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EXPONENTIALLY STABLE INTEGRAL MANIFOLDS, AVERAGING PRINCIPLE AND CONTINUOUS DEPENDENCE ON A PARAMETER

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(*Continuation*)

4. A BOUNDARY VALUE PROBLEM

The results of the preceeding two sections will be applied to the boundary value problem

$$u_{tt} - u_{xx} = \varepsilon g(x, t, u, u_x, u_t)\,, \tag{1,4}$$

$$0 \leqq x \leqq 1,\quad t \geqq 0,\quad u(0, t) = u(1, t) = 0,\quad u(x, 0) = \varphi(x),\quad u_t(x, 0) = \psi(x),\quad \varepsilon > 0\,.$$

The following conditions guarantee the existence and uniqueness of classical solution of (1,4) on some finite interval:

(i) $g = g(x, t, u, v, w)$ and the derivatives g_x, g_u, g_v, g_w are continuous for $x \in \langle 0, 1\rangle$, $t \geqq 0$, $(u, v, w) \in E_3$,

(ii) g, g_x, g_u, g_v, g_w are bounded and fulfil a Lipschitz condition with respect to u, v, w on the set $0 \leqq x \leqq 1$, $t \geqq 0$, $u^2 + v^2 + w^2 \leqq R^2$ for every $R > 0$,

(iii) φ is twice continuously differentiable and ψ is once continuously differentiable on $\langle 0, 1\rangle$, $\varphi(0) = \varphi(1) = \psi(0) = \psi(1) = 0$,

(iv) $$g(0, t, 0, v, 0) = 0 = g(1, t, 0, v, 0) \quad \text{for} \quad t \geqq 0,\ v \in E_1\,,$$
$$\varphi''(0) = \varphi''(1) = 0\,.$$

We shall assume that conditions (i)–(iv) are fulfilled and that in addition

(v) the derivatives of g of the second order with respect to x, u, v, w are continuous, bounded and fulfil a Hölder condition on the set $0 \leqq x \leqq 1$, $t \geqq 0$, $u^2 + v^2 + w^2 \leqq R^2$ for every $R > 0$,

(vi) g is periodic in t with the period 2.

Let us extend the domain of definition of φ and ψ to E_1 so that

$$(2,4) \quad \varphi(x) = -\varphi(-x) = \varphi(x+2), \; \psi(x) = -\psi(-x) = \psi(x+2), \quad x \in E_1 .$$

The domain of definition of g let us change in such a way that g will be defined for $x \neq n$, $n = \ldots -1, 0, 1, \ldots$, $t \geqq 0$, $(u, v, w) \in E_3$ and for $x = n$, $t \geqq 0$, $u = 0 = w$, $v \in E_1$ and that

$$(3,4) \qquad g(-x, t, -u, v, -w) = -g(x, t, u, v, w) = g(2 - x, t, -u, v, -w) .$$

Obviously $g(n, t, 0, v, 0) = 0$ and g is continuous in its domain of definition. The problem

$$(4,4) \qquad u_{tt} - u_{xx} = \varepsilon g(x, t, u, u_x, u_t), \quad x \in E_1, \; t \geqq 0 ,$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

is equivalent to the equation

$$(5,4) \qquad 2u(x, t) = \varphi(x + t) + \varphi(x - t) + \int_{x-t}^{x+t} \psi(\sigma)\, \mathrm{d}\sigma +$$
$$+ \varepsilon \int_0^t \int_{x-t+\tau}^{x+t-\tau} g(\sigma, \tau, u(\sigma, \tau), u_x(\sigma, \tau), u_t(\sigma, \tau))\, \mathrm{d}\sigma\, \mathrm{d}\tau$$

and it follows by the method of successive approximations that the solution of (5,4) exists (and is unique) on $\langle 0, L/\varepsilon \rangle$ and L does not depend on ε. It is a consequence of the uniqueness that

$$(6,4) \qquad u(-x, t) = -u(x, t) = u(2 - x, t) ,$$

u being the solution of (5,4). The existence of a solution of (4,4) and the uniqueness in the class of functions fulfilling (6,4) follow from the existence and uniqueness of solutions of (18,4) also.

If the domain of definition of the solution u of (5,4) is restricted to $0 \leqq x \leqq 1$, then u is the solution of (1,4). If (iv) is not fulfilled, then there exist a unique solution of (5,4) with continuous derivatives of the first order and the derivatives of the second order are continuous for $x + t \neq n$, $-x + t \neq n$, $x \neq n$, $n = \ldots -1, 0, 1, \ldots$

If $v = v(x)$ is locally integrable, $v(x) = v(x + 2)$, $x \in E_1$, $\int_0^2 v \,\mathrm{d}x = 0$, let $I_x v = V = V(x)$ be defined by the conditions $\mathrm{d}V/\mathrm{d}x = v$, $\int_0^2 V \mathrm{d}x = 0$.

Hence

$$(7,4) \quad V(x) = \int_0^x v(\xi)\, \mathrm{d}\xi + \frac{1}{2} \int_0^2 (\xi - 2)\, v(\xi)\, \mathrm{d}\xi = \frac{1}{2} \int_x^{x+2} (\xi - x - 1)\, v(\xi)\, \mathrm{d}\xi$$

and

$$(8,4) \qquad \max_x |V(x)| \leqq \tfrac{1}{2} \operatorname*{sup\,ess}_\xi |v(\xi)| .$$

Let us define the functions $\tilde{S} = \tilde{S}(\xi, \sigma)$, $\tilde{R} = (\xi, \sigma)$, $\xi \in E_1$, $0 \leqq \sigma \leqq L/\varepsilon$ by the relations

$$\tilde{S}(x + t, t) = \tfrac{1}{2}u(x, t) + \tfrac{1}{2}(I_x u_t)(x, t), \tag{9,4}$$
$$\tilde{R}(x - t, t) = \tfrac{1}{2}u(x, t) - \tfrac{1}{2}(I_x u_t)(x, t),$$

u being a (classical) solution of (4,4) for $x \in E_1$, $t \in \langle 0, L/\varepsilon \rangle$. The derivatives $\partial^2/\partial\xi^2$, $\partial^2/\partial\xi\, \partial\sigma$ of $\tilde{S}$, $\tilde{R}$ are continuous and it follows from (8,4), (6,4) and (4,4) that

$$\tilde{S}(-x, t) = -\tilde{R}(x, t) = \tilde{S}(-x + 2, t), \quad \tilde{S}_\sigma(x + t, t) + \tilde{R}_\sigma(x - t, t) = 0, \tag{10,4}$$

$$u(x, t) = \tilde{S}(x + t, t) + \tilde{R}(x - t, t) = \tilde{S}(x + t, t) - \tilde{S}(-x + t, t), \tag{11,4}$$
$$u_t(x, t) = \tilde{S}_\xi(x + t, t) - \tilde{R}_\xi(x - t, t) = \tilde{S}_\xi(x + t, t) - \tilde{S}_\xi(-x + t, t),$$

$$\tilde{S}_{\xi,\sigma}(x + t, t) = \frac{\varepsilon}{2}\, g(x, t, \tilde{S}(x + t, t) - \tilde{S}(-x + t, t), \tag{12,4}$$
$$\tilde{S}_\xi(x + t, t) + \tilde{S}_\xi(-x + t, t), \tilde{S}_\xi(x + t, t) - \tilde{S}_\xi(-x + t, t)).$$

Put $\xi = x + t$, $\varepsilon t = \tau$, $S(\xi, \tau) = \tilde{S}(\xi, \tau/\varepsilon)$ then

$$S_{\xi,\tau} = \tfrac{1}{2}g\,(\xi - \tau/\varepsilon, \tau/\varepsilon\,,\ S(\xi, \tau) - S\,(-\xi + 2\,\tau/\varepsilon,\ \tau),\ S_\xi(\xi, \tau) + \tag{13,4}$$
$$+ S_\xi\,(-\xi + 2\,\tau/\varepsilon, \tau)\,,\ S_\xi(\xi, \tau) - S_\xi\,(-\xi + 2\,\tau/\varepsilon, \tau))\,,\ \xi \in E_1,\ \tau \in \langle 0, L \rangle$$

and

$$u(x, t) = S(x + t, \varepsilon t) - S(-x + t, \varepsilon t), \tag{14,4}$$

$$u_t(x, t) = S_\xi(x + t, \varepsilon t) - S_\xi(-x + t, \varepsilon t). \tag{15,4}$$

It follows from (6,4) that

$$\int_0^2 S(\xi, \tau)\, d\xi = 0\,. \tag{16,4}$$

On the other hand if S, S_ξ, S_τ, $S_{\xi\xi}$, $S_{\xi\tau}$ are continuous, if S fulfils (13,4), (16,4), if

$$S(\xi, 0) = \tfrac{1}{2}\,\varphi(\xi) + \tfrac{1}{2}(I_\xi \psi)(\xi)$$

and if u is defined by (14,4), then u is a solution of (4,4).

Equations (13,4), (16,4) may be examined as an ordinary differential equation. Let M be the Banach space the elements of which are of bounded measurable functions $y = y(\xi)$, $\xi \in E_1$, $y(\xi) = y(\xi + 2)$, $\int_0^2 y\, d\xi = 0$, $\|y\| = \sup_\xi |y(\xi)|$. Let M_{1C} be the Banach space of such functions $y = y(\xi)$, $\xi \in E_1$ that $dy/d\xi$ is continuous, $y(\xi) = y(\xi + 2)$, $\int_0^2 y \, d\xi = 0$ with the norm $\|y\|_1 = \max_\xi |dy/d\xi(\xi)|$. The natural map of M_{1C} into M we shall interpret as the inclusion $M_{1C} \subset M$; obviously $\|y\| \leqq \frac{1}{2}\|y\|_1$ for $y \in M_{1C}$ (cf. (8,4)). For $y \in M$, $\tau \geqq 0$ put $Y = I_\xi y$ and

$$(17,4) \qquad f(y, \tau, \varepsilon)(\xi) = \tfrac{1}{2} g(\xi - \tau/\varepsilon, \tau/\varepsilon, Y(\xi) - Y(-\xi + 2\tau/\varepsilon), \\ y(\xi) + y(-\xi + 2\tau/\varepsilon), y(\xi) - y(-\xi + 2\tau/\varepsilon))$$

and examine the equation

$$(18,4) \qquad \frac{dy}{d\tau} = f(y, \tau, \varepsilon).$$

As g is periodic in t with the period 2, let us define

$$(19,4) \quad f_0(y)(\xi) = \frac{1}{4}\int_0^2 g(\xi - \sigma, \sigma, Y(\xi) - Y(-\xi + 2\sigma), \quad y(\xi) + y(-\xi + 2\sigma), \\ y(\xi) - y(-\xi + 2\sigma)) \, d\sigma.$$

Equation (18,4) will be called the transformed equation of the problem (1,4) and equation

$$\frac{dy}{d\tau} = f_0(y)$$

will be called the averaged equation of the problem (1,4).

Lemma 1,4. *If $y \in M$, then $f(y, \tau, \varepsilon) \in M$, $f_0(y) \in M$.*

Proof. Let $y \in M$, then

$$(21,4) \qquad \int_{-1+\tau/\varepsilon}^{1+\tau/\varepsilon} f(y, \tau, \varepsilon)(\xi) \, d\xi = \frac{1}{2}\int_{-1}^{1} \chi(\xi) \, d\xi = 0,$$

$$\chi(\xi) = g\,(\xi, \tau/\varepsilon, Y(\xi + \tau/\varepsilon - Y(-\xi + \varepsilon/\tau, y(\xi + \tau/\varepsilon + y(-\xi + \tau/\varepsilon), \\ y(\xi + \tau/\varepsilon) - y(-\xi + \tau/\varepsilon)),$$

as χ is odd. χ is bounded and therefore $f(y, \tau, \varepsilon) \in M$. Similarly $f_0(y) \in M$.

Lemma 2,4. *For every $R > 0$ there exists a $K > 0$ and a $\mu > 0$ in such a way that*

$$(22,4) \qquad \|f(y, \tau, \varepsilon)\| \leqq K, \quad \|f_0(y)\| \leqq K,$$

$$(23,4) \qquad \|f(y_2, \tau, \varepsilon) - f(y_1, \tau, \varepsilon)\| \leqq K\|y_2 - y_1\|, \\ \|f_0(y_2) - f_0(y_1)\| \leqq K\|y_2 - y_1\|,$$

(24,4)
$$\left\|\frac{\partial f}{\partial y}(y_2, \tau, \varepsilon) - \frac{\partial f}{\partial y}(y_1, \tau, \varepsilon)\right\| \leqq K\|y_2 - y_1\|^{\mu},$$
$$\left\|\frac{\partial f_0}{\partial y}(y_2) - \frac{\partial f_0}{\partial y}(y_1)\right\| \leqq K\,\|y_2 - y_1\|^{\mu}$$

for $y, y_1, y_2 \in M$, $\|y\|, \|y_1\|, \|y_2\| \leqq R$, $\tau \geqq 0$, $\varepsilon > 0$.

Proof. (22,4) and (23,4) follow immediately from (17,4), (19,4) and from the conditions (i) and (ii). Let $z \in M$; (24,4) follows from the relations

$$\left(\frac{\partial f}{\partial y}(y_1, \tau, \varepsilon)\, z\right)(\xi) = \frac{1}{2}\left(\frac{\partial g}{\partial u}\right)(Z(\xi) - Z(-\xi + 2\tau/\varepsilon)) +$$
$$+ \frac{1}{2}\left(\frac{\partial g}{\partial v}\right)(z(\xi) + z(-\xi + 2\tau/\varepsilon)) + \frac{1}{2}\left(\frac{\partial g}{\partial w}\right)(z(\xi) - z(-\xi + 2\tau/\varepsilon)),$$

$$\left(\frac{\partial g}{\partial u}\right) = \frac{\partial g}{\partial u}(\xi - \tau/\varepsilon,\ \tau/\varepsilon\ Y_1(\xi) - Y_1(-\xi + 2\tau/\varepsilon,\ \ y_1(\zeta) + y_1(-\xi + 2\tau/\varepsilon),$$
$$y_1(\xi) - y_1(-\xi + 2\tau/\varepsilon)),$$

$$\left(\frac{\partial g}{\partial v}\right) = \ldots$$

$$\left\|\frac{\partial f}{\partial y}(y_1, \tau, \varepsilon)\right\| = \sup_{\|z\| \leqq 1}\ \operatorname{sup\,ess}_{\xi} \left|\frac{\partial f}{\partial y}(y_1, \tau, \varepsilon)\, z\,(\xi)\right|,$$

from similar relations for f_0 and from the conditions (i), (ii), (v).

Lemma 3,4. *To every* $\zeta > 0$ *and* $R > 0$ *there exists such an* $\varepsilon_0 > 0$ *that*

(25,4)
$$\left\|\int_{\tau_1}^{\tau_2}[f(y, \tau, \varepsilon)(\xi) - f_0(y)(\xi)]\right\| \mathrm{d}\tau \leqq \zeta$$

for

$$y \in M\,, \quad \|y\| \leqq R\,, \quad 0 \leqq \tau_1 \leqq \tau_2\,, \quad 0 < \varepsilon \leqq \varepsilon_0\,.$$

Proof. If $r = r(\tau)$, $\tau \in E_1$ is measurable, $|r(\tau)| \leqq K$, $r(\tau) = r(\tau + 2\varepsilon)$, then $|\int_{\tau_1}^{\tau_2} r(\tau)\,\mathrm{d}\tau - (\tau_2 - \tau_1)\frac{1}{2}\int_0^2 r(\varepsilon\sigma)\,\mathrm{d}\sigma| \leqq 4K\varepsilon$ and (25,4) follows from

(26,4)
$$2\int_{\tau_1}^{\tau_2}[f(y, \tau, \varepsilon)(\xi) - f_0(y)(\xi)]\,\mathrm{d}\tau =$$
$$= \int_{\tau_1}^{\tau_2} g\,(\xi - \tau/\varepsilon, \tau/\varepsilon, Y(\xi) - Y(-\xi + 2\tau/\varepsilon),\ \ y(\xi) + y(-\xi + 2\tau/\varepsilon),$$
$$y(\xi) - y(-\xi + 2\tau/\varepsilon))\,\mathrm{d}\tau - (\tau_2 - \tau_1)\frac{1}{2}\int_0^2 g(\xi - \sigma, \sigma, Y(\xi) -$$
$$- Y(-\xi + 2\sigma),\ \ y(\xi) + y(-\xi + 2\sigma),\ \ y(\xi) - y(-\xi + 2\sigma))\,\mathrm{d}\sigma\,.$$

Lemma 4,4. *Lemmas* 1,4, 2,4 *and* 3,4 *hold, if* M *is replaced by* M_{1C} *and the norm* $\| \ \|$ *is replaced by* $\| \ \|_1$.

We shall not prove Lemma 4,4, as it is too laborious, we shall only indicate several points.

If $y \in M_{1C}$ then $\mathrm{d}\chi/\mathrm{d}\xi$ is continuous (χ was introduced in (21,3); $\mathrm{d}\chi/\mathrm{d}\xi$ is continuous for $\xi \neq n$, $n = \ldots -1, 0, 1, \ldots$ and $\lim_{\xi \to 0+} \mathrm{d}\chi/\mathrm{d}\xi$ and $\lim_{\xi \to 0-} \mathrm{d}\chi/\mathrm{d}\xi$ exist and as χ is odd, these limits are equal. $\mathrm{d}\chi/\mathrm{d}\xi$ exists and is continuous near $\xi = 0$ and similar reasoning for holds $\xi = n$, $n = \ldots -1, 0, 1, \ldots$). Therefore $f(y, \tau, \varepsilon) \in M_{1C}$ and it follows that $f_0(y) \in M_{1C}$. (22,4) follows from (i) and (ii); the estimate for $|(\mathrm{d}/\mathrm{d}\xi) f((y, \tau, \varepsilon)(\xi)|$ is found at first for $\xi \neq \tau/\varepsilon + n$, $n = \ldots -1, 0, 1, \ldots$ and then it is extended for $\xi = \tau/\varepsilon + n$. In order to prove (23,4) and (24,4) the assumption (v) is needed, (25,4) follows from (26,4) (differentiated with respect to ξ).

If $y \in M_{1C}$ then $\int_{\tau_1}^{\tau_2} f(y, \tau, \varepsilon)\, \mathrm{d}\tau$ exists and

$$\left(\int_{\tau_1}^{\tau_2} f(y, \tau, \varepsilon)\, \mathrm{d}\tau\right)(\xi) = \frac{1}{2}\int_{\tau_1}^{\tau_2} g(\xi - \tau/\varepsilon, \tau/\varepsilon, Y(\xi) - Y(-\xi + 2\tau/\varepsilon),$$
$$y(\xi) + y(-\xi + 2\tau/\varepsilon), \ y(\xi) - y(-\xi + 2\tau/\varepsilon))\, \mathrm{d}\tau .$$

The same assertion holds for the above integral with f replaced by f_0. Therefore Lemma 4,4 implies that the conditions (1,1), (2,1), (8,1), (13,1), (17,1) and (6,2) are fulfilled, if equations (18,4) and (20,4) are examined in M_{1C} so that Theorems 1,1 and 1,2 may be applied and ζ in (10,1) may be chosen small, if ε_0 is sufficiently small. In order that Theorem 1,2 might be applied it is necessary to verify (7,2) and (9,2) only. The facts that (18,3) is examined in M_{1C} and that y is a solution will be expressed briefly that y is a solution of (18,4) in M_{1C}; in this case $Y(\xi, \tau) = Y(\tau)(\xi)$ satisfies (13,4) and (16,4), Y, $Y_\xi = y$, Y_τ, $Y_{\xi\xi}$, $Y_{\xi\tau}$ being continuous and u defined by (14,4) (with $S = Y$) is a classical solution of (4,4).

Note 1,4. If (18,4) is to be examined in M, a slight difficulty arises from the fact that $\int_{\tau_1}^{\tau_2} f(y, \tau, \varepsilon)\, \mathrm{d}\tau$ need not exist (in the sense of Bochner) for $y \in M$. Therefore let us define the functions $F(y, \tau, \varepsilon)$, $F_0(y, \tau)$ for $y \in M$, $\tau \geqq 0$, $\varepsilon > 0$ by the relations

$$\text{(27,4)} \qquad F(y, \tau, \varepsilon) = \int_0^\tau \tfrac{1}{2} g\,(\xi - \sigma/\varepsilon, \sigma/\varepsilon, \ Y(\xi) - Y(-\xi + 2\sigma/\varepsilon),$$
$$y(\xi) + y(-\xi + 2\sigma/\varepsilon), \ y(\xi) - y(-\xi + 2\sigma/\varepsilon))\, \mathrm{d}\sigma ,$$
$$F_0(y, \tau) = f_0(y)\, \tau .$$

It follows from Lemma 1,4 that $F(y, \tau, \varepsilon) \in M$ if $y \in M$, $\tau \geqq 0$, $\varepsilon > 0$. Lemma 2,4 implies that

$$\text{(28,4)} \qquad \|\Delta_\tau^\sigma F(y, \tau, \varepsilon)\| \leqq K\sigma ,$$

(29,4) $$\|\Delta_\tau^\sigma \Delta_y^z F(y, \tau, \varepsilon)\| \leqq K\|z\| \sigma,$$

for

$$y, y + z \in M, \ \|y\|, \|y + z\| \leqq R, \quad \tau \geqq 0, \sigma \geqq 0.$$

It follows from (24,4) that $\|\Delta_y^{z_1}\Delta_y^{z_2} f(y, \tau, \varepsilon)\| \leqq K\|z_1\| \cdot \|z_2\|^\mu$, for $y, y + z_1, y + z_2, y + z_1 + z_2 \in M$, $\|y\|, \|y + z_1\|, \|y + z_2\|, \|y + z_1 + z_2\| \leqq R$, $\tau \geqq 0$ and therefore

(30,4) $$\|\Delta_\tau^\sigma \Delta_y^{z_1} \Delta_y^{z_2} F(y, \tau, \varepsilon)\| \leqq K\|z_1\| \cdot \|z_2\|^\mu \cdot \sigma, \quad \sigma \geqq 0.$$

It follows from Lemma 3,4 that to every $\zeta > 0$ and $R > 0$ there exists such an $\varepsilon_0 > 0$ that

(31,4) $$\|\Delta_\tau^\sigma[F(y, \tau, \varepsilon) - F_0(y, \tau)]\| \leqq \zeta,$$

for $y \in M$, $\|y\| \leqq R$, $\tau \geqq 0$, $0 \leqq \sigma \leqq 1$, $0 < \varepsilon \leqq \varepsilon_0$. Therefore F_0 fulfils (28,4), (29,4) and (30,4). Instead of (18,4) let us examine the generalized equation

(32,4) $$\frac{dy}{d\tau} = D_\tau F(y, \tau, \varepsilon)$$

in M; it follows (cf. (28,4)–(31,4) and (28,4)–(31,4) with F replaced by F_0) that Theorems 7,1 and 9,1 may be applied to (32,4) and

(33,4) $$\frac{dy}{d\tau} = D_\tau F_0(y, \tau)$$

((33,4) being equivalent to (20,4), cf. section 1) and that ζ may be chosen small if ε_0 is sufficiently small. In order that Theorem 5,2 might be applied it is necessary to verify (7,2) and (9,2) only.

Note 2,4. The solution y of (32,4) in M may be regarded as a generalized solution of (18,4). This is a consequence of the following Lemmas:

Lemma 5,4. *Let $y_i = y_i(\tau)$, $i = 1, 2, \ldots$ and $z = z(\tau)$ be solutions of* (32,4) *in M defined on $\langle 0, L\rangle$, $\|y_i(\tau)\|$, $\|z(\tau)\| \leqq R$, $\tau \in \langle 0, L\rangle$, $i = 1, 2, \ldots$ For $u \in M$ put $\|u\|_{L_1} = \int_0^2 |u(\xi)| \, d\xi$. If $\|y_i(0) - z(0)\|_{L_1} \to 0$, with $i \to \infty$ then $\|y_i(\tau) - z(\tau)\|_{L_1} \to 0$ with $i \to \infty$ uniformly on $\langle 0, L\rangle$.*

Proof. For every $R > 0$ there exists a $K > 0$ that

$$\|\Delta_\tau^\sigma[F(u, \tau) - F(v, \tau)]\|_{L_1} \leqq K\sigma\|u - v\|_{L_1}$$

if $u, v \in M$, $\|u\|, \|v\| \leqq R$, $\tau \in E_1$, $\sigma \geqq 0$. As $\|u\|_{L_1} \leqq 2\|u\|$, the integral in

$$y_i(\tau_1) - z(\tau_1) = y_i(0) - z(0) + \int_0^{\tau_1} D_\sigma[F(y_i(\tau), \sigma) - F(z(\tau), \sigma)]$$

exists, if considered in L_1 (cf. the definition of the integral $\int_\alpha^\beta \mathrm{D}_\sigma F(x(\tau)\sigma$ in section 1). Hence

$$\|y_i(\tau) - z(\tau)\|_{L_1} \leqq \|y_i(0) - z(0)\|_{L_1} + K \int_0^\tau \|y_i(\sigma) - z(\sigma)\|_{L_1} \, \mathrm{d}\sigma$$

and Lemma 5,4 follows.

Lemma 6,4. *Let* $y = y(\tau)$, $\tau \in \langle 0, L \rangle$ *be a solution of* (32,4) *in* M, $y(0) = \tilde{y} \in M_{1C}$. *Then* y *is a solution of* (18,4) *in* M_{1C}.

Proof. Denote by $y_1 = y_1(\tau)$ the solution of (18,4) in M_{1C}, $y_1(0) = \tilde{y}$; we shall show that y_1 exists on $\langle 0, L \rangle$. Suppose that y_1 exists on $\langle 0, L_1)$, $0 < L_1 < L$ and that y_1 does not exist on $\langle 0, L_2)$ if $L_2 > L_1$. Put $y_1(\xi, \tau) = y_1(\tau)(\xi)$. $y_1(\xi, \tau)$, $\partial y_1/\partial \xi$ and $\partial y_1/\partial \tau$ are continuous on $E_1 \times \langle 0, L_1)$ and fulfil the equations

$$(34,4) \qquad \frac{\partial}{\partial \tau} y_1(\xi, \tau) = \tfrac{1}{2}(g),$$

$$(35,4) \qquad \frac{\partial}{\partial \tau}\frac{\partial y_1}{\partial \xi}(\xi, \tau) = \frac{1}{2}\left(\frac{\partial g}{\partial \xi}\right) + \frac{1}{2}\left(\frac{\partial g}{\partial u}\right)(y_1(\xi, \tau) - y_1(-\xi + 2\tau/\varepsilon, \tau)) +$$
$$+ \frac{1}{2}\left(\frac{\partial g}{\partial v}\right)\frac{\partial y_1}{\partial \xi}(\xi, \tau) - \frac{\partial y_1}{\partial \xi}(-\xi + 2\tau/\varepsilon, \tau) + \frac{1}{2}\left(\frac{\partial g}{\partial w}\right)\left(\frac{\partial y_1}{\partial \xi}(\xi, \tau) + \right.$$
$$\left. + \frac{\partial y_1}{\partial \xi}(-\xi + 2\tau/\varepsilon, \tau)\right),$$

$$(g) = g(\xi - \tau/\varepsilon, \tau/\varepsilon, Y_1(\xi,\tau) - Y_1(-\xi + 2\tau/\varepsilon, \tau)\ ,\ \ y_1(\xi, \tau) + y_1(-\xi + 2\tau/\varepsilon, \tau),$$
$$y_1(\xi, \tau) - y_1(-\xi + 2\tau/\varepsilon, \tau)),$$
$$\left(\frac{\partial g}{\partial \xi}\right) = \ldots \qquad \frac{\partial y_1}{\partial \xi}(\alpha, \tau) = \frac{\partial}{\partial \xi} y_1(\xi, \tau)|_{\xi=\alpha}.$$

As $y_1(\tau) = y(\tau)$ for $\tau \in (0, L_1)$, it follows from (35,4) (regarded as a linear vector equation for $\partial y_1/\partial \xi$ in M) that $\partial y_1/\partial \xi$ is bounded on $E_1 \times (0, L_1)$. Therefore y_1 fulfils a Lipschitz condition with respect to ξ on $E_1 \times (0, L_1)$ and $\lim y_1(\tau)$ in M exists and is continuous in ξ. We shall denote this limit by $y_1(L_1)$ so that $y_1(\tau)$ is continuous on $\langle 0, L_1 \rangle$ in the norm of the space M and $y_1(\xi, \tau)$ is continuous on $E_1 \times \langle 0, L_1 \rangle$. Again from (34,4) and (35,4) we deduce that the solution y_1 in M_{1C} exists on $\langle 0, L_1 \rangle$ and the existence theorem implies that y_1 exists on $\langle 0, L_2)$ for some $L_2 > L_1$. This contradiction proves that $L_1 = L$ and the above argument shows that y_1 exists on $\langle 0, L \rangle$.

It follows from Lemmas 5,4 and 6,4 that a solution $y = y(\tau)$ of (32,4) in M is a limit in the norm $\| \ \|_{L_1}$ of a sequence of solutions of (18,4) in M_{1C}.

The theory developed in sections 1 and 2 will be applied to the problem (1,4) with special functions g. In examples 1,5–3,5 the existence of an asymptotically stable periodic solution is proved; example 4,5 is an autonomous equation and it is proved that there exists an asymptotically stable integral manifold; this manifold is formed by a periodic solution $u(x, t)$ and its translations $u(x, t + \sigma)$.

In this section the transformed equation and the averaged equation are examined in the space M_{1C}. If $s(\xi)$ is locally integrable, $s(\xi + 2) = s(\xi)$, let us denote

$$Ps = \frac{1}{2}\int_0^2 s(\xi)\,\mathrm{d}\xi\,. \tag{1,5}$$

Example 1,5. Let

$$g = 2[-u_t^3 + \vartheta(x + t) - \vartheta(-x + t)]\,, \tag{2,5}$$

$$\vartheta \in M_{1C},\ \vartheta(\xi + 1) = -\vartheta(\xi) \quad \text{for} \quad \xi \in E_1,\ \vartheta \neq 0\,.$$

The averaged equation is

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_0(y) \tag{3,5}$$

with

$$f_0(y)(\xi) = -y^3(\xi) - 3y(\xi)\,Py^2 + Py^3 + \vartheta(\xi)\,. \tag{4,5}$$

Equation

$$-y^3(\xi) - 3y(\xi)\,R + Q + \vartheta(\xi) = 0 \tag{5,5}$$

has a unique solution $y = y(\xi)$ if $R > 0$ and $Q \in E_1$ are fixed. This solution has a continuous derivative of the first order and it belongs to M_{1C} iff $Q = 0$.

Let $Q = 0$, $R > 0$ and denote by y_R the solution of (5,5). Obviously $y_R(\xi + 1) = -y_R(\xi)$ for $\xi \in E_1$ and if $0 < R_1 < R_2$, $\xi \in E_1$, $\vartheta(\xi) \neq 0$, then $|y_{R_1}(\xi)| > |y_{R_1}(\xi)|$. Therefore there exists a unique $\bar{R}$ such that $\bar{R} = Py_{\bar{R}}^2$ and $\bar{y} = y_{\bar{R}}$ is the only solution of

$$f_0(y) = 0 \tag{6,5}$$

in M_{1C}. Put $y = \bar{y} + c$ and apply Theorems 1,2 and 3,2 with

$$C = M_{1C}\,, \quad \mathscr{C} = (0)\,, \quad G = \mathscr{E}[c \in M_{1C};\ \|c\| < 1]\,.$$

It was shown in section 4 that it is necessary to verify (7,2) and (9,2) only. As $X = C$, $\mathscr{C} = (0)$, (7,2) is fulfilled and it remains to verify (9,2). A is defined by the relation

$$(Ac)(\xi) = -3\,\bar{y}^2(\xi)\,c(\xi) - 3\,c(\xi)\,P\bar{y}^2 - 6\,\bar{y}(\xi)\,P(\bar{y}c) + 3\,P(\bar{y}^2c)\,. \tag{7,5}$$

If c is a solution of

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = Ac\,, \quad c(\tilde{\tau}) = \tilde{c}\,, \tag{8,5}$$

then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}Pc^2 = P(c\,.\,Ac) = -3P(\bar{y}^2c^2) - 3Pc^2P\bar{y}^2 - 6(P(\bar{y}c))^2 \leqq -3P\bar{y}^2\,.\,Pc^2\,,$$

$$(Pc^2)(\tau) \leqq (P\tilde{c}^2)\,e^{--6(\tau-\tilde{\tau})P\bar{y}^2}\,, \quad \tau \geqq \tilde{\tau}\,.$$

The function $c(\xi, \tau)$ fulfils equation

$$\frac{\partial}{\partial\tau}c(\xi,\tau) = (-3P\bar{y}^2 - 3\bar{y}^2(\xi))\,c(\xi,\tau) - 6\bar{y}(\xi)\,P(\bar{y}c) + 3P(\bar{y}^2c)\,.$$

It is obvious that

$$|P(\bar{y}c)| \leqq (P\bar{y}^2)^{\frac{1}{2}}\,(Pc^2)^{\frac{1}{2}} \leqq (P\bar{y}^2)^{\frac{1}{2}}\,\|\tilde{c}\|\,e^{-3(\tau-\tilde{\tau})P\bar{y}^2}$$

$$(\|\tilde{c}\| = \max_{\xi}|\tilde{c}(\xi)|,\ P\tilde{c}^2 \leqq \|\tilde{c}\|^2)\,,$$

$$|P(\bar{y}^2c)| \leqq (P\bar{y}^4)^{\frac{1}{2}}\,(Pc^2)^{\frac{1}{2}} \leqq (P\bar{y}^4)^{\frac{1}{2}}\,\|\tilde{c}\|\,e^{-3(\tau-\tilde{\tau})P\bar{y}^2}\,, \quad \tau \geqq \tilde{\tau}\,.$$

By means of standard methods it may be found that $|c(\xi, \tau)| \leqq r(\tau)$, r being the solution of

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -3P\bar{y}^2\,.\,r + K_1\|\tilde{c}\|\,e^{-3(\tau-\tilde{\tau})P\bar{y}^2}\,, \quad r(\tilde{\tau}) = \|\tilde{c}\|\,,$$

$$K_1 = 6\max_{\xi}|\bar{y}(\xi)|\,(P\bar{y}^2)^{\frac{1}{2}} + 3(P\bar{y}^4)^{\frac{1}{2}}\,, \quad \tau \geqq \tilde{\tau}\,.$$

As $r(\tau) = \|\tilde{c}\|\,[1 + K_1(\tau - \tilde{\tau})]\,e^{-6(\tau-\tilde{\tau})P\bar{y}^2}$ and $\|\tilde{c}\| \leqq \frac{1}{2}\|\tilde{c}\|_1$, we obtain

$$|c(\xi,\tau)| \leqq K_2 e^{-3(\tau-\tilde{\tau})P\bar{y}^2}\|\tilde{c}\|_1\,, \quad \tau \geqq \tilde{\tau} \tag{9,5}$$

K_2 being a positive constant. c is a solution of (8,5) in M_{1C}, hence the function $\partial c/\partial\xi$ fulfils

$$\frac{\partial}{\partial\tau}\frac{\partial c}{\partial\xi}(\xi,\tau) = (-3P\bar{y}^2 - 3\bar{y}^2(\xi))\frac{\partial c}{\partial\xi}(\xi,\tau) - 6\bar{y}(\xi)\frac{\partial\bar{y}}{\partial\xi}(\xi)\,c(\xi,\tau) - 6\frac{\partial\bar{y}}{\partial\xi}(\xi)\,P(\bar{y}c)\,. \tag{10,5}$$

Starting from (10,5) we obtain in the same manner that

$$\left|\frac{\partial c}{\partial\xi}(\xi,\tau)\right| \leqq K_3 e^{-(\tau-\tilde{\tau})P\bar{y}^2}\|\tilde{c}\|_1$$

K_3 being a positive constant. Therefore (9,2) is fulfilled and Theorems 1,2 and 3,2 may be applied. It follows that for $0 < \varepsilon < \varepsilon_0$, ε_0 being sufficiently small the problem (1,4) with g defined by (2,5) has a periodic solution with the period 2. This solution is asymptotically stable with respect to the norm

$$\|u(.,t)\| = \max_{0 \leq x \leq 1} \left(\left| \frac{\partial^2 u}{\partial t\, \partial x}(x,t) \right|, \left| \frac{\partial^2 u}{\partial x^2}(x,t) \right| \right). \tag{11,5}$$

Example 2,5. Let

$$g = 2[-u^2 u_t + \vartheta(x+t) - \vartheta(-x+t)], \tag{12,5}$$

$$\vartheta \in M_{1C},\ \vartheta(\xi + 1) = -\vartheta(\xi) \quad \text{for} \quad \xi \in E_1,\ \vartheta \neq 0.$$

The averaged equation is (3,4), f_0 being defined by

$$f_0(y) = -\,Y^2(\xi) y(\xi) - y(\xi) P Y^2 + \vartheta(\xi). \tag{13,5}$$

Let M_{2C} be the space of functions $Y \in M_{1C}$, which have a continuous derivative of the second order with the norm $\|Y\|_2 = \max_{\xi} |(\mathrm{d}^2 Y/\mathrm{d}\xi^2)(\xi)|$ and examine

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = f_0^*(Y) \tag{14,5}$$

in M_{2C}, f_0^* being defined by

$$f_0^*(Y)(\xi) = -\tfrac{1}{3}Y^3(\xi) - Y(\xi)\, PY^2 + \tfrac{1}{3}PY^3 + \Theta(\xi), \quad \Theta = I_\xi \vartheta$$

(14,5) goes over in (3,5) with f_0 defined by (4,5) if we put $3^{1/2}\,\hat{y} = Y$, $3^{1/2}\,\hat{\vartheta} = \Theta$. Therefore there exists a unique solution $\bar{Y} \in M_{2C}$ of $f_0^*(Y) = 0$. As $(\mathrm{d}/\mathrm{d}\xi) f_0^*(Y)(\xi) = f_0(y)(\xi)$ for $Y \in M_{2C}$, $\bar{y} \in M_{1C}$ is a solution of $f_0(y) = 0$. Put $y = \bar{y} + c$, $c \in M_{1C}$ and apply Theorems 1,2 and 3,2 with $C = M_{1C}$, $\mathscr{C} = (0)$, $G = \mathscr{E}[c \in M_{1C};\ \|c\|_1 < 1]$. As (7,2) is fulfilled in an obvious way, we have to verify (9,2). A is defined by the relation

$$(Ac)(\xi) = -\,(P\bar{Y}^2 + \bar{Y}^2(\xi))\, c(\xi) - 2Y(\xi)\, C(\xi)\, y(\xi) - 2y(\xi)\, P(YC).$$

Put

$$(A^*C)(\xi) = -\,(P\bar{Y}^2 + \bar{Y}^2(\xi))\, C(\xi) - 2Y(\xi)\, P(YC) + P(Y^2 C).$$

Equation

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = A^* C \tag{15,5}$$

may be examined in a similar way as (8,5). Therefore the solutions of (15,5) in M_{2C} may be estimated by

$$\|C(\tau)\|_2 \leq \|\tilde{C}\|_2\, K_3 e^{-(\tau - \tilde{\tau}) P\bar{Y}^2/6}, \quad \tau \geq \tilde{\tau}.$$

As $(\mathrm{d}/\mathrm{d}\xi)(A^*C)(\xi) = (Ac)(\xi)$ for every $c \in M_{1C}$, $C = I_\xi c$, the solutions of $\mathrm{d}c/\mathrm{d}\tau = Ac$ may be estimated by

$$\|c(\tau)\|_1 \leqq \|\tilde{c}\|_1 K_3 e^{-(\tau-\tilde{\tau})P\gamma^2/6}$$

and (9,2) holds, Theorems 1,2 and 3,2 may be applied and the same conclusions are valid as in the case of Example 1,5.

Example 3,5. Let

$$g = 2[d_1 u_t^3 + d_2 u_x^2 u_t + d_3 u_t + \vartheta(x+t) - \vartheta(-x+t)],$$
$$\vartheta \in M_{1C},\ \vartheta(\xi+1) = -\vartheta(\xi) \quad \text{for} \quad \xi \in E_1,\ \vartheta \neq 0.$$

The averaged equation is (3,4) with

$$f_0(y)(\xi) = (d_1 + d_2)\, y^3(\xi) + (3d_1 - d_2)\, y(\xi)\, Py^2 + d_3\, y(\xi) - (d_1 + d_2)\, Py^3 + \vartheta(\xi)$$

If $d_1 < 0$, $3d_1 \leqq d_2 < -d_1$, $d_3 \leqq 0$, then the same method as in Example 1,5 may be applied and the same conclusions are valid.

Note 1,5. If $g = u^{m_1} u_x^{m_2} u_t^{m_3}$ for $0 \leqq x \leqq 1$, $t \geqq 0$, m_1, m_2, m_3 being nonnegative integers, $m_1 + m_3$ being positive and even, then having performed the extension described in section 4 we have to deal with the function $u^{m_1} u_x^{m_2} u_t^{m_3} \,.\, s$, $s(x) = 1$ for $2i < x < 2i + 1$, $s(x) = -1$ for $2i - 1 < x < 2i$, $i = \dots -1, 0, 1, \dots$ and the averaged equation is $\mathrm{d}y/\mathrm{d}\tau = 0$. Therefore the averaged equation of the problem (1,4) does not change, if a linear combination of terms of the above type is added to the right hand side of (1,4).

Similarly if $g = \Theta(x, t)$, $\Theta(x, t) = -\Theta(-x, t) = \Theta(x + 2, t) = \Theta(x, t + 2)$, Θ and Θ_x being continuous, then the averaged equation is $\mathrm{d}y/\mathrm{d}\tau = \Theta^*$, $\Theta^*(\xi) = \frac{1}{2}\int_0^2 \Theta(\xi - \sigma, \sigma)\,\mathrm{d}\sigma$ and the averaged equation of the problem (1,4) does not change, if such a function Θ is added to the right hand side of (1,4) that $\Theta^* = 0$.

Example 4,5. Let

$$g = -2\left[h\left(\frac{u_x + u_t}{2}\right) - h\left(\frac{u_x - u_t}{2}\right)\right]\cos 2\pi x - 2u_t\,. \tag{16,5}$$

Suppose that $h = h(\lambda)$, $\lambda \in E_1$ is odd, has a continuous derivative of the second order, $\mathrm{d}h/\mathrm{d}\lambda$ is positive, $\mathrm{d}^2h/\mathrm{d}\lambda^2$ is negative for $\lambda > 0$, $(\mathrm{d}h/\mathrm{d}\lambda)(0) > 2$, $\mathrm{d}h/\mathrm{d}\lambda \to 0$ with $\lambda \to \infty$ and that $\mathrm{d}^2h/\mathrm{d}\lambda^2$ satisfies a Hölder condition on every bounded interval. The averaged equation is (3,5), f_0 being defined by

$$f_0(y)(\xi) = -y(\xi) + \frac{1}{2}\int_0^2 h(y(\sigma))\cos\pi(\xi - \sigma)\,\mathrm{d}\sigma\,. \tag{17,5}$$

The solution of

$$f_0(y) = 0 \tag{18,5}$$

is necessarilly of the form

$$y(\xi) = R\cos\pi(\xi - \varphi)\,, \quad R \geqq 0\,, \quad \varphi \in E_1 \tag{19,5}$$

and y defined by (19,5) is a solution of (18,5) if

$$\Omega(R) = 0\,, \quad \Omega(R) = \frac{1}{2}\int_0^2 h(R\cos\pi\sigma)\cos\pi\sigma\,\mathrm{d}\sigma - R\,. \tag{20,5}$$

As $\mathrm{d}\Omega/\mathrm{d}R$ is decreasing, $(\mathrm{d}\Omega/\mathrm{d}R)(0) > 0$, $\mathrm{d}\Omega/\mathrm{d}R < 0$ for large values of R, there exists a unique solution $\bar{R}$ of (20,5) so that $y(\xi) = \bar{R}\cos\pi(\xi - \varphi)$, $\varphi \in E_1$ are all solutions of (18,5).

It was proved in section 3 that conditions (4,2), (5,2) and (6,2) are fulfilled on every bounded subset of $Y = M_{1C}$ and that to every $\zeta > 0$ there exists an $\varepsilon_0 > 0$ that (10,1) hold for $0 < \varepsilon \leqq \varepsilon_0$. Let Z be the space of such $z \in M_{1C}$ that $\int_0^2 z(\xi)\cos\pi\xi\,\mathrm{d}\xi = \int_0^2 z(\xi)\sin\pi\xi\,\mathrm{d}\xi = 0$, let $C = Z \times E_1$, $\mathscr{C} = E_1$, $X = C \times \mathscr{C}$,

$$\|c\| = \|z\| + |r| \quad \text{for} \quad c = (z, r),\ z \in Z,\ r \in E_1\,,$$

$$\|x\| = \|c\| + |\gamma| \quad \text{for} \quad x = (c, \gamma),\ c \in C,\ \gamma \in \mathscr{C}\,,$$

$$G = \mathscr{E}[(c, \gamma) = (z, r, \gamma) \in X\,;\ \ \|z\| < \tfrac{1}{2}\bar{R},\ |r| < \tfrac{1}{2}\bar{R}]$$

and let the transformation T from G to Y be defined by

$$y(\xi) = (\bar{R} + r)\cos\pi(\xi - \gamma) + z(\xi) \tag{21,5}$$

T is smooth and periodic in γ with the period 2, the correspondence of y to (c, γ) is one to one up to an even number which may be added to γ and the local inverse of T is smooth. Therefore there exist functions $\hat{f} = \hat{f}(x, \tau, \varepsilon)$ and $\hat{f}_0 = \hat{f}_0(x)$ that T transforms equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \hat{f}(x, \tau, \varepsilon) \tag{22,5}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \hat{f}_0(x) \tag{23,5}$$

to (18,4) and (3,5). The functions $\hat{f}, \hat{f}_0$ are defined uniquely $\hat{f}(x_1, \tau, \varepsilon) = \hat{f}(x_2, \tau, \varepsilon)$, $\hat{f}_0(x_1) = f_0(x_2)$, $x_1 = (c, \gamma)$, $x_2 = (c, \gamma + 2) \in G$ and fulfil the conditions (4,2), (5,2) and (6,2). To every $\zeta > 0$ there exists an $\varepsilon_0 > 0$ that (10,1) hold in G for $0 < \varepsilon \leqq \varepsilon_0$

(with $f = \hat{f}$, $f_0 = \hat{f}_0$). It remains to verify (7,2) and (9,2). Equation (23,5) may be written in the following form

$$(24{,}5)\qquad \frac{\mathrm{d}z}{\mathrm{d}\tau} = -z\,,$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\bar{R} - r + \frac{1}{2}\int_0^2 h((\bar{R} + r)\cos\pi(\sigma - \gamma) + z(\sigma))\cos\pi(\sigma - \gamma)\,\mathrm{d}\sigma\,,$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = (2\pi(\bar{R} + r))^{-1}\int_0^2 h((\bar{R} + r)\cos\pi(\sigma - \gamma) + z(\sigma))\sin\pi(\sigma - \gamma)\,\mathrm{d}\sigma\,.$$

Therefore $\hat{f}_0(x) = 0$, if $x = (z, r, \gamma)$, $z = 0$, $r = 0$ and (7,2) is satisfied. Equation (8,2) may be given the following form

$$(25{,}5)\qquad \frac{\mathrm{d}z}{\mathrm{d}\tau} = -z$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \left(-1 + \frac{1}{2}\int_0^2 \frac{\mathrm{d}h}{\mathrm{d}\lambda}(\bar{R}\cos\pi\sigma)\cos\pi\sigma\,\mathrm{d}\sigma\right) r +$$

$$+ \frac{1}{2}\int_0^2 \frac{\mathrm{d}h}{\mathrm{d}\lambda}(\bar{R}\cos\pi(\sigma - \delta))\,z(\sigma)\cos\pi(\sigma - \delta)\,\mathrm{d}\sigma\,,\quad \delta \in E_1\,.$$

As $\mathrm{d}\Omega/\mathrm{d}R$ is decreasing, $(\mathrm{d}\Omega/\mathrm{d}R)(0) > 0$, $(\mathrm{d}\Omega/\mathrm{d}R)(R) < 0$ for large values of R, there exists a unique $R_1 > 0$, $(\mathrm{d}\Omega/\mathrm{d}R)(R_1) = 0$; from $\Omega(0) = 0$ it follows that $\Omega(R_1) > 0$, $R_1 < \bar{R}$ and

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R}(\bar{R}) = -1 + \frac{1}{2}\int_0^2 \frac{\mathrm{d}h}{\mathrm{d}\lambda}(\bar{R}\cos\sigma)\cos\pi\sigma\,\mathrm{d}\sigma < 0\,.$$

It may be verified in a similar manner as in Example 1,5 that (9,2) is fulfilled. Theorems 1,2, 2,2 and 3,2 may be applied. It follows (cf. (21,5), (14,4)) that there exist functions

$$(26{,}5)\qquad \hat{u}(x, t, \gamma, \varepsilon) = \frac{2}{\pi}(\bar{R} + r(\varepsilon t, \gamma, \varepsilon))\sin\pi x\cos\pi(t - \gamma) +$$

$$+ (Iz)(x + t, \varepsilon t, \gamma, \varepsilon) - (Iz)(x - t, \varepsilon t, \gamma, \varepsilon)\,,$$

$$\hat{w}(x, t, \gamma, \varepsilon) = 2(\bar{R} + r(\varepsilon t, \gamma, \varepsilon))\sin\pi x\sin\pi(t - \gamma) +$$

$$+ z(x + t, \varepsilon t, \gamma, \varepsilon) - z(-x + t, \varepsilon t, \gamma, \varepsilon)\,,$$

$z(., \tau, \gamma, \varepsilon) \in Z$, $0 < \varepsilon \leqq \varepsilon_0$, ε_0 sufficiently small that the following assertion takes place: if u is a solution of (1,4) (with g defined by (17,5)),

$$(27{,}5)\qquad u(x, \tilde{t}) = \hat{u}(x, \tilde{t}, \tilde{\gamma}, \varepsilon),\ u_t(x, \tilde{t}) = \hat{w}(x, \tilde{t}, \tilde{\gamma}, \varepsilon)\quad \text{for}\quad 0 \leqq x \leqq 1\,,$$

then there exists a function $\gamma(\tau)$ that

$$(28,5) \qquad u(x, t) = \hat{u}(x, t, \gamma(t), \varepsilon)\,,$$
$$u_t(x, t) = \hat{w}(x, t, \gamma(t), \varepsilon) \quad \text{for} \quad 0 \leqq x \leqq 1,\ t \geqq \tilde{t}\,.$$

The functions $\hat{u}$, $\hat{w}$ may be interpreted as an integral manifold of the problem (1,4). It follows from (26,5) that

$$\hat{u}(-x, t, \gamma, \varepsilon) = -\hat{u}(x, t, \gamma, \varepsilon) = \hat{u}(-x + 2, t, \gamma, \varepsilon)\,,$$
$$\hat{w}(-x, t, \gamma, \varepsilon) = -\hat{w}(x, t, \gamma, \varepsilon) = \hat{w}(-x + 2, t, \gamma, \varepsilon) \quad \text{for} \quad x, t, \gamma \in E_1$$

and (26,5), (21,5) and (17,4) imply (cf. Note 7,2) that

$$(29,5) \qquad \hat{u}(x, t, \gamma, + 2, \varepsilon) = \hat{u}(x, t, \gamma, \varepsilon) = \hat{u}(x, t + 2, \gamma, \varepsilon)\,,$$
$$\hat{w}(x, t, \gamma + 2, \varepsilon) = \hat{w}(x, t, \gamma, \varepsilon) = \hat{w}(x, t + 2, \gamma, \varepsilon) \quad \text{for} \quad x, t, \gamma \in E_1\,.$$

If a solution u of (1,4), which fulfils (27,5), is extended as to fulfil (6,4), then u fulfils (28,5) for $x \in E_1$. The correspondence between r, z and $\hat{u}$, $\hat{w}$ (defined by (26,5)) is one to one (cf. the definition of Z); as (1,4) is an autonomous problem, it follows from the uniqueness of the integral manifold of (18,4) (Theorem 1,2, assertion (v)) that to every t_1, t_2 and γ_1 there exists a γ_2 that

$$(30,5) \qquad \hat{u}(x, t_1, \gamma_1, \varepsilon) = \hat{u}(x, t_2, \gamma_2, \varepsilon)\,, \quad \hat{w}(x, t_1, \gamma_1, \varepsilon) = \hat{w}(x, t_2, \gamma_2, \varepsilon)\,.$$

According to the definition of z $\pi^{-1}\, 2(\bar{R} + r(\varepsilon t, \gamma, \varepsilon)) \cos \pi(t - \gamma)$ is the coefficient of $\sin x$ in the Fourier expansion of $\hat{u}(\,.\,, t, \gamma, \varepsilon)$; therefore (26,5) implies

$$(31,5) \qquad (\bar{R} + r(\varepsilon t_1, \gamma_1, \varepsilon)) \cos \pi(t_1 - \gamma_1) = (\bar{R} + r(\varepsilon t_2, \gamma_2, \varepsilon)) \cos \pi(t_2 - \gamma_2)\,,$$
$$(\bar{R} + r(\varepsilon t_1, \gamma_1, \varepsilon)) \sin \pi(t_1 - \gamma_1) = (\bar{R} + r(\varepsilon t_2, \gamma_2, \varepsilon)) \sin \pi(t_2 - \gamma_2)\,.$$

As r and z are periodic in γ with the period 2, (31,5) is equivalent to

$$(32,5) \qquad r(\varepsilon t_1, \gamma_1, \varepsilon) = r(\varepsilon t_2, \gamma_2, \varepsilon),\ t_1 - \gamma_1 = t_2 - \gamma_2 + 2k\,, \quad k = \dots -1, 0, 1, \dots$$

Put $t_1 = t$, $\gamma_1 = \gamma$, $t_2 = 0$, $k = 0$ in the last equation; (28,5) may be rewritten as

$$(33,5) \qquad u(x, t) = \hat{u}(x, 0, \gamma(t) - t, \varepsilon)\,, \quad u_t(x, t) = \hat{w}(x, 0, \gamma(t) - t, \varepsilon)\,.$$

It follows from (33,5) and (29,5) that

$$u(x, t_1) = u(x, t_2),\ u_t(x, t_1) = u_t(x, t_2) \quad \text{for} \quad x \in E_1\,,$$

if

$$(34,5) \qquad \gamma(t_1) - t_1 = \gamma(t_2) - t_2 + 2k\,, \quad k = \dots -1, 0, 1, \dots$$

and as (1,5) is an autonomous problem, $t_2 - t_1$ is a period of u, if (34,5) holds. γ fulfils the equation from Note 1,2. Therefore (cf. $\varepsilon t = \tau$) $|\gamma(t_2) - \gamma(t_1)| \leqq K_1 \varepsilon |t_2 - t_1|$ and it may be shown that u is periodic in t and its period differs from 2 by $O(\varepsilon)$ (here $O(\varepsilon)$ may be replaced by $o(\varepsilon)$, if we take into account that $\alpha_0(0, \gamma, \tau) = 0$ and use some elementary estimates) and that the integral manifold is formed by the periodic solution $u(x, t)$, $u_t(x, t)$ and all its translations $u(x, t + \lambda)$, $u_t(x, t + \lambda)$, $\lambda \in E_1$.

6. THE LOSS OF SMOOTHNESS OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM

In this section the following problem will be examined

$$u_{tt} - u_{xx} = \varepsilon(1 - u_t^2)\, u_t\,, \quad 0 \leqq x \leqq 1\,, \quad t \geqq 0\,, \quad \varepsilon > 0\,, \qquad (1,6)$$
$$u(0, t) = u(1, t) = 0\,.$$

Let $q(\xi) = \frac{1}{2}$ for $0 \leqq \xi < 1$, $q(\xi + 1) = -q(\xi)$ for $\xi \in E_1$ and let $Q = I_\xi q$ i.e. $Q(\xi) = \frac{1}{2}\xi - \frac{1}{4}$ for $0 \leqq \xi < 1$, $Q(\xi + 1) = -Q(\xi)$ for $\xi \in E_1$. The results of this section which concern the classical solutions of (1,6) are formulated in the following Theorem:

Theorem 1,6. *There exist functions φ and ψ, which fulfil the conditions* (iii), (iv) *of section 4 and to every $\delta > 0$ there exists an $\varepsilon_0 > 0$ that for $0 < \varepsilon \leqq \varepsilon_0$ the following assertions take place*:

(i) *the solution u of* (1,6), *$u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$ exists for $t \geqq 0$ and u, u_x, u_t are bounded,*

(ii) *there exists the limit*

$$\lim_{i \to \infty} u(x, t + 2i) = v(x, t)\,, \quad i = 1, 2, 3, \ldots \qquad (2,6)$$

uniformly with respect to x and t.

(iii) *v is continuous, $v(x, t + 2) = v(x, t)$ and there exists a $\vartheta = \vartheta(\varepsilon)$ that the derivatives $\partial v/\partial x$, $\partial v/\partial t$ exist if $t + x$, $t - x \neq j + \vartheta$, $j = \ldots -1, 0, 1, \ldots$ and*

$$\lim_{i \to \infty} u_t(x, t + 2i) = \frac{\partial v}{\partial t}(x, t)\,, \quad \lim_{i \to \infty} u_x(x, t + 2i) = v_x(x, t)\,, \quad i = 1, 2, 3, \ldots \qquad (3,6)$$
$$t + x,\ t - x \neq j + \vartheta\,, \quad j = \ldots -1, 0, 1, \ldots$$

$$\left|\frac{\partial v}{\partial t}(x, t) - q(x + t - \vartheta) + q(-x + t - \vartheta)\right| \leqq \delta\,, \qquad (4,6)$$
$$t + x,\ t - x \neq j + \vartheta\,, \quad j = \ldots -1, 0, 1, \ldots$$

$$(5,6) \qquad \left|\frac{\partial v}{\partial x}(x, t) - q(x + t - \vartheta) - q(-x + t - \vartheta)\right| \leqq \delta\,,$$

$$t + x,\ t - x \neq j + \vartheta\,,\quad j = \ldots -1, 0, 1, \ldots$$

$$(6,6) \quad |v(x, t) - Q(x + t - \vartheta) + Q(-x + t - \vartheta)| \leqq \tfrac{1}{2}\delta\,,\quad t \geqq 0\,,\quad x \in \langle 0, 1\rangle\,.$$

Note 1,6. It follows from (4,6) and (5,6) that $\partial v/\partial t$, $\partial v/\partial x$ are discontinuous at (x, t) if $x + t = j + \vartheta$ or $-x + t = j + \vartheta$, $j = \ldots -1, 0, 1, \ldots$ and $\delta < \frac{1}{2}$. v may be regarded as a generalized periodic solution of (1,6), which is the limit for $t \to \infty$ of the classical solution u and the function $Q(x + t - \vartheta) - Q(-x + t - \vartheta)$ is an approximation of v.

Note 2,6. The same results are valid, if q, Q are replaced by q^*, Q^* which are defined as follows: there exist numbers $0 = \xi_0 < \xi_1 < \ldots < \xi_l = 1$,

$$q^*(\xi) = \tfrac{1}{2} \text{ for } \xi_{2i} \leqq \xi < \xi_{2i+1}\,,\quad i = 0, 1, \ldots, 2i + 1 \leqq l\,,$$

$$q^*(\xi) = -\tfrac{1}{2} \text{ for } \xi_{2i+1} \leqq \xi < \xi_{2i+2}\,,\quad i = 0, 1, \ldots, 2i + 2 \leqq l\,,$$

$$q^*(\xi + 1) = -q^*(\xi) \text{ for } \xi \in E_1,\ Q^* = I_\xi q^*.$$

We shall use the method developed in section 4. The transformed equation (cf. (18,4)) is

$$(7,6) \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = f(y, \tau, \varepsilon)$$

$$f(y, \tau, \varepsilon)(\xi) = \tfrac{1}{2}\left[1 - (y(\xi) - y(-\xi + 2\tau/\varepsilon))^2\right](y(\xi) - y(-\xi + 2\tau/\varepsilon))\,,$$

and the averaged equation (cf. (20,4)) is

$$(8,6) \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = f_0(y)\,,$$

$$f_0(y)(\xi) = \tfrac{1}{2}[y(\xi)(1 - y^2(\xi) - 3Py^2) + Py^3]\,.$$

q is obviously the solution of $f_0(y) = 0$ in M.

Note 3,6. Define for $y \in M$, $F(y, \tau, \varepsilon)$ by the relation

$$(9,6) \qquad F(y, \tau, \varepsilon) = \frac{1}{2}\int_0^\tau [1 - (y(\xi) - y(-\xi + 2\sigma/\varepsilon))^2]\,)y(\xi) - y(-\xi + 2\sigma/\varepsilon))\,\mathrm{d}\sigma\,,$$

put $c = y - \bar{q}$, $\bar{q}(\xi) = q(\xi - \vartheta)$ for $\xi \in E_1$, $C = M$, $\mathscr{C} = (0)$ and consider the generalized equation

$$(10,6) \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = \mathrm{D}_\tau F(y, \tau, \varepsilon)$$

instead of (7,6) (cf. Note 1,4). We find that the equation (8,2) (corresponding to (8,6)) has the form

$$\frac{dc}{d\tau} = Ac\,, \quad (Ac)(\xi) = -\tfrac{1}{4}\,c(\xi) - 3\,\bar{q}(\xi)\,P(\bar{q}c).$$

As $(d/d\tau)\,P(\bar{q}c) = -\left(\frac{1}{4} + 3P\bar{q}^2\right)P(\bar{q}c)$, we deduce in a similar manner as in Example 1,4 that the estimate (9,2) is satisfied. Theorem 6,2 and the modified Theorem 4,2 (cf. Note 6,2) imply that there exist $\varepsilon_0' > 0$ and $\varkappa_2 > 0$ that equation (10,6) has a periodic solution $\hat{y}$, $\hat{y}(\tau + \varepsilon) = \hat{y}(\tau)$, $\|\hat{y}(\tau) - q\| \leqq \varkappa_2$ for $\tau \geqq 0$ and

$$(11,6) \qquad \|y(\tau) - \hat{y}(\tau)\| \leqq K' e^{-\nu'(\tau - \tilde{\tau})} \|\tilde{y} - \hat{y}(\tilde{\tau})\|\,, \; (\nu' > 0)\,, \quad \text{for} \quad \tau \geqq \tilde{\tau}$$

for every solution y of (10,6) in M, $y(\tilde{\tau}) = \tilde{y}$, $\|\tilde{y} - \bar{q}\| \leqq \varkappa_2$. Put $\hat{y}(\xi, \tau) = \hat{y}(\tau)(\xi)$, $\hat{Y}(\xi, \tau) = I_\xi\,\hat{y}(\xi, \tau)$, $\hat{u}(x, t) = \hat{Y}(x + t, \varepsilon t) - \hat{Y}(-x + t, \varepsilon t)$. According to (13,4) and Note 2,4 $\hat{u}$ may be regarded as a generalized solution of (1,6), which is exponentially stable in a suitable set of generalized solutions. It will be shown that $\hat{u}(x, t) = v(x, t + \vartheta)$ so that $\hat{u}$ is the limit for $t \to \infty$ of a classical solution (cf. (2,6)) (ε being sufficiently small).

It was proved in section 4 that the function $u(x, t) = Y(x + t, \varepsilon t) - Y(-x + t, \varepsilon t)$ is a classical solution of the problem (1,6) and that $u_t(x, t) = y(x + t, \varepsilon t) - y(-x + t, \varepsilon t)$, $u_x(x, t) = y(x + t, \varepsilon t) + y(-x + t, \varepsilon t)$, if $Y(\xi, \tau) = I_\xi\, y(\xi, \tau)$, $y(\xi, \tau) = y(\tau)(\xi)$ and $y = y(\tau)$ is a solution of (7,6) in M_{1C}. Theorem 1,6 is a consequence of the following Theorem:

Theorem 2,6. *There exists a function $\tilde{y} \in M_{1C}$ ($\tilde{y}$ may be chosen analytic) and to every $\delta > 0$ there exists an $\varepsilon_0 > 0$ that for $0 < \varepsilon \leqq \varepsilon_0$ the following assertions take place:*

(i) *the solution y of (7,6) in M_{1C}, $y(0) = \tilde{y}$ exists for $\tau \geqq 0$ and is bounded in the norm of the space M,*

(ii) *put $y(\xi, \tau) = y(\tau)(\xi)$; there exists such a $\vartheta = \vartheta(\varepsilon)$ that the limit*

$$(12,6) \qquad \lim_{i \to \infty} y(\xi, \tau + i\varepsilon) = z(\xi, \tau)\,, \quad i = 1, 2, 3, \ldots$$

exists for $\xi \neq \vartheta + j$, $j = \ldots -1, 0, 1, \ldots$ and

$$(13,6) \qquad |z(\xi, \tau) - q(\xi - \vartheta)| \leqq \delta \quad \textit{for} \quad \xi \neq \vartheta + j\,, \quad j = \ldots -1, 0, 1, \ldots$$

Note 4,6. It follows from (12,6) that $z(\xi, \tau + \varepsilon) = z(\xi, \tau)$, $\xi \neq \vartheta + j$, $j = \ldots - 1, 0, 1, \ldots$ As $y(\xi, \tau)$ and $(\partial y/\partial \tau)(\xi, \tau)$ are bounded, (12,6) together with (7,4) implies that $Y(\xi, \tau + i\varepsilon) - Z(\xi, \tau) \to 0$ with $i \to \infty$, $i = 1, 2, \ldots$ uniformly with respect to $\xi, \tau, \tau \geqq 0$, $Z = I_\xi z$ and hence the uniform convergence in (2,6) follows ($v(x, t) = Z(x + t, \varepsilon t) - Z(-x + t, \varepsilon t)$). The assertion on the convergence in (12,6) may be strengthened as follows: if J is a closed interval, $J \subset (\vartheta, \vartheta + 1)$, then there exist positive constants $\mathring{K}$, $\mathring{\nu}$ that

$$(14{,}6) \qquad |y(\xi, \tau) - z(\xi, \tau)| \leqq \mathring{K} e^{-\mathring{\gamma}\tau} \quad \text{for} \quad \tau \geqq 0, \ \xi \in J .$$

Note 5,6. y is a classical solution,

$$y(\xi, \tau + i\varepsilon) = y(\xi, i\varepsilon) + \int_{i\varepsilon}^{\tau} f(y(\sigma), \sigma, \varepsilon)(\xi) \, d\sigma$$

As $y(\xi, \tau)$ and $(\partial y / \partial \tau)(\xi, \tau)$ are bounded, (12,6) (or (14,6)) and (7,4) imply that

$$(15{,}6) \qquad z(\xi, \tau) = z(\xi, 0) + \int_0^{\tau} f(z(\sigma), \sigma, \varepsilon)(\xi) \, d\sigma , \quad \xi \neq \vartheta + j , \quad j = -1, 0, 1, \ldots$$

As $f(z(\sigma), \sigma, \varepsilon)(\xi)$ is bounded, $z(\xi, \tau)$ fulfils a Lipschitz condition with respect to τ and followingly (15,6) may be rewritten in the form

$$z(\tau_1) = z(0) + \int_0^{\tau_1} D_\sigma F(z(\tau), \sigma) , \quad \tau_1 \geqq 0 ,$$

F being defined by (9,6) and $z = z(\tau) \in M$ being defined by $z(\tau)(\xi) \doteq z(\xi, \tau)$. Therefore z is a periodic solution of (10,6) in M, $\|z - \bar{q}\| \leqq \delta$. It follows from (11,6) that $\hat{y} = z$ if $\delta \leqq \varkappa_2$, $\varepsilon \leqq \varepsilon_0$ and therefore $\hat{u}(x, t) = v(x, t + \vartheta)$ as stated in Note 3,6.

In order to prove Theorem 2,6 several Lemmas will be needed. The positive number ε_0 will not be fixed; it may be diminished in the course of the considerations. For $y \in M_{1C}$ let $y' \in M$, $y'(\xi) = (\partial y / \partial \xi)(\xi)$. Let $\hat{M}$ be the set of such elements $y \in M$ that $y(\xi + 1) = -y(\xi)$ a.e. on E_1 and let $\hat{M}_{1C} = M_{1C} \cap \hat{M}$. If S is a measurable subset of E_1, let us denote by $|S|$ the measure of S.

Lemma 1,6. *Let $y(y_0)$ be the solution of (7,5) ((8,5)) in M on $\langle \tilde{\tau}, \tau_1 \rangle$, $y(\tilde{\tau}) = y_0(\tilde{\tau}) = \tilde{y} \in \hat{M}$. Then $y(\tau), y_0(\tau) \in \hat{M}$ for $\tau \in \langle \tilde{\tau}, \tau_1 \rangle$.*

Lemma 1,6 follows from the uniqueness of the solutions of (7,6) ((8,6)) and from the relation $f(y, \tau, \varepsilon)(\xi + 1) = -f(y, \tau, \varepsilon)(\xi)$, $f_0(y)(\xi + 1) = -f_0(y)(\xi)$ a.e. for $y \in \hat{M}$, $\xi \in E_1$.

Lemma 2,6. *There exist $\varepsilon_0 > 0$ and K_1 that $\tilde{y} \in \hat{M}$, $\|\tilde{y}\| \leqq \frac{3}{4}$, $\tilde{\tau} \in E_1$ implies that the solution y of (7,6) in M, $y(\tilde{\tau}) = \tilde{y}$ (y_0 of (8,6) in M, $y_0(\tilde{\tau}) = \tilde{y}$), $0 < \varepsilon \leqq \varepsilon_0$ is defined on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon_0 \rangle$, $\|y(\tau)\|$, $\|y_0(\tau)\| < 1$ on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon_0 \rangle$ and*

$$(16{,}6) \qquad \begin{aligned} y(\tilde{\tau} + \varepsilon) &= \tilde{y} + f_0(\tilde{y})\, \varepsilon + Z , \\ y_0(\tilde{\tau} + \varepsilon) &= \tilde{y} + f_0(\tilde{y})\, \varepsilon + Z_0 , \quad \|Z\|, \|Z_0\| \leqq K_1 \varepsilon^2 . \end{aligned}$$

The proof is elementary (cf. (12,1)).

Lemma 3,6. *Let a, b, v be continuous real functions on* $\langle\tilde{\tau}, \tau_1\rangle$ *and let*

$$v(\tau) = a(\tau) + \int_{\tilde{\tau}}^{\tau} b(\sigma)\, v(\sigma)\, \mathrm{d}\sigma\,, \quad \tau \in \langle\tilde{\tau}, \tau_1\rangle\,. \tag{17,6}$$

Then

$$v(\tau) = a(\tau) + \int_{\tilde{\tau}}^{\tau} \exp\left\{\int_{\sigma}^{\tau} b(\lambda)\, \mathrm{d}\lambda\right\} b(\sigma)\, a(\sigma)\, \mathrm{d}\sigma\,, \quad \tau \in \langle\tilde{\tau}, \tau_1\rangle\,. \tag{18,6}$$

It follows by substituting (18,6) into (17,6), substracting a and differentiating that v defined by (18,6) satisfies (17,6); there are no other solutions, as (17,6) is an equation of Volterra's type.

Lemma 4,6. *Let y be a solution of* (7,6) *in* M_{1C} *on* $\langle\tilde{\tau}, \tau_1\rangle$, $y(\tilde{\tau}) = \tilde{y}$, $y(\xi, \tau) = y(\tau)(\xi)$, $y'(\xi, \tau) = (\partial y/\partial \xi)(\xi, \tau)$. *Then*

$$\begin{aligned} y'(\xi, \tau) = {} & \tilde{y}'(\xi) + r(\xi, \tau) + \\ & + \int_{\tilde{\tau}}^{\tau} \exp\left\{\int_{\sigma_1}^{\tau} [1 - 3(y(\xi, \sigma_2) - y(-\xi + 2\sigma_2/\varepsilon, \sigma_2))^2]\, \mathrm{d}\sigma_2\right\} . \\ & . [1 - 3y(\xi, \sigma_1) - y(-\xi + 2\sigma_1/\varepsilon\,, \sigma_1))^2]\, (\tilde{y}'(\xi) + r(\xi, \sigma_1))\, \mathrm{d}\sigma_1\,, \end{aligned} \tag{19,6}$$

$$\xi \in E_1,\ \tau \in \langle\tilde{\tau}, \tau_1\rangle\,,$$

r being defined by

$$\begin{aligned} r(\xi, \tau) = {} & \tfrac{1}{2}\varepsilon\{y(-\xi + 2\tau/\varepsilon, \tau) - \tilde{y}(-\xi + 2\tilde{\tau}/\varepsilon) - \\ & - [y(\xi, \tau) - y(-\xi + 2\tau/\varepsilon, \tau)]^3 + [\tilde{y}(\xi) - \tilde{y}(-\xi + 2\tilde{\tau}/\varepsilon)]^3 - \\ & - \int_{\tilde{\tau}}^{\tau} [1 - 3(y(\xi, \sigma) - y(-\xi + 2\sigma/\varepsilon, \sigma))^2]\, [(\partial y/\partial \tau)(\xi, \sigma) - (\partial y/\partial \tau)(-\xi + 2\sigma/\varepsilon, \sigma)]\, \mathrm{d}\sigma\,, \end{aligned} \tag{20,6}$$

$$(\partial y/\partial \tau)(\alpha, \beta) = \partial/\partial \tau\ y(\xi, \tau)|_{\xi = \alpha, \tau = \beta}\,.$$

Proof. It follows from

$$y(\xi, \tau) = \tilde{y}(\xi) + \int_{\tilde{\tau}}^{\tau} f(y, \sigma, \varepsilon)\, (\xi)\, \mathrm{d}\tau$$

that

(21,6)

$$y'(\xi, \tau) = \tilde{y}'(\xi) + r(\xi, \tau) + \int_{\tilde{\tau}}^{\tau} [1 - 3(y(\xi, \sigma) - y(-\xi + 2\sigma/\varepsilon, \sigma))^2]\, y'(\xi, \sigma)\, \mathrm{d}\sigma\,,$$

$$r(\xi, \tau) = \int_{\tilde{\tau}}^{\tau} [1 - 3(y(\xi, \sigma) - y\,(-\xi + 2\sigma/\varepsilon, \sigma))^2]\, y'(-\xi + 2\sigma/\varepsilon, \sigma)\, \mathrm{d}\sigma =$$

$$= \tfrac{1}{2}\varepsilon\{y(-\xi + 2\tau/\varepsilon, \tau) - \tilde{y}(-\xi + 2\tilde{\tau}/\varepsilon) - [y(\xi, \tau) - y(-\xi + 2\tau/\varepsilon, \tau)]^3 +$$

$$+ [\tilde{y}(\xi) - \tilde{y}(-\xi + 2\tilde{\tau}/\varepsilon)]^3 - \int_{\tilde{\tau}}^{\tau} [1 - 3(y(\xi, \sigma) - y(-\xi + 2\sigma/\varepsilon, \sigma))^2] \,.$$

$$. [(\partial y/\partial \tau)(\xi, \sigma) - (\partial y/\partial \tau)(-\xi + 2\sigma/\varepsilon, \sigma)] \, d\sigma\} \,.$$

The last equality may be verified by differentiating with respect to τ. Applying Lemma 3,6 to (21,6) we obtain (19,6).

Lemma 5,6 *Let $\varrho > 0$. There exists an ε_0 that the following assertion holds:*

if $$\tilde{y} \in M, \ \|\tilde{y}\| \leqq 1, \ P\tilde{y}^2 \leqq \tfrac{1}{4} + \varrho, \ \tilde{\tau} \in E_1, \ 0 < \varepsilon \leqq \varepsilon_0$$

then the solution $y(y_0)$ of (7,6) ((8,6)) *in M, $y(\tilde{\tau}) = \tilde{y}$ $(y_0(\tilde{\tau}) = \tilde{y})$ exists on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$ and*

$$(22,6) \qquad Py^2(., \tilde{\tau} + \varepsilon) \leqq \tfrac{1}{4} + \varrho \,, \quad Py_0^2(., \tilde{\tau} + \varepsilon) \leqq \tfrac{1}{4} + \varrho \,.$$

Proof. Let ε_0 be defined by Lemma 2,6; y and y_0 are defined on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$. As y_0 fulfils (8,6), it may be proved by a standard argument that $(d/d\tau) Py_0^2 = Py_0^2(1 - 3Py_0^2) - Py_0^4$. Obviously $Py_0^4 \geqq (Py_0^2)^2$, hence $(d/d\tau) Py_0^2 \leqq Py_0^2(1 - 4Py_0^2)$. Let γ be the solution of $d\gamma/d\tau = \gamma(1 - 4\gamma)$, $\gamma(0) = \tfrac{1}{4} + \varrho$. Then $Py_0^2(., \tilde{\tau} + \varepsilon) \leqq \gamma(\varepsilon) \leqq \tfrac{1}{4} + \varrho - \varrho\varepsilon + K_2\varepsilon^2$, K_2 being a positive constant and the second inequality (22,6) is fulfilled, if $K_2\varepsilon_0 \leqq \varrho$. It follows from (16,6) that $Py^2(., \tilde{\tau} + \varepsilon) \leqq \tfrac{1}{4} + \varrho - \varrho\varepsilon + K_3\varepsilon^2$. If necessary, let us diminish ε_0 so that $K_2\varepsilon_0$, $K_3\varepsilon_0 \leqq \varrho$. Then (22,6) is satisfied.

Lemma 6,6. *Let the positive numbers ϱ, μ, ν, Ω satisfy*

$$(23,6) \qquad \mu + \nu < (20)^{-1}, \ \mu > 3\mu^2 + 9\varrho, \ \Omega < 1, \ \nu > 3\mu + 1 - \Omega \,.$$

There exists such an $\varepsilon_0 > 0$ that the assertions (i)–(iv) *hold.*

Suppose that $0 < \varepsilon \leqq \varepsilon_0$, $\tilde{y} \in \hat{M}$, $\tilde{\tau} \in E_1$ and that

$$(24,6) \qquad |\tilde{y}(\xi)| \leqq \tfrac{1}{2} + \nu \quad \text{a.e. in } E_1 \,,$$

$$(25,6) \qquad P\tilde{y}^2 \leqq \tfrac{1}{4} + \varrho \,,$$

$$(26,6) \qquad |\mathscr{E}[\xi \in (0, 1); \ |\tilde{y}(\xi)| \geqq \tfrac{1}{2} - \mu]| \geqq \Omega \,.$$

Let $y(y_0)$ be the solutions of (7,6) ((8,6)) *in M, $y(\tilde{\tau}) = \tilde{y} = y_0(\tilde{\tau})$, $y(\xi, \tau) = y(\tau)(\xi)$, $y_0(\xi, \tau) = y_0(\tau)(\xi)$. Then*

(i) *y and y_0 exist on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$, $\|y(\tau)\|$, $\|y_0(\tau)\| < 1$ on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$, $y(., \tilde{\tau} + \varepsilon)$ and $y_0(., \tilde{\tau} + \varepsilon)$ fulfil* (24,6)–(26,6),

(ii) *let* $\mu \leqq \lambda \leqq \frac{1}{2} - \mu$, $\tilde{y}(\xi_1) \geqq \lambda\,(\tilde{y}(\xi_1) \leqq -\lambda)$ *for* $\xi_1 \in S \subset E_1$, *S measurable then*

$$y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \lambda + \tfrac{1}{3}\varepsilon\mu^2 \,(y(\xi_1, \tilde{\tau} + \varepsilon) \leqq -\lambda - \tfrac{1}{3}\varepsilon\mu^2) \quad \textit{a.e. in } S$$

and y_0 *fulfils the same inequality.*

(iii) *let* $\tilde{y} \in \hat{M}_{1C}$, $\tilde{y}'(\xi_1) \geqq 10^3\,(\tilde{y}'(\xi_1) \leqq -10^3)$, $|\tilde{y}(\xi_1)| \leqq \mu$ *for some* $\xi_1 \in E_1$; *then* y *is a solution of* (7,6) *in* M_{1C} *and* $y'(\xi_1, \tilde{\tau} + \varepsilon) \geqq \tilde{y}'(\xi_1)\,[1 + \varepsilon 10^{-2}]$ $(y'(\xi_1, \tilde{\tau} + \varepsilon) \leqq \tilde{y}'(\xi_1)\,[1 + \varepsilon 10^{-2}])$,

(iv) *let* $\tilde{y} \in \hat{M}_{1C}$, $|\tilde{y}'(\xi_1)| \geqq 10^3$, $|\tilde{y}(\xi_1)| \geqq \frac{1}{2} - \mu$ *for some* $\xi_1 \in E_1$; *then*

$$|y'(\xi_1, \tilde{\tau} + \varepsilon)| \leqq |\tilde{y}'(\xi_1)|\,(1 - \varepsilon 10^{-2})\,.$$

Proof. As ε_0 may be chosen arbitrarily small, we shall use Lemmas 2,6 and 5,6 and

$$(27,6) \qquad \mu + \nu + 6\varepsilon_0 \leqq 20^{-1},\ 2K_1\varepsilon_0 \leqq \nu - 3\mu - (1 - \Omega),\ 3K_1\varepsilon_0 < \mu^2\,.$$

Lemmas 2,6 and 6,4 imply that y, y_0 exist on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $\|y(\tau)\|$, $\|y_0(\tau)\| < 1$ on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ and that (16,6) holds. As $\varepsilon_0 < 120^{-1}$, $\|\tilde{y}\| < 1$, the function $\sigma + \varepsilon\sigma[1 - -\sigma^2 - 3P\tilde{y}^2]$ is increasing on $\langle -1, 1\rangle$ and it follows from (16,6) that

$$\begin{aligned}|y(\xi, \tilde{\tau} + \varepsilon)| &\leqq \tfrac{1}{2} + \nu + \varepsilon(\tfrac{1}{2} + \nu)\,[1 - (\tfrac{1}{2} + \nu)^2 - 3P\tilde{y}^2] + K_1\varepsilon^2 \leqq \\ &\leqq \tfrac{1}{2} + \nu + \tfrac{1}{2}\varepsilon[1 - \tfrac{1}{4} - \nu - 3\Omega(\tfrac{1}{4} - \mu)] + K_1\varepsilon^2 \leqq \\ &\leqq \tfrac{1}{2} + \nu + \tfrac{1}{2}\varepsilon[\tfrac{3}{4}(1 - \Omega) + 3\mu - \nu] + K_1\varepsilon^2 \leqq \tfrac{1}{2} + \nu\end{aligned}$$

and $y(., \tilde{\tau} + \varepsilon)$ fulfils (24,6). Similarly $y_0(., \tilde{\tau} + \varepsilon)$ fulfils (24,6). It follows from Lemma 5,6 that $y(., \tilde{\tau} + \varepsilon)$ and $y_0(., \tilde{\tau} + \varepsilon)$ fulfil (25,6) and it is a consequence of the assertion (ii) that $y(., \tilde{\tau} + \varepsilon)$ and $y_0(., \tilde{\tau} + \varepsilon)$ satisfy (26,6); let us prove the assertion (ii).

Let $\tilde{y}(\xi_1) \geqq \lambda$, $\mu \leqq \lambda \leqq \frac{1}{2} - \mu$. As the function $\sigma + \varepsilon\sigma[1 - \sigma^2 - 3P\tilde{y}^2]$ is increasing on $\langle -1, 1\rangle$, it follows from (16,6) that

$$\begin{aligned}y(\xi_1, \tilde{\tau} + \varepsilon) &\geqq \lambda + \varepsilon\lambda[1 - \lambda^2 - 3P\tilde{y}^2] - K_1\varepsilon^2 \geqq \\ &\geqq \lambda + \varepsilon\mu[1 - (\tfrac{1}{2} - \mu)^2 - 3(\tfrac{1}{4} + \varrho)] - K_1\varepsilon^2 \geqq \\ &\geqq \lambda + \varepsilon\mu[\mu - \mu^2 - 3\varrho] - K_1\varepsilon^2 \geqq \lambda + \tfrac{2}{3}\varepsilon\mu^2 - K_1\varepsilon^2 \geqq \lambda + \tfrac{1}{3}\varepsilon\mu^2\,.\end{aligned}$$

The case that $y(\xi_1, \tau) \leqq -\lambda$ is similar. The assertions (i), (ii) are proved.

In order to prove the assertions (iii) and (iv) (19,6) will be needed. If $\tilde{y} \in \hat{M}_{1C}$, then y is a solution of (7,6) in M_{1C} according to Lemma 6.4. If $\|y\| \leqq 1$ then $|f(y, \tau, \varepsilon)(\xi)| \leqq 3$. As $\|\tilde{y}\| \leqq \frac{1}{2} + \nu < \frac{3}{4}$, it follows from (7,6) that $|y(\xi, \tau)| \leqq 1$ for $\xi \in E_1$, $\tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$; therefore $r(\xi, \tau)$ may be estimated by (cf. (20,6))

$$|r(\xi, \tau)| \leqq \varepsilon\tfrac{1}{2}[2 + \tfrac{2}{3}8 + (120)^{-1}\,.\,4\,.\,6] < 5\varepsilon\,.$$

As $\tilde{\tau} \leqq \sigma_1 \leqq \tau \leqq \tilde{\tau} + \varepsilon$ we obtain that

$$\exp\left\{\int_{\sigma_1}^{\sigma} 1 - (y(\xi, \sigma_2) - y(-\xi + 2\sigma_2/\varepsilon, \sigma_2))^2] \,\mathrm{d}\sigma_2\right\} \geqq e^{-3\varepsilon} \geqq e^{-1/40} > \tfrac{1}{2}.$$

Let $|\tilde{y}(\xi_1)| \leqq \mu$, $\tilde{y}'(\xi_1) \geqq 10^3$. Then $|y(\xi_1, \tau)| \leqq \mu + 3\varepsilon$ and $|y(\xi, \tau)| \leqq \frac{1}{2} + \nu + 3\varepsilon$ for $\tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $\xi \in E_1$. It follows from (91,6) that

$$y'(\xi_1, \tilde{\tau} + \varepsilon) \geqq \tilde{y}'(\xi_1) - 5\varepsilon + \int_{\tilde{\tau}}^{\tilde{\tau}+\varepsilon} \tfrac{1}{2}[1 - 3(\mu + 3\varepsilon + \tfrac{1}{2} + \nu + 3\varepsilon)^2] \,.$$

$$.\, (\tilde{y}'(\xi_1) - 5\varepsilon)\,\mathrm{d}\tau \geqq (\tilde{y}'(\xi_1) - 5\varepsilon)\left[1 + \tfrac{1}{2}\varepsilon(1 - 3(\tfrac{1}{2} + \tfrac{1}{20})^2\right] \geqq \tilde{y}'(\xi_1)\left[1 + \varepsilon\,.\,10^{-2}\right] +$$

$$+ 10^3\,.\,\varepsilon\,.\,10^{-2} - 5\varepsilon(1 + 120^{-1}) \geqq \tilde{y}'(\xi_1)\left[1 + \varepsilon\,.\,10^{-2}\right].$$

The case that $\tilde{y}'(\xi_1) \leqq -10^3$ is similar and assertion (iii) holds.

Let $\tilde{y}'(\xi_1) \geqq 10^3$, $\tilde{y}(\xi_1) \geqq \frac{1}{2} - \mu$,

$$S_1 = \mathscr{E}\left[\sigma \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle;\ 1 - 3(y(\xi_1, \sigma) - y(-\xi_1 + 2\sigma/\varepsilon, \sigma))^2 \geqq 0\right],$$

$$S_2 = \mathscr{E}\left[\sigma \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle;\ 1 - 3(y(\xi_1, \sigma) - y(-\xi_1 + 2\sigma/\varepsilon, \sigma))^2 < 0\right],$$

$$S_3 = \mathscr{E}\left[\sigma \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle;\ \tilde{y}(-\xi_1 + 2\sigma/\varepsilon) \leqq -\tfrac{1}{2} + \mu\right].$$

As $\tilde{y}(\xi) = -\tilde{y}(\xi + 1)$, it follows that $|S_3| \geqq \frac{1}{2}\varepsilon\Omega$. Taking into account that

$$|y(\xi_1, \sigma) - \tilde{y}(\xi_1)| \leqq 3\varepsilon,\quad |y(-\xi_1 + 2\sigma/\varepsilon, \sigma) - \tilde{y}(-\xi_1 + 2\sigma/\varepsilon)| \leqq 3\varepsilon$$

for $\sigma \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $2\mu + 6\varepsilon \leqq \mu + \nu + 6\varepsilon_0 \leqq 20^{-1}$, we obtain that $1 - 3[y(\xi_1, \sigma) - y(-\xi_1 + 2(\sigma/\varepsilon), \sigma)]^2 = 1 - 3[1 - 2\mu - 6\varepsilon]^2 \leqq -1{,}7$ for $\sigma \in S_3$, therefore $S_3 \subset S_2$, $|S_2| \geqq \frac{1}{2}\varepsilon\Omega$, $|S_1| \leqq \varepsilon(1 - \frac{1}{2}\Omega)$. Applying (19,6) with $\xi = \xi_1$, $\tau = \tilde{\tau} + \varepsilon$ we find that (remind that $|r| \leqq 5\varepsilon$)

$$y'(\xi_1, \tilde{\tau} + \varepsilon) \leqq \tilde{y}'(\xi_1) + 5\varepsilon + \int_{S_1} + \int_{S_2} \leqq \tilde{y}'(\xi_1) + 5\varepsilon + \int_{S_1} + \int_{S_3},$$

$$\int_{S_1} \leqq \int_{S_3} e^{\varepsilon}\,.\,1\,.\,(\tilde{y}'(\xi_1) + 5\varepsilon)\,\mathrm{d}\sigma_1 \leqq \varepsilon\,(1 - \tfrac{1}{2}\Omega)\,\tfrac{13}{12}\,(\tilde{y}'(\xi_1) + 5\varepsilon),$$

$$\int_{S_3} \leqq \int_{S_3} e^{-11\varepsilon}\,.\,(-\tfrac{18}{10})\,.\,(\tilde{y}'(\xi_1) - 5\varepsilon)\,\mathrm{d}\sigma_1 \leqq$$

$$\leqq -\tfrac{1}{2}\varepsilon\Omega\,.\,\tfrac{10}{12}\,.\,\tfrac{18}{10}(\tilde{y}'(\xi_1) - 5\varepsilon) \leqq -\varepsilon\Omega\,\tfrac{17}{24}(\tilde{y}'(\xi_1) - 5\varepsilon).$$

Therefore

$$y'(\xi_1, \tilde{\tau} + \varepsilon) \leqq \tilde{y}'(\xi_1)\left[1 + \varepsilon(1 - \tfrac{1}{2}\Omega)\tfrac{13}{12} - \varepsilon\Omega\tfrac{17}{24}\right] + 5\varepsilon\left[1 + \varepsilon(1 - \tfrac{1}{2}\Omega)\tfrac{13}{12} + \varepsilon\Omega\tfrac{17}{24}\right] \leqq$$

$$\leqq \tilde{y}(\xi_1)\left[1 + \varepsilon(\tfrac{13}{12} - \Omega\tfrac{15}{12})\right] + 10\varepsilon \leqq \tilde{y}'(\xi_1)\left[1 - \varepsilon 10^{-2}\right] - 10^3\varepsilon 10^{-2} + 10\varepsilon \leqq$$

$$\leqq \tilde{y}'(\xi_1)\left[1 - \varepsilon 10^{-2}\right].$$

On the other hand (19,6) implies that $y'(\xi_1, \tilde{\tau} + \varepsilon) > 0$. As the cases that $\tilde{y}'(\xi_1) \leqq -10^3$ or $\tilde{y}(\xi_1) \leqq -\frac{1}{2} + \mu$ are similar, the assertion (iv) holds and Lemma 6,6 is proved completely.

Lemma 7,6. *Let (23,6) be fulfilled, let ε_0 be defined by Lemma 6,6, let $\tilde{y} \in \hat{M}_{1C}$ fulfil (24,6)–(26,6), and let there exist such numbers $\alpha, \beta, -\frac{1}{2} < \alpha < \beta < \frac{1}{2}$ that $\tilde{y}(\xi) \leqq -\mu$ for $\xi \in \langle -\frac{1}{2}, \alpha \rangle$, $\tilde{y}(\xi) \geqq \mu$ for $\xi \in \langle \beta, \frac{1}{2} \rangle$ and $(\mathrm{d}\tilde{y}/\mathrm{d}\xi)(\xi) \geqq \varkappa \geqq 10^3$ for $\xi \in \langle \alpha, \beta \rangle$. Let $y, y(\tau), y(\xi, \tau)$ have the usual meaning.*

Then there exist such numbers $\alpha^*, \beta^*, \alpha \leqq \alpha^* < \beta^* \leqq \beta$ that $y(\xi, \tilde{\tau} + \varepsilon) \leqq -\mu$ for $\xi \in \langle -\frac{1}{2}, \alpha^* \rangle$, $y(\xi, \tilde{\tau} + \varepsilon) \geqq \mu$ for $\xi \in \langle \beta^*, \frac{1}{2} \rangle$ and $(\partial y/\partial \xi)(\xi, \tilde{\tau}, + \varepsilon) \geqq \varkappa(1 + \varepsilon 10^{-2})$ for $\xi \in \langle \alpha^*, \beta^* \rangle$.

Lemma 7,6 is a consequence of assertions (i)–(iii) of Lemma 6,6.

For $y \in M$ put $\|y\|_{L_2} = (\int_0^2 y^2(\xi)\, \mathrm{d}\xi)^{\frac{1}{2}}$.

Lemma 8,6. *Let $\tilde{y}_1, \tilde{y}_2 \in \hat{M}$ fulfil (24,6), (25,6) and*

$$|\tilde{y}_i(\xi)| \geqq \tfrac{1}{2} - \mu\,, \quad i = 1, 2, \quad \operatorname{sgn} \tilde{y}_1(\xi) = \operatorname{sgn} \tilde{y}_2(\xi) \quad \textit{a.e. in } E_1 \tag{28,6}$$

($\varepsilon, \mu, \nu, \ldots$ having the same meaning as in Lemma 6,6). Let y_{i0} be the solutions of (8,6) in M, $y_{i0}(\tilde{\tau}) = \tilde{y}_i$, $i = 1, 2$.

Then y_{i0} exist on $\langle \tilde{\tau}, \infty)$, $\|y_{i0}(\tau)\| < 1$ for $\tau \in \langle \tilde{\tau}, \infty)$ and

$$\|y_{20}(\tau) - y_{10}(\tau)\|_{L_2} \leqq e^{-(\tau - \tilde{\tau}) 10^{-1}} \,.\, \|\tilde{y}_2 - \tilde{y}_1\|_{L_2}\,, \quad \tau \geqq \tilde{\tau}\,. \tag{29,6}$$

Proof. As f_0 does not depend on ε, assertions (i), (ii) of Lemma 6,6 together with equation (8,6) imply that y_{i0}, $i = 1, 2$ are defined on $\langle \tilde{\tau}, \infty)$, $\|y_{i0}(\tau)\| < 1$ on $\langle \tilde{\tau}, \infty)$ and that $y_{i0}(\tau)$, $i = 1, 2$ fulfil (24,6), (25,6) and (28,6) for $\tau \geqq \tilde{\tau}$. The functions $y_{i0}(\xi, \tau)$ are bounded, measurable in (ξ, τ) and

$$y_{i0}(\xi, \tau) = \tilde{y}_i(\xi) + \int_{\tilde{\tau}}^{\tau} f_0(y_{i0}(\sigma))(\xi)\, \mathrm{d}\sigma \quad \text{a.e. in} \quad E_1 \times \langle \tilde{\tau}, \infty)\,, \quad i = 1, 2\,.$$

Therefore

$$e^{(\tau - \tilde{\tau})/5}(y_{20}(\xi, \tau) - y_{10}(\xi, \tau))^2 \leqq (\tilde{y}_2(\xi) - \tilde{y}_1(\xi))^2 +$$
$$+ \tfrac{1}{5}\int_{\tilde{\tau}}^{\tau} e^{(\sigma - \tilde{\tau})/5}(y_{20}(\xi, \sigma) - y_{10}(\xi, \sigma))^2\, \mathrm{d}\sigma + 2\int_{\tilde{\tau}}^{\tau} e^{(\sigma - \tilde{\tau})/5}(y_{20}(\xi, \sigma) - y_{10}(\xi, \sigma))\,.$$
$$.\,[f_0(y_{20}(\sigma))(\xi) - f_0(y_{10}(\sigma))(\xi)]\, \mathrm{d}\sigma\,.$$

Hence, from the identity

$$(y_{20} - y_{10})\,[y_{20} - y_{10} - y_{20}^3 + y_{10}^3 - 3y_{20}Py_{20}^2 + 3y_{10}Py_{10}^2] =$$
$$= (y_{20} - y_{10})^2 - (y_{20} - y_{10})^2\,(y_{20}^2 + y_{20}y_{10} + y_{10}^2) -$$
$$- \tfrac{3}{2}(y_{20} - y_{10})^2\,(Py_{20}^2 + Py_{10}^2) - \tfrac{3}{2}(y_{20}^2 - y_{10}^2)\,(Py_{20}^2 - Py_{10}^2)$$

($Py_{i0}^3 = 0$ as $y_{i0}(\tau) \in \hat{M}$ for $\tau \geqq \tilde{\tau}$ according to Lemma 1,6) and from (28,6) with $\tilde{y}_i$ replaced by $y_i(., \tau)$ it follows that

$$(30,6) \qquad e^{(\tau-\tilde{\tau})/5} \|y_{20}(\tau) - y_{10}(\tau)\|_{L_2}^2 \leqq \|\tilde{y}_2 - \tilde{y}_1\|_{L_2}^2 +$$

$$+ \int_{\tilde{\tau}}^{\tau} e^{(\sigma-\tau)/5} \|y_{20}(\sigma) - y_{10}(\sigma)\|_{L_2}^2 \left[\tfrac{1}{5} + 1 - 3(\tfrac{1}{2} - \mu)^2 - \tfrac{3}{2} \cdot 2(\tfrac{1}{2} - \mu)^2\right] \mathrm{d}\sigma \leqq$$

$$\leqq \|\tilde{y}_2 - \tilde{y}_1\|_{L_2}^2 ,$$

as $\mu < 20^{-1}$ and (29,6) holds.

Let us prove Theorem 2,6. Let us choose the positive numbers $\varrho, \mu, \nu, \Omega$ so that (23,6) holds, let $\mu < \delta$ $\nu < \delta$ and let ε_0 be defined by Lemma 6,6 or smaller, if necessary; let $\mu + 3\varepsilon_0$, $\nu + 3\varepsilon_0 < \delta$. Let $\tilde{y} \in M_{1C}$ fulfil the assumptions of Lemma 7,6 ($\tilde{y}$ may be chosen analytic).

Let us put $y_i = y(i\varepsilon)$, $i = 0, 1, 2, \ldots$ y being the solution of (7,6) in M_{1C}, $y(0) = \tilde{y}$. It follows by induction from Lemma 6,6 that y_i fulfils (24,6)–(26,6), $i = 0, 1, 2, \ldots$ that the solution y is defined for $\tau \geqq 0$ and that $\|y(\tau)\| < 1$ for $\tau \geqq 0$. The assertion (i) of Theorem 2,6 holds.

Let us put $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_1 = \alpha^*$, $\beta_1 = \beta^*$. It follows from Lemma 7,6 that there exist sequences α_i, β_i, $\alpha_0 \leqq \alpha_1 \leqq \alpha_2 \leqq \ldots$, $\beta_0 \geqq \beta_1 \geqq \beta_2 \geqq \ldots$, $\alpha_i < \beta_i$, $i = 0, 1, 2, \ldots$ and that $y_i(\xi) \leqq -\mu$ for $\xi \in \langle -\frac{1}{2}, \alpha_i \rangle$, $y_i(\xi) \geqq \mu$ for $\xi \in \langle \beta_i, \frac{1}{2} \rangle$, $(\mathrm{d}y_i/\mathrm{d}\xi)(\xi) \geqq 10^3(1 + \varepsilon \cdot 10^{-2})$ for $\xi \in \langle \alpha_i, \beta_i \rangle$, $i = 0, 1, 2, \ldots$ Hence

$$(31,6) \qquad \beta_i - \alpha_i \leqq 10^{-3}(1 + \varepsilon \cdot 10^{-2})^{-i}, \quad i = 0, 1, 2, \ldots$$

The assertion (ii) of Lemma 6,6 implies that

$$(32,6) \qquad y_i(\xi) \leqq -\tfrac{1}{2} + \mu \quad \text{for} \quad \xi \in \langle -\tfrac{1}{2}, \alpha_{i-k} \rangle, \quad i \geqq k,$$

$$y_i(\xi) \geqq \tfrac{1}{2} - \mu \quad \text{for} \quad \xi \in \langle \beta_{i-k}, \tfrac{1}{2} \rangle, \quad i \geqq k,$$

$k \geqq 3(2\varepsilon\mu^2)^{-1}$, $k = k(\varepsilon)$ being the least integer.

Put $\vartheta = \lim_{i\to\infty} \alpha_i$ and define the function

$$\tilde{s}_i(\xi) = y_i(\xi) \quad \text{for} \quad \xi \in (-\tfrac{1}{2}, \alpha_{i-k}\rangle \cup \langle \beta_{i-k}, \tfrac{1}{2}),$$

$$\tilde{s}_i(\xi) = -\tfrac{1}{2} \quad \text{for} \quad \xi \in (\alpha_{i-k}, \vartheta),$$

$$\tilde{s}_i(\xi) = \tfrac{1}{2} \quad \text{for} \quad \xi \in (\vartheta, \beta_{i-k}), \quad i = k, k+1, \ldots$$

Let $s_i(s_{i0})$ be the solution of (10,6) ((8,6)) in M, $s_i(i\varepsilon) = \tilde{s}_i = s_{i,0}(i\varepsilon)$. As equation (8,6) is autonomous, Lemma 8,6 implies that

$$(33,6) \quad \|s_{i+l,0}((i + l)\varepsilon + \sigma) - s_{i,0}(i\varepsilon + \sigma)\|_{L_2} \leqq e^{-\sigma/10}\|\tilde{s}_{i+l} - \tilde{s}_i\|_{L_2} \quad \text{for} \quad \sigma \geqq 0 .$$

Theorem 8,1 may be applied to $s_{i+l}, s_i, s_{i+l,0}, s_{i,0}$. Let X be the space of such classes of equivalent measurable functions s that $s(\xi + 2) = s(\xi)$ a.e. in E_1, $\int_0^2 s^2(\xi)\,d\xi < \infty$, the norm being defined by $\|s\|_{L_2} = (\int_0^2 s^2(\xi)\,d\xi)^{\frac{1}{2}}$. Let G be the set of such $s \in X$ that $|s(\xi)| < 2$ a.e. in E_1. Let us verify all that assumptions of Theorem 8,1 are fulfilled. Lemma 8,6 implies that the solutions s_{i0} of (8,6) in M exist on $\langle i\varepsilon, \infty)$ and that $\|s_{i0}(i\varepsilon + \lambda)\| \leqq 1$ for $\lambda \geqq 0$, $i = k, k+1, \ldots$ It was shown in section 1 that every solution of (3,1) is simultaneously a solution of (52,1), F being defined by $F(x, \tau) = \int_0^\tau f(x, \sigma)\,d\sigma$. Therefore s_{i0} are solutions of

$$\frac{dy}{d\tau} = D_\tau F_0(y, \tau)\,, \tag{34,6}$$

F_0 being defined by

$$F_0(y, \tau)(\xi) = \tfrac{1}{2}\tau[y(\xi)(1 - y^2(\xi) - 3Py^2) + Py^3]\,. \tag{35,6}$$

Theorem 7,1 may be applied to equations (10,6) and (34,6) (in the space M, cf. Note 1,4); therefore solutions s_i of (10,6) exist on $\langle i\varepsilon, i\varepsilon + 1\rangle$ and $\|s_i(i\varepsilon + \lambda)\| < 2$ for $0 \leqq \lambda \leqq 1$, if $0 < \varepsilon \leqq \varepsilon_0$ and ε_0 is small enough, $i = k, k+1, \ldots$

It may be verified that there exists a $K_1 > 0$ that F (defined by (9,6)) and F_0 (defined by (35,6)) fulfil (51,1), if the norm $\|\ \|$ is replaced by the norm $\|\ \|_{L_2}$ in (51,1). As $\|y\|_{L_2} \leqq 2\|y\|$ for $y \in M$, it follows that the integrals $\int_{i\varepsilon}^{i\varepsilon+\lambda} D_\sigma F(s_i(\tau), \sigma)$, $\int_{i\varepsilon}^{i\varepsilon+\lambda} D_\sigma F_0(s_{i0}(\tau), \sigma)$ exist in X, $0 \leqq \lambda \leqq 1$ and s_i, s_{i0} are solutions of (10,6) and (34,6) in X. Obviously $s_i(\tau), s_{i0}(\tau) \in G$ for $i\varepsilon \leqq \tau \leqq i\varepsilon + 1$. It is also verified easily that (55,1) is fulfilled (ω may be chosen linear). In order that Theorem 8,1 might be applied it remains to verify (54,1) and (56,1) (in X). (54,1) is a consequence of Lemma 3,4 and inequality $\|y\|_{L_2} \leqq 2\|y\|$ for $y \in M$. Put $g(\lambda) = \frac{1}{2}\lambda(1 - \lambda^2)$, $y^* = y + z$, $(y, y + z \in G)$

$$H_1(\tau)(\xi) = \int_0^\tau [g(y(\xi) - y(-\xi + 2\lambda/\varepsilon)) - g(y^*(\xi) - y^*(-\xi + 2\lambda/\varepsilon))]\,d\lambda\,,$$

$$H_2(\tau)(\xi) = \tau\int_0^1 [g(y(\xi) - y(-\xi + 2\lambda/\varepsilon)) - g(y^*(\xi) - y^*(-\xi + 2\lambda/\varepsilon))]\,d\lambda\,.$$

In order to verify (56,1) we have to estimate (in X)

$$\Delta_\tau^\sigma \Delta_y^z[F(y, \tau) - F_0(y, \tau)] = H_1(\tau + \sigma) - H_2(\tau + \sigma) - H_1(\tau) + H_2(\tau)\,.$$

As $H_1(\tau + \varepsilon) - H_2(\tau + \varepsilon) - H_1(\tau) + H_2(\tau) = 0$, we may restrict ourselves to $0 < \sigma < \varepsilon$ and estimate $H_1(\tau + \sigma) - H_1(\tau)$ and $H_2(\tau + \sigma) - H_2(\sigma)$ separately.

$$\|H_1(\tau + \sigma) - H_1(\tau)\|_{L_2}^2 = \int_0^2 \int_\tau^{\tau+\sigma} [g(y(\xi) - y(-\xi + 2\lambda_1/\varepsilon)) - g(y^*(\xi) -$$

$$- y^*(-\xi + 2\lambda_1/\varepsilon))]\,d\lambda_1 \int_\tau^{\tau+\sigma} [g(y(\xi) - y(-\xi + 2\lambda_2/\varepsilon)) - g(y^*(\xi) -$$

$$- y^*(-\xi + 2\lambda_2/\varepsilon)] \, d\lambda_2 \, d\xi \leqq \int_\tau^{\tau+\sigma} \int_\tau^{\tau+\sigma} \int_0^2 [g(y(\xi) - y(-\xi + 2\lambda_1/\varepsilon)) -$$

$$- g(y^*(\xi) - y^*(-\xi + 2\lambda_1/\varepsilon))] \, [g(y(\xi) - y(-\xi + 2\lambda_2/\varepsilon)) - g(y^*(\xi) -$$

$$- y^*(-\xi + 2\lambda_2/\varepsilon))] \, d\xi \, d\lambda_1 \, d\lambda_2 \leqq \varepsilon^2 C \|z\|_{L_2}^2 .$$

$\|H_2(\tau + \sigma) - H_2(\tau)\|_{L_2}$ may be estimated in a similar way. Therefore all assumptions of Theorem 8,1 are satisfied. Let l be an integer, $\frac{1}{2} \leqq l\varepsilon \leqq 1$. As $F(y, \tau, \varepsilon)$ is periodic in τ with the period ε and as f_0 does not depend on τ, we deduce from (57,1) that

$$\begin{aligned} (36,6) \qquad & \|s_{i+l}((i + l)\,\varepsilon + \sigma) - s_i(i\varepsilon + \sigma)\|_{L_2} \leqq \\ & \leqq \|s_{i+l,0}((i + l)\,\varepsilon + \sigma) - s_{i,0}(i\varepsilon + \sigma)\|_{L_2} + \|\tilde{s}_{i+l} - \tilde{s}_i\| \, \chi_2(\zeta, 1) \leqq \\ & \leqq \gamma_1 \|\tilde{s}_{i+l} - \tilde{s}_i\| , \quad \gamma_1 = e^{-1/20} + \chi_2(\zeta, 1) , \quad \tfrac{1}{2} \leqq \sigma \leqq 1 . \end{aligned}$$

If ε_0 is sufficiently small, conditions (10,1) and (17,1) are fulfilled and we shall suppose that ζ is so small that $\gamma_1 < 1$.

Put $\sigma = l\varepsilon$ in (36,6). It follows that

$$\begin{aligned} (37,6) \qquad & \|s_{i+l}((i + 2l)\,\varepsilon) - s_i((i + l)\,\varepsilon)\|_{L_2} \leqq \gamma_1 \|\tilde{s}_{i+l} - \tilde{s}_i\|_{L_2} , \\ & \|y_{i+2l} - y_{i+l}\|_{L_2} \leqq \|y_{i+2l} - s_{i+l}((i + 2l)\varepsilon)\|_{L_2} + \\ & + \|s_{i+l}((i + 2l)\,\varepsilon) - s_i((i + l)\,\varepsilon)\|_{L_2} + \|s_i((i + l)\,\varepsilon) - y_{i+l}\|_{L_2} \leqq \\ & \leqq \|y_{i+2l} - s_{i+l}((i + 2l)\,\varepsilon)\|_{L_2} + \|y_{i+l} - s_i((i + l)\,\varepsilon)\|_{L_2} + \\ & + \gamma_1 \|y_{i+l} - \tilde{s}_{i+l}\|_{L_2} + \gamma_1 \|y_i - \tilde{s}_i\|_{L_2} + \gamma_1 \|y_{i+l} - y_i\|_{L_2} . \end{aligned}$$

(31,6) and (32,6) imply that

$$(38,6) \qquad \|y_i - \tilde{s}_i\|_{L_2} \leqq (2 \,.\, 10^{-3} \,.\, (1 + \varepsilon \,.\, 10^{-2})^{-i+k})^{\frac{1}{2}} , \quad i = k, k + 1, \ldots$$

y and s_i are solutions of (7,6) in X and $y(\tau)$, $s_i(\tau) \in G$ for $\tau \geqq i\varepsilon$, $i = 0, 1, \ldots$; therefore (f satisfies (2,1), $l\varepsilon \leqq 1$) there exists such a $K_4 > 0$ that

$$(39,6) \qquad \|y_{i+l} - s_i((i + l)\,\varepsilon)\|_{L_2} \leqq K_4 \|y_i - \tilde{s}_i\|_{L_2} , \quad i = 0, 1, 2, \ldots$$

(37,6), (38,6) and (39,6) imply that

$$(40,6) \qquad \|y_{i+2l} - y_{i+l}\|_{L_2} \leqq \gamma_1 \|y_{i+l} - y_i\|_{L_2} + K_5(1 + \varepsilon \,.\, 10^{-2})^{-i/2} , \quad i = k, k + 1, \ldots$$

It follows by induction from (40,6) that

$$\begin{aligned} & \|y_{i+(j+1)l} - y_{i+jl}\|_{L_2} \leqq \gamma_1^j \|y_{i+l} - y_i\|_{L_2} + \\ & + K_5[(1 + \varepsilon \,.\, 10^{-2})^{(-i-(j-1)l)/2} + \gamma_1(1 + \varepsilon \,.\, 10^{-2})^{(-i-(j-2)l)/2} + \ldots + \\ & + \gamma_1^{j-2}(1 + \varepsilon \,.\, 10^{-2})^{-i/2}] , \quad i = k, k + 1, \ldots , \quad j = 1, 2, \ldots \end{aligned}$$

As $l\varepsilon \geqq \frac{1}{2}$, there exist such numbers $0 < \gamma_2 < 1$ and $K_6 > 0$ that

$$\|y_{i+(j+1)l} - y_{i+jl}\|_{L_2} \leqq K_6 \gamma_2^j\,, \quad i = k, k+1, \ldots, \quad j = 1, 2, \ldots \tag{41,6}$$

Hence

$$\|y_{i+ml} - y_{i+nl}\|_{L_2} \leqq K_6(1 - \gamma_2)^{-1} \gamma_2^n\,, \quad i = k, k+1, \ldots, \quad m > n \geqq 1\,. \tag{42,6}$$

Put $w = \lim_{n \to \infty} y_{i+nl}$ in X. As (42,6) holds for every integer l, $\frac{1}{2} \leqq l\varepsilon \leqq 1$ and as $\varepsilon < (120)^{-1}$, w does not depend on i and

$$\|w - y_{i+nl}\|_{L_2} \leqq K_6(1 - \gamma_2)^{-1} \gamma_2^n\,, \quad i = k, k+1, \ldots, \quad n = 1, 2, \ldots$$

Therefore there exist such $K_7 > 0$ and $v_1 > 0$ that

$$\|w - y(i\varepsilon)\|_{L_2} \leqq K_7 e^{-\gamma_1 i\varepsilon}\,, \quad i = 0, 1, 2, \ldots \tag{43,6}$$

As $y(i\varepsilon + \tau)$, $y(j\varepsilon + \tau)$ are solutions of (7,6) in X, $y(i\varepsilon + \tau)$, $y(j\varepsilon + \tau) \in G$ for $\tau \geqq 0$, $i, j = 0, 1, 2, \ldots$, there exists a $K_8 > 0$ that

$$\|y(i\varepsilon + \sigma) - y(j\varepsilon + \sigma)\|_{L_2} \leqq e^{K_8 \sigma} \|y(i\varepsilon) - y(j\varepsilon)\|_{L_2}\,,$$
$$\sigma \geqq 0\,, \quad i, j = 0, 1, 2, \ldots$$

hence the limit $z(\tau) = \lim_{i \to \infty} y(i\varepsilon + \tau)$ exists, $z(\tau + \varepsilon) = z(\tau)$ and

$$\|z(\tau) - y(\tau)\|_{L_2} \leqq K_9 e^{-\gamma_1 \tau}\,, \quad \tau \geqq 0\,, \tag{44,6}$$

K_9 being a positive constant.

Let J be a closed interval, $J \subset (\vartheta, \vartheta + 1)$. It follows from the definition of ϑ and from the assertions (i), (ii) of Lemma 6,6 that there exists a $T > 0$ in such a way that

$$y_i(\xi) \geqq \tfrac{1}{2} - \mu \quad \text{for} \quad \xi \in J,\ i\varepsilon \geqq T\,. \tag{45,6}$$

y is a solution of (7,6) in M_{1C}; put $G = \mathscr{E}[y \in M_{1C};\ \|y\| < 1]$; as $y(\tau) \in G$ for $\tau \geqq 0$, there exist $K_{10} > 0$, $K_{11} > 0$ in such a way that $\|y(\tau)\|_1 \leqq K_{10} e^{K_{11}\tau}$, $\tau \geqq 0$; followingly

$$\left|\frac{\partial y}{\partial \xi}(\xi, \tau)\right| \leqq K_{10} e^{K_{11}(T+1)} \quad \text{for} \quad \xi \in E_1,\ 0 \leqq \tau \leqq T + 1\,.$$

Assertion (iv) of Lemma 6,6 implies that there exists a $K_{12} > 0$ that

$$\left|\frac{\partial y}{\partial \xi}(\xi, i\varepsilon)\right| \leqq K_{12} \quad \text{for} \quad \xi \in J,\ i = 0, 1, 2, \ldots$$

and (19,6) and (20,6) imply that

$$\left|\frac{\partial y}{\partial \xi}(\xi, \tau)\right| \leqq K_{13} \quad \text{for} \quad \xi \in J,\ \tau \geqq 0\,, \tag{46,6}$$

K_{13} being a positive constant. It follows from (44,6) and (49,6) that

$$z(\xi, \tau) = \lim_{i \to \infty} y(\xi, \tau + i\varepsilon) \quad \text{for} \quad \xi \in J,\ \tau \in E_1\,,$$

$$|z(\xi, \tau) - y(\xi, \tau)| \leqq \overset{\circ}{K} e^{-\overset{\circ}{\gamma}\tau} \quad \text{for} \quad \xi \in J,\ \tau \in E_1\,,$$

$\overset{\circ}{K}$ being a positive constant, $\overset{\circ}{\nu} = \frac{1}{2}\nu_1$ and (14,6) is satisfied. As J is an arbitrary closed interval, $J \subset (\vartheta, \vartheta + 1)$ the limit $z(\xi, \tau) \lim_{i \to \infty} y(\xi, \tau + i\varepsilon)$ exists for $\xi \neq \vartheta$, $\tau \geqq 0$. The functions y_i fulfil (24,6); hence and from (45,6) it follows that

$$\tfrac{1}{2} - \mu \leqq y(\xi, i\varepsilon) \leqq \tfrac{1}{2} + \nu \quad \text{for} \quad \xi \in J,\ i\varepsilon \geqq T\,. \tag{47,6}$$

As y is the solution of (7,6) in M, $\|y(\tau)\| < 1$ for $\tau \geqq 0$ and $\|f(y, \tau, \varepsilon)\| \leqq 3$ for $y \in M$, $\|y\| \leqq 1$, $\tau \geqq 0$, it follows that $\|y(\tau_2) - y(\tau_1)\| \leqq 3|\tau_2 - \tau_1|$. Therefore (47,6) implies that $\frac{1}{2} - \mu - 3\varepsilon \leqq y(\xi, \tau) \leqq \frac{1}{2} + \nu + 3\varepsilon$ for $\xi \in J$, $\tau \geqq T$ and consequently

$$\tfrac{1}{2} - \delta \leqq z(\xi, \tau) \leqq \tfrac{1}{2} + \delta \quad \text{for} \quad \xi \in (\vartheta, \vartheta + 1),\ \tau \geqq 0\,, \tag{48,6}$$

as J is an arbitrary closed interval, $J \subset (\vartheta, \vartheta + 1)$ and $\mu + 3\varepsilon_0$, $\nu + 3\varepsilon_0 < \delta$. $z(.,\tau) \in \hat{M}$ as $y(.,\tau) \in \hat{M}$ and therefore (13,6) holds. Theorem 2,6 (and (14,6)) is proved completely.

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Резюме

ПОКАЗАТЕЛЬНО УСТОЙЧИВЫЕ ИНТЕГРАЛЬНЫЕ МНОГООБРАЗИЯ, ПРИНЦИП УСРЕДНЕНИЯ И НЕПРЕРЫВНАЯ ЗАВИСИМОСТЬ ОТ ПАРАМЕТРА

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Развивается теория показательно устойчивых инвариантных многообразий для токов-понятие тока является более общим, чем понятие динамической системы. Находятся условия для тока X, обеспечивающие существование и единственность показательно устойчивого инвариантного многообразия для всякого тока Y, достаточно близкого к X (Теорема 2,2).

Общя теория применяется для дифференциальных уравнений в пространствах Банаха, для дифференциальных уравнений с запаздываниями и для возмущенного волнового уравнения с одним пространственным переменным.

Для того, чтобы специализация общей теории для обыкновенных дифференциальных уравнений охватывала принцип усреднения, являются существенными некоторые теоремы (Теоремы 1,1 и 3,1) о непрерывной зависимости решений дифференциальных уравнений от параметра, содержащиеся в гл. 1. Эти теоремы обладают следующими основными чертами:

(i) условие, что разность правых частей дифференциальных уравнений является малой, заменено более слабым условием, что эта разность станет малой после интегрирования по независимой переменной.

(ii) пусть x и y – решения дифференциального уравнения, зависящего от параметра, выполняющие начальные условия $x(t_0) = \tilde{x}$, $y(t_0) = \tilde{y}$. Дается оценка для изменения $x(t) - y(t)$ в зависимости от параметра, и это изменение является малым по сравнению с $\|\tilde{x} - \tilde{y}\|$.

В главе 2 развивается общая теория инвариантных многообразий. В главе 3 доказывается – грубо говоря – существование и единственность показательно устойчивого инвариантного многообразия для всякого дифференциального уравнения с запаздываниями, которое является достаточно близким дифференциальному уравнению (без запаздываний), обладающему показательно устойчивым инвариантным многообразием и размерности инвариантных многообразий равны.

В главах 4, 5 и 6 общая теория применяется к краевой задаче для возмущенного волнового уравнения с одним пространственным переменным. Глава 4 содержит предварительные сведения, в главе 5 рассмотрены некоторые специальные случаи возмущений, и в главе 6 доказывается для одного частного возмущения существование гладкого решения, которое, оставаясь ограниченным для $t \to \infty$, стремится равномерно к периодической функции, обладающей разрывными производными первого порядка.