}(\tau, \omega) f_{jk}(\tau, \omega))\, \mathrm{d}F_{lk}(\tau)$$

$\mathscr{M}$ is called the class of functions $x(t, \omega)$ which are $\mathscr{L} \times \mathscr{F}$ measurable in both variables ($\mathscr{L}$ denotes the σ-field of Lebesgue measurable sets on the real line), $x(t, \omega)$ is $\mathscr{F}(t)$-measurable for every t and $\|x(.)\|_t = \sqrt{E \sup_{\langle 0,t\rangle} |x(\tau, \omega)|^2} < \infty$. With respect to i), ii), we obtain that

$$\left\|\int_0^t a(\tau, x(\tau))\, \mathrm{d}\tau\right\|_t \leqq Kt + K \int_0^t \|x(.)\|_\tau\, \mathrm{d}\tau$$

$$\left\|\int_0^t B(\tau, x(\tau))\, \mathrm{d}w(\tau)\right\|_t = \sqrt{E \sup_{\langle 0,t\rangle} \left|\int_0^\tau B\, \mathrm{d}w\right|^2} \leqq 2\sqrt{E\left|\int_0^t B\, \mathrm{d}w\right|^2} \leqq$$

$$\leqq 2\sqrt{\left(n \int_0^t E|B|^2\, \mathrm{d}F\right)} \leqq 2K\left[\sqrt{(n(F(t) - F(0)))} + \sqrt{\left(n \int_0^t \|x(.)\|_\tau^2\, \mathrm{d}F(\tau)\right)}\right].$$

The first inequality in the second row is satisfied according to Theorem 3,4, Chap. VII of [2]. From these estimates it follows that if $x \in \mathcal{M}$ and $\hat{x}$ is given by

$$\hat{x}(t) = x_0 + \int_0^t a(\tau, x(\tau))\, \mathrm{d}\tau + \int_0^t B(\tau, x(\tau))\, \mathrm{d}w(\tau)\,,$$

then $\hat{x}(t)$ can be chosen so that $\hat{x} \in \mathcal{M}$ also. Since $x_0 \in \mathcal{M}$, there is $x_m(t) \in \mathcal{M}$ where $x_m(t)$ are successive approximations of

$$x(t) = x_0 + \int_0^t a(\tau, x(\tau))\, \mathrm{d}\tau + \int_0^t B(\tau, x(\tau))\, \mathrm{d}w(\tau)\,. \tag{6,7}$$

These approximations fulfil

$$\|x_{m+1}(.) - x_m(.)\|_t \leqq K \int_0^t \|x_m(.) - x_{m-1}(.)\|_\tau\, \mathrm{d}\tau +$$
$$+ 2K \sqrt{\left(n \int_0^t \|x_m(.) - x_{m-1}(.)\|_\tau^2\, \mathrm{d}F(\tau)\right)}$$

thus x_m converge to some function x in the norm $\| \;\|_t$. This function x is a solution of (6,7) in the sense that

$$\left\| x(t) - x_0 - \int_0^t a(\tau, x(\tau))\, \mathrm{d}\tau - \int_0^t B(\tau, x(\tau))\, \mathrm{d}w(\tau) \right\|_t = 0\,.$$

The following estimates can be proved:

$$\sqrt{E \sup_{\langle t_0, t\rangle} |x(\tau) - x(t_0)|^2} \leqq (1 + \|x(.)\|_t)\, K(t - t_0 + 2\sqrt{(n(F(t) - F(t_0)))}) \tag{7,7}$$

$$\|x(.)\|_t \leqq \sqrt{2}(\|x_0\| + Kt + 2K\sqrt{(n(F(t) - F(0)))}\,. \tag{8,7}$$
$$. \exp\{K^2 n(2 + \sqrt{t})^2 (F(t) - F(0) + t)\}\,.$$

References

[1] *Боголюбов Н. Н., Митропольский Ю. А.:* Асимптотические методы в теории нелинейных колебаний ГОС. ИЗД. ТЕХ.-ТЕОР. ЛИТ. 1955.

[2] *Doob J. L.:* Stochastic Processes. New York 1953.

[3] *Дынкин Е. В.:* Марковские процессы. ФИЗМАТГИЗ. 1963.

[4] *Courant R.:* Partial Differential Equations. New York 1962.

[5] *Wang Jia-Gang:* Properties and limit theorems of sample functions of the Ito stochastic integral. Chinese Mathematics, 1964, Vol *V*, N 4, 556—570.

[6] *Agmon S., Douglis A., Nirenberg L.:* Estimate near boundary for solutions of elliptic partial differential equations. Satisfying general boundary conditions I. Communications on pure and applied Math. 1959, Vol *12*, 623—727.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).

Резюме

О РАСПРОСТРАНЕНИИ МЕТОДА УСРЕДНЕНИЯ НА СТОХАСТИЧЕСКИЕ УРАВНЕНИЯ

ИВО ВРКОЧ, (Ivo Vrkoč), Прага

Пусть выполняются следующие предположения:

1) Вектор $a(t, x)$ и квадратная матрица $B(t, x)$ непрерывны по t, x и выполняют условие Липшица по x.

2) $|a(t, 0)| \leqq K$, $|B(t, 0)| \leqq K$, где $|B| = \sqrt{(\sum b_{ij}^2)}$ и b_{ij} – составляющие матрицы B.

3) Заданно семейство процессов $w_\varepsilon(t)$ с независимыми приращениями, для которых $E(w_\varepsilon(t_2) - w_\varepsilon(t_1)) = 0$, $E|w_\varepsilon(t_2) - w_\varepsilon(t_1)|^2 = F_\varepsilon(t_2) - F_\varepsilon(t_1)$ и функции $F_\varepsilon(t)$ непрерывны.

4) Существует функция $\varphi(\varepsilon)$, $\varphi(\varepsilon) > 0$ для $\varepsilon > 0$ так, что $\underset{\varepsilon \to 0}{\mathrm{l.i.m.}}\, (\bar{w}_\varepsilon(t_2) - \bar{w}_\varepsilon(t_1)) = w_0(t_2) - w_0(t_1)$ равномерно на каждом компактном множестве значений t_1, t_2, где $\bar{w}_\varepsilon(t) = \sqrt{(\varphi(\varepsilon))}\, w(t/\varepsilon)$ (l.i.m. обозначает предел в среднем квадратическом).

5) $w_\varepsilon^*(t) = \bar{w}_\varepsilon(t) - w_0(t)$ – процессы с независимыми приращениями.

6) Пусть $\mathscr{F}(t)$ соотв. $\overline{\mathscr{F}}_\varepsilon(t)$ обозначают самые малые σ-поля, индуцированные случайными величинами $w_0(t_2) - w_0(t_1)$ соотв. $\bar{w}_\varepsilon(t_2) - \bar{w}_\varepsilon(t_1)$ для $0 \leqq t_1 < t_2 \leqq t$. $\mathscr{F}_\varepsilon^*(t)$ обозначает σ-поле, содержащее самое малое σ-поле, которое индуцировано случайными величинами $w_\varepsilon^*(t_2) - w_\varepsilon^*(t_1)$ для $0 \leqq t_1 < t_2 \leqq t$; $\mathscr{F}_\varepsilon^*(t)$ такое, что величины $w_\varepsilon^*(t_4) - w_\varepsilon^*(t_3)$ независимы от $\mathscr{F}_\varepsilon^*(t)$ для $t \leqq t_3 < t_4$. Эти поля удовлетворяют соотношениям $\overline{\mathscr{F}}_\varepsilon(t) \subset \mathscr{F}(t)$, $\overline{\mathscr{F}}_\varepsilon(t) \subset \mathscr{F}_\varepsilon^*(t)$.

7) Существует вектор $\bar{a}(x)$ так, что

$$\lim_{T \to \infty} \frac{1}{T} \int_0^t a(t, x)\, \mathrm{d}t = \bar{a}(x)$$

равномерно по x.

8) Существует матрица $\bar{B}(x)$ так, что

$$\lim_{T \to \infty} \int_{\alpha T}^{\alpha T + \beta T} |B(t, x) - \bar{B}(x)|^2\, \mathrm{d}F(t/T) = 0$$

равномерно по x при всех α, β, $0 \leqq \alpha \leqq L$, $0 < \beta < L$, где L данное положительное число.

9) Случайная величина $x_0(\omega)$ независима от всех приращений $w_\varepsilon(t_2) - w_\varepsilon(t_1)$ и $E|x_0(\omega)|^2 < \infty$.

В работе доказывается основная теорема

Теорема 1. *Пусть предположения* 1), ..., 9) *выполняются и пусть* $x(t)$, $y(t)$ – *решения уравнений*

$$(1) \qquad x(t) = x_0 + \varepsilon \int_0^t a(\tau, x(\tau))\, d\tau + \sqrt{(\varphi(\varepsilon))} \int_0^t B(\tau, x(\tau))\, dw_\varepsilon(\tau)$$

$$(2) \qquad y(t) = x_0 + \int_0^t \bar{a}(y(\tau))\, d\tau + \int_0^t \bar{B}(y(\tau))\, dw_0(\tau)\,.$$

Тогда к данным числам $\eta > 0$, $L > 0$ *существует* $\varepsilon_0 > 0$ *так, что*

$$E(\sup_{\langle 0, L/\varepsilon\rangle} |x(t) - y(\varepsilon t)|^2) < \eta \quad \text{для} \quad 0 < \varepsilon \leqq \varepsilon_0\,.$$

В случае, когда $w_0(t)$ – процесс Винера (по [3]) условие 8) перейдет в условие $\lim_{T\to\infty} (1/T) \int_0^T |B(t, x) - \bar{B}(x)|^2\, dt = 0$ равномерно по x. Показан пример, доказывающий, что это условие невозможно ослабить в условие $\lim_{T\to\infty} (1/T) \int_0^T B(t, x)\, dt = \bar{B}(x)$.

Теорему 1 нельзя непосредственно применить в влучае процессов с поглощающей стенкой, но следующая теорема показывает, что при более ограничающих предположениях аналогичное утверждение выполняется. Пусть процесс $x^*(t, \omega)$ определен в пространстве E_n и пусть G – данная область из E_n. Обозначим $\tau(\omega)$ момент первого выхода процесса x^* из G и будем предполагать, что $\tau(\omega)$ – случайная величина. Положим $x(t, \omega) = x^*(\tau(\omega), \omega)$ для $t \geqq \tau(\omega)$ и $x(t, \omega) = x^*(t, \omega)$ для $t < \tau(\omega)$. Процесс $x(t, \omega)$ назовем процессом с поглощающей стенкой G', соответствующий процессу $x^*(t, \omega)$.

Граница G' области G гладкая, если её можно локально выразить при помощи функций, вторые производные которых выполняют условие Гелдера.

Скажем, что матрица $\bar{B}(x)$ каноническая, если существует постоянная $m > 0$ так, что $\sum_{i,j,k=1}^{n} \bar{b}_{ik}(x)\, \bar{b}_{jk}(x)\, \lambda_i \lambda_j \geqq m \sum_{i=1}^{n} \lambda_1^2$ для любых действительных чисел λ_i.

Теорема 2. *Пусть дана область* G *с компактным замыканием* $\bar{G}$ *и гладкой границей* G'. $a(t, x)$, $B(t, x)$ *выполняют условия теоремы* 1. *и* $\bar{B}(x)$ *каноническая в* $\bar{G}$. *Предположим, что* $w_0(t)$ – *процесс Винера* [3] *и* $\overline{\mathscr{F}}_\varepsilon(t) = \mathscr{F}(t)$. *Если* $x^*(t)$, $y^*(t)$ – *решения уравнений* (1), (2) *и* $x(t)$, $y(t)$ – *процессы с поглощающей стенкой* G', *соответствующие процессам* $x^*(t)$, $y^*(t)$, *то для процессов* $x(t)$, $y(t)$ *выполняется утверждение теоремы* 1.

Для следующих теорем нужно определить устойчивость в среднем и асимптотическую устойчивость в среднем.

Определение 1. Решение $z(t)$ уравнения

$$z(t) = z_0 + \int_{t_0}^{t} a(\tau, z(\tau))\,\mathrm{d}\tau + \int_{t_0}^{t} B(\tau, z(\tau))\,\mathrm{d}w(\tau) \tag{3}$$

устойчиво в среднем, если существует функция $\delta(\varepsilon)$, $\delta(\varepsilon) > 0$, $\delta(\varepsilon) \to 0$ для $\varepsilon \to 0$ так, что $\sup_{\langle t_0,\infty)} E|z(t) - \tilde{z}(t)|^2 < \varepsilon$ для всякого $t_0 \geqq 0$ и всякого решения

$$\tilde{z}(t) = \tilde{z}(t_0) + \int_{t_0}^{t} a(\tau, \tilde{z}(\tau))\,\mathrm{d}\tau + \int_{t_0}^{t} B(\tau, \tilde{z}(\tau))\,\mathrm{d}w(\tau)\,,$$

для которого $E|z(t_0) - \tilde{z}(t_0)|^2 < \delta$.

Определение 2. Решение $z(t)$ уравнения (3) асимптотически устойчиво в среднем если: 1) устойчиво в среднем 2) существует число $A > 0$ и функция $T(\delta, \eta) > 0$, определенная для $0 < \delta < A$, $0 < \eta < A$, так, что $E|z(t) - \tilde{z}(t)|^2 < \eta$ для $t \geqq t_0 + T(\delta, \eta)$, если $E|z(t_0) - \tilde{z}(t_0)|^2 < \delta$.

Имеет место

Теорема 3. *Пусть предположения теоремы* 1 *выполняются, причем*

$$\lim_{T\to\infty} \frac{1}{T} \int_{\alpha T}^{\alpha T + T} a(t, x)\,\mathrm{d}t = \bar{a}(x)\,, \quad \lim_{T\to\infty} \frac{1}{T} \int_{\alpha T}^{\alpha T + \beta T} |B(t, x) - \bar{B}(x)|^2\,\mathrm{d}t = 0$$

равномерно по x *и* α *при любом* β, *и пусть* $w(t)$ – *процесс Винера. Если* (2) *имеет постоянное асимптотически устойчивое в среднем решение* $y(t) = y_0$ *для* $t \geqq t_0$, *то к данному числу* $\eta > 0$ *существуют числа* $\varepsilon_0 > 0$ *и* $\delta > 0$ *так, что* $\sup_{\langle 0,\infty)} E|x(t) - y_0|^2 < \eta$ *для* $0 < \varepsilon \leqq \varepsilon_0$, *где* $x(t)$ – *решение уравнения* (1) *с начальным условием* $E|x(0) - y_0|^2 < \delta$.

Для устойчивости в среднем и асимптотической устойчивости в среднем приводятся достаточные условия.

Теорема 4. *Пусть уравнение* (3) *имеет решение* $z(t) \equiv 0$; $a(t, z)$, $B(t, z)$ *выполняют условие* 1), $w(t)$ *выполняет* 3), *причем функция* $F(t)$ *абсолютно непрерывна. Пусть существует квадратическая форма* $V(t, z) = \sum c_{ij}(t)\, z_i z_j$ *такая, что функции* $c_{ij}(t)$ *имеют непрерывные вторые производные, существуют числа* $d_1 > 0$, $d_2 > 0$ *так, что*

$$d_1|z|^2 \leqq V(t, z) \leqq d_2|z|^2 \tag{4}$$

$$W(t, z) = \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial z_i} a_i + \sum_{i,j,k,l} c_{ij}(t)\, b_{ik}(t, z)\, b_{jl}(t, z)\, f_{kl}(t) \leqq 0 \tag{5}$$

для почти всех t (в смысле меры Лебега). Тогда решение $z(t) \equiv 0$ устойчиво в среднем.

Функции $f_{kl}(t)$ суть производные функций $F_{kl}(t)$, причём

$$F_{kl}(t_2) - F_{kl}(t_1) = E((w_k(t_2) - w_k(t_1))(w_l(t_2) - w_l(t_1))).$$

Аналогичная теорема выполняется в случае асимптотической устойчивости в среднем.

Теорема 5. *Пусть выполняются предположения теоремы 4 кроме (5), которое видоизменено к условию $W(t, z) \leqq -d_3|z|^2$ для почти всех t в смысле Лебега и $d_3 > 0$. Тогда решение $z(t) \equiv 0$ асимптотически устойчиво в среднем.*

Следующая теорема показывает, как применить эту теорию к дифференциальным уравнениям, которые не имеют форму (1).

Пусть дана система двух стохастических уравнений

$$x(t) = x_0 + \int_{t_0}^{t} y(\tau)\,\mathrm{d}\tau \tag{6}$$

$$y(t) = y_0 - \omega^2 \int_{t_0}^{t} x(\tau)\,\mathrm{d}\tau - \mu \int_{t_0}^{t} y(\tau) + \varepsilon \int_{t_0}^{t} a(\tau, x(\tau))\,\mathrm{d}\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} B(\tau, x(\tau))\,\mathrm{d}w_\varepsilon(\tau),$$

которая соответствует обыкновенному уравнению второй степени $\ddot{x} + \mu\dot{x} + \omega^2 x = \varepsilon f(t, x)$.

Теорема 6. *Пусть $x(t), y(t)$ — решение (6) и $Q_1(t), Q_1'(t), Q_2(t), Q_2'(t)$ — решения (6) при $\varepsilon = 0$, $Q_1(0) = Q_2'(0) = 1$, $Q_2(0) = Q_1'(0) = 0$. Решение $x(t), y(t)$ можно представить в виде*

$$x(t) = Q_1(t - t_0)\,u(t) + Q_2(t - t_0)\,v(t)$$
$$y(t) = -\omega^2\, Q_2(t - t_0)\,u(t) + (Q_1(t - t_0) - \mu\, Q_2(t - t_0))\,v(t),$$

где $u(t), v(t)$ — решение

$$u(t) = x_0 + \varepsilon \int_{t_0}^{t} Q_2(t_0 - \tau)\,a(\tau, u(\tau)\,Q_1(\tau - t_0) + v(\tau)\,Q_2(\tau - t_0))\,\mathrm{d}\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2(t_0 - \tau)\,B(\tau, u(\tau)\,Q_1(\tau - t_0) + v(\tau)\,Q_2(\tau - t_0))\,\mathrm{d}w_\varepsilon(\tau), \tag{7}$$

$$v(t) = y_0 + \varepsilon \int_{t_0}^{t} Q_2'(t_0 - \tau)\,a(\tau, u(\tau)\,Q_1(\tau - t_0) + v(\tau)\,Q_2(\tau - t_0))\,\mathrm{d}\tau + \sqrt{(\varphi(\varepsilon))} \int_{t_0}^{t} Q_2'(t_0 - \tau)\,B(\tau, u(\tau)\,Q_1(\tau - t_0) + v(\tau)\,Q_2(\tau - t_0))\,\mathrm{d}w_\varepsilon(\tau).$$

К системе (7) можно применить предшествующие теоремы. Подобные теоремы имеют место и для систем (6) высших степеней.