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NEW KINDS OF THEOREMS ON NON-NEGATIVE MATRICES

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In this paper we shall establish some results of a new kind concerning non-negative matrices. In particular we shall study what can be said about the sum $A + B$ and product AB of two non-negative matrices knowing some properties of the matrices A and B.

I have been led to these problems in a quite natural manner by the methods developed in the paper [7].

For convenience of the reader we recall some notions introduced in [7] so that the present paper becomes independent of a detailed knowledge of the paper [7].

Let $N = \{1, 2, \ldots, n\}$. Consider the set of all "$n \times n$ matrix units", i.e. the set of symbols $\{e_{ij} \mid i \in N, j \in N\}$ together with the zero 0 adjoined.

We define in $S_n = \{0\} \cup \{e_{ij} \mid i \in N, j \in N\}$ a multiplication by

$$e_{ij}e_{ml} = \begin{cases} 0 & \text{for } j \neq m, \\ e_{il} & \text{for } j = m, \end{cases}$$

where 0 has the usual properties of a multiplicative zero. The set $S = S_n$ with this multiplication is a 0-simple semigroup containing exactly n non-zero idempotents $E = \{e_{11}, e_{22}, \ldots, e_{nn}\}$.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support C_A of A we shall mean the subset of S containing 0 and all elements e_{ij} for which $a_{ij} > 0$.

For any non-negative $n \times n$ matrices A, B we clearly have $C_{A+B} = C_A \cup C_B$ and $C_{AB} = C_A \cdot C_B$. Hereby the multiplication of subsets of S has the obvious meaning used in the theory of semigroups. In particular the supports of the elements of the sequence

$$(1) \qquad A, A^2, A^3, \ldots$$

are the following subsets of S:

$$(2) \qquad C_A, C_A^2, C_A^3, \ldots$$

Clearly the sequence (2) contains only a finite number of different elements.

The following statements follow from the elements of the theory of finite semigroups.

Let $k = k(A)$ be the least positive integer such that $C_A^k = C_A^l$ for some $l > k$. Let further $l = k + d$, $d = d(A) \geqq 1$, be the least positive integer for which $C_A^k = C_A^{k+d}$ holds. The sequence (2) is then of the form

$$C_A, C_A^2, \ldots, C_A^{k-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid \ldots$$

The system of sets $\mathfrak{S}(A) = \{C_A, C_A^2, \ldots, C_A^{k+d-1}\}$ with respect to the multiplication of subsets of S forms a finite semigroup. For every $\alpha \geqq k$ and every $\beta \geqq 0$ we have $C_A^\alpha = C_A^{\alpha+\beta d}$. Further it is well known that

$$\mathfrak{G}(A) = \{C_A^k, C_A^{k+1}, \ldots, C_A^{k+d-1}\}$$

is a cyclic group of order d. Every element of the sequence (2) which occurs more than once is an element $\in \mathfrak{G}(A)$.

The unit element of the group $\mathfrak{G}(A)$ is C_A^ϱ with suitably chosen $\varrho = \varrho(A)$ satisfying $k \leqq \varrho \leqq k + d - 1$. Note that for any $k_0 \geqq k$ we may clearly write

$$\mathfrak{G}(A) = \{C_A^{k_0}, C_A^{k_0+1}, \ldots, C_A^{k_0+d-1}\}.$$

In this manner we have associated to any non-negative $n \times n$ matrix A three integers $k = k(A)$, $d = d(A)$, $\varrho = \varrho(A)$ which clearly depend only on the distribution of zeros and "non-zeros" in A.

For further purposes we also mention that for any $h > n$ we always have

$$C_A^h \subset C_A \cup C_A^2 \cup \ldots \cup C_A^n$$

so that $C_A \cup C_A^2 \cup \ldots \cup C_A^n$ is always a semigroup (subsemigroup of S generated by C_A).

1. PRELIMINARIES

In the paper [7] I have used the notions introduced above to a semigroup treatment of some theorems on non-negative matrices, in particular, theorems concerning the reducibility of the powers of an irreducible matrix.

Instead of referring to the classical notion of the irreducibility of a non-negative matrix I introduce the following definition (which is, of course, equivalent to the classical one; see [7], Theorem 1):

Definition. A non-negative $n \times n$ matrix A is called irreducible if and only if

$$C_A \cup C_A^2 \cup \ldots \cup C_A^n = S.$$

It turns out (see [7], Theorem 2) that A is irreducible if and only if

$$C_A^{k(A)} \cup C_A^{k(A)+1} \cup \ldots \cup C_A^{k(A)+d(A)-1} = S\,, \tag{3}$$

or – what is the same –

$$C_A^{\varrho(A)} \cup C_A^{\varrho(A)+1} \cup \ldots \cup C_A^{\varrho(A)+d(A)-1} = S\,.$$

For an irreducible matrix A the number $d(A)$ is simply the classical index of imprimitivity of A.[1]) In this case (the case of an irreducible matrix) the sets $C_A^{k(A)}, \ldots, \ldots, C_A^{k(A)+d(A)-1}$, are pairwise quasidisjoint, i.e. the intersection of any two of them is the zero element 0. (See [7], Theorem 4.) (It should be noted that for reducible matrices this statement need not hold. See [8].)

The result formulated in (3) may be reformulated in the following manner:

Lemma 1. *A non-negative matrix A is irreducible if and only if there exist two positive integers k_1, d_1 such that*

$$C_A^{k_1} \cup C_A^{k_1+1} \cup \ldots \cup C_A^{k_1+d_1-1} = S\,. \tag{4}$$

In this case we have $d_1 \geqq d(A)$.

Proof. a) If A is irreducible, then (4) is satisfied by putting $k_1 = k(A)$, $d_1 = d(A)$. The relation $d_1 \geqq d(A)$ follows from the fact that $C_A^{k(A)}, \ldots, C_A^{k(A)+d(A)-1}$ are quasidisjoint.

b) Suppose that (4) is satisfied. Then A contains in each row and each column at least one element different from zero. (For, if e.g. the entries of the first row were all zeros, the same would hold for $A^{k_1}, A^{k_1+1}, \ldots$ and the union $C_A^{k_1} \cup C_A^{k_1+1} \cup \ldots$ could not be S.)

If $k_1 \geqq k(A)$ the irreducibility follows from (3) and we necessarily have $d_1 \geqq d(A)$, since the summands are quasidisjoint.

If $k_1 < k(A)$, multiply (4) by $C_A^{k(A)-k_1}$ so that we have

$$C_A^{k(A)} \cup C_A^{k(A)+1} \cup \ldots \cup C_A^{k(A)+d_1-1} = S\,.\,C_A^{k(A)-k_1} = S.$$

The fact that A is irreducible (and $d_1 \geqq d(A)$) follows again from (3).

Lemma 2. *Suppose that A is irreducible. If for some $k_1 > 0$ we have*

$$C_A^{k_1} \cup C_A^{k_1+1} \cup \ldots \cup C_A^{k_1+d(A)-1} = S\,.$$

then $k_1 \geqq k(A)$.

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[1]) For an irreducible matrix the number $d(A)$ may be characterized also as the greatest common divisor of all positive integers α such that $E \cap C_A^{\alpha} \neq \emptyset$. (See [7], Theorem 7, or [11], p. 49. See also [3], where graph-theoretical methods are used.)

For an arbitrary non-negative matrix A I have proved in [8] that $d(A)$ is the greatest common divisor of all positive integers s such that C_A^s is itself a semigroup (subsemigroup of S).

Proof. Multiply this relation by $C_A^{\varrho(A)}$. We then have

$$C_A^{k_1+\varrho(A)} \cup \ldots \cup C_A^{k_1+\varrho(A)+d(A)-1} = S .$$

On the other hand, since $E \subset C_A^{\varrho(A)}$, (See [7], Theorem 3) we have $C_A^{k_1} = C_A^{k_1} . E \subset \subset C_A^{k_1+\varrho(A)}$, and consequently

$$C_A^{k_1+u} \subset C_A^{k_1+\varrho(A)+u} \quad (\text{for } u = 0, 1, \ldots, d(A) - 1) . \tag{5}$$

Now

$$S = C_A^{k_1} \cup \ldots \cup C_A^{k_1+d(A)-1} \subset C_A^{\varrho(A)+k_1} \cup \ldots \cup C_A^{\varrho(A)+k_1+d(A)-1} = S .$$

Since the sets $C_A^{\varrho(A)+k_1}, \ldots, C_A^{\varrho(A)+k_1+d(A)-1}$ are quasidisjoint and (5) holds, we necessarily have $C_A^{k_1} = C_A^{\varrho(A)+k_1}$. Hence $C_A^{k_1}$ appears in the sequence (2) more than once. Since $k(A)$ is the least exponent with this property, we have $k_1 \geqq k(A)$, which proves our assertion.

We shall need also the following

Definition. An irreducible matrix A is called primitive if there is a positive integer w such that $C_A^w = S$.

This is the case if and only if $d(A) = 1$. In this case the sequence (2) is of the form

$$C_A, C_A^2, \ldots, C_A^k = C_A^{k+1} = \ldots ,$$

where $C_A^k = C_A^\varrho = S$. The number $k = k(A)$ is then called the exponent of the primitive matrix A. [2])

The method sketched above leads in a quite natural way to the following question: Knowing the numbers $k(A)$, $d(A)$, $k(B)$, $d(B)$, what can be said about the numbers $k(A + B)$, $k(AB)$, $d(A + B)$, $d(AB)$? The purpose of this paper is to give a more or less complete answer to this question.

We close this introduction with the following

Lemma 3. *Suppose that A is non-negative and A^2 is irreducible. Then*

a) $d(A) = d(A^2)$;

b) $k(A^2) = \begin{cases} \frac{1}{2} k(A) & \textit{if } k(A) \textit{ is even}, \\ \frac{1}{2}[k(A) + 1] & \textit{if } k(A) \textit{ is odd}. \end{cases}$

Proof. Since A^2 is irreducible, so is A. Consider the sequence

$$C_A^2, C_A^4, C_A^6, \ldots$$

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[2]) It should be noted that in the last years a series of papers was concerned with the problem to find estimations for the exponent $k(A)$ of a primitive matrix. (See, e.g. [1], [3], [7], [11], and the bibliography therein.) The corresponding problem for an arbitrary non-negative matrix, i.e. estimations for the numbers $k(A)$, $d(A)$ and $\varrho(A)$ are treated in [5], [6], [9], [10].

The least power in this sequence which occurs in $\mathfrak{G}(A)$ is C_A^{2h}, where $2h$ is either $k(A)$ or $k(A)+1$ according as $k(A)$ is even or odd. Hence $k(A^2)$ is either $\frac{1}{2}k(A)$ or $\frac{1}{2}[k(A)+1]$.

Since $\mathfrak{G}(A^2)$ is clearly a subgroup of $\mathfrak{G}(A)$, we have $d(A^2) \leqq d(A)$. On the other hand by supposition

$$S = C_{A^2}^{k(A^2)} \cup C_{A^2}^{k(A^2)+1} \cup \ldots \cup C_{A^2}^{k(A^2)+d(A^2)-1} =$$
$$= C_A^{2k(A^2)} \cup C_A^{2k(A^2)+2} \cup \ldots \cup C_A^{2k(A^2)+2d(A^2)-2}.$$

Each summand on the right hand side is equal to one of the sets

$$C_A^{k(A)},\ C_A^{k(A)+1},\ \ldots,\ C_A^{k(A)+d(A)-1} \tag{6}$$

the union of which is equal to S. Since the sets (6) are quasidisjoint, we necessarily have $d(A^2) \geqq d(A)$. This proves our assertion.

2. THE SUM OF TWO NON-NEGATIVE MATRICES

All matrices considered below are non-negative $n \times n$ square matrices, $n > 1$.

If A is irreducible and B any non-negative matrix it follows immediately from the customary definition of irreducibility that $A + B$ is also irreducible. This is (among others) formally proved (on the basis of our definition of irreducibility) in the following theorem:

Theorem 1. *If A is irreducible and B any non-negative matrix, then $A + B$ is irreducible and $d(A + B) \leqq d(A)$. If $d(A + B) = d(A)$, then $k(A + B) \leqq k(A)$.*

Proof. We have

$$C_{A+B}^{k(A)} \cup C_{A+B}^{k(A)+1} \cup \ldots \cup C_{A+B}^{k(A)+d(A)-1} =$$
$$= (C_A \cup C_B)^{k(A)} \cup \ldots \cup (C_A \cup C_B)^{k(A)+d(A)-1} \supset C_A^{k(A)} \cup \ldots \cup C_A^{k(A)+d(A)-1} = S\,.$$

Hence

$$(C_{A+B})^{k(A)} \cup \ldots \cup (C_{A+B})^{k(A)+d(A)-1} = S. \tag{7}$$

By Lemma 1 $A + B$ is irreducible and $d(A + B) \leqq d(A)$.

In general we cannot use Lemma 2 and (7) to find an estimation for $k(A + B)$. For, we only know that

$$C_{A+B}^{k(A)} \cup \ldots \cup C_{A+B}^{k(A)+d(A+B)-1} \subseteqq$$
$$\subseteqq C_{A+B}^{k(A)} \cup \ldots \cup C_{A+B}^{k(A)+d(A+B)-1} \cup \ldots \cup C_{A+B}^{k(A)+d(A)-1} = S\,.$$

But if $d(A + B) = d(A)$, we have

$$C_{A+B}^{k(A)} \cup \ldots \cup C_{A+B}^{k(A)+d(A+B)-1} = S,$$

and now we can use Lemma 2 which implies $k(A) \geqq k(A + B)$. This proves Theorem 1.

The last case occurs in particular if $d(A) = 1$, for then we necessarily have $d(A + B) = 1$. This implies:

Theorem 1a. *If A is primitive and B any non-negative matrix, then $A + B$ is primitive and $k(A + B) \leqq k(A)$.*

We also have:

Corollary 1. *If A, B are both irreducible, then $d(A + B) \leqq \min(d(A), d(B))$.*

Remark. We have defined the number $d(A)$ for any non-negative matrix A. The relation $d(A + B) \leqq d(A)$ need not hold if A is not irreducible. Take, for instance,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $d(A) = 1$, $d(B) = 1$, while $d(A + B) = 3$.

Also Theorem 1a need not hold if A is merely irreducible. Take, for instance,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $k(A) = 1$, $k(A + B) = 5$.

The next theorem is very close to a result of HERSTEIN [4] which forms a useful lemma for general considerations. (See [2], p. 322.)

Theorem 2. *A non-negative matrix A is irreducible if and only if $A + A^2$ is primitive.*

Proof. a) Suppose that A is irreducible. We always have

$$(C_{A+A^2})^{n-1} = (C_A \cup C_A^2)^{n-1} = C_A^{n-1} \cup C_A^n \cup \ldots \cup C_A^{2n-2}.$$

If $n = 2$, we have $C_A \cup C_A^2 = S$, so that $A + A^2$ is positive. If $n > 2$, the union on the right hand side may be written in the form $C_A^{n-2}(C_A \cup C_A^2 \cup \ldots \cup C_A^n)$, which is equal to $C_A^{n-2} \cdot S = S$. Hence $A + A^2$ is primitive (and $k(A + A^2) \leqq n - 1$).

b) Conversely: If $A + A^2$ is primitive, there is a positive integer w such that $(C_A \cup C_A^2)^w = S$. Hence $C_A^w \cup C_A^{w+1} \cup \ldots \cup C_A^{2w} = S$. By Lemma 1 A is irreducible.

Remark. By the same argument we may prove that a non-negative matrix A is irreducible if and only if $E + A$ is primitive.

The next theorem gives some informations concerning the relation between AB and $A + B$.

Theorem 3. *Let AB be irreducible. Then $A + B$ is irreducible and $d(A + B) \leqq d(AB)$. If $d(A + B) = d(AB)$, then $k(A + B) \leqq 2k(AB)$.*

Proof. Since AB is irreducible, it follows by Theorem 1 that $AB + (BA + A^2 + B^2) = (A + B)^2$ is irreducible. The more $A + B$ is irreducible and

$$S = C_{AB}^{k(AB)} \cup \ldots \cup C_{AB}^{k(AB)+d(AB)-1} \subset C_{(A+B)^2}^{k(AB)} \cup \ldots \cup C_{(A+B)^2}^{k(AB)+d(AB)-1} .$$

Hence

$$S = C_{(A+B)^2}^{k(AB)} \cup \ldots \cup C_{(A+B)^2}^{k(AB)+d(AB)-1} . \tag{8}$$

By Lemma 1 $d((A + B)^2) \leqq d(AB)$. By Lemma 3 $d((A + B)^2) = d(A + B)$, hence $d(A + B) \leqq d(AB)$.

If $d(A + B) = d((A + B)^2) = d(AB)$, then (8) implies (by Lemma 2) $k(AB) \geqq k((A + B)^2)$. By Lemma 3 we have $k(AB) \geqq \frac{1}{2} k(A + B)$. This proves our Theorem.

Corollary 3. *If AB is primitive, then $A + B$ is primitive and $k(A + B) \leqq 2k(AB)$.*

3. PRODUCTS OF NON-NEGATIVE MATRICES

We now turn to the study of the product of two non-negative matrices.

The product of two irreducible matrices need not be irreducible.[3])

Also (what is perhaps more surprising) the product of two primitive matrices need not be irreducible (and hence need not be primitive). To show this consider the following primitive matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Their product

$$AB = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is reducible (and not completely reducible).

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[3]) In this connection we mention that if A is irreducible but not primitive, then some power of A is (completely) reducible.

We shall see that this last case cannot occur if the matrices A, B commute.

If two matrices A, B commute (i.e. $AB = BA$), then we also have $C_A C_B = C_B C_A$. But conversely if $C_A C_B = C_B C_A$, we need not have $AB = BA$. Let, e.g., $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq BA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, while $C_A C_B = C_B C_A$.

Instead of supposing in the following that the matrices commute we shall make the weaker assumption that C_A and C_B commute.

Remark. The last example shows that the product of two reducible matrices A, B can be primitive (hence irreducible) even if C_A, C_B commute.

In the case that C_A, C_B commute we first have the following general result:

Theorem 4. *If C_A, C_B commute, then*

a) $d(AB) \leqq \dfrac{d(A) \,.\, d(B)}{(d(A), d(B))}$,

b) $k(AB) \leqq \max\,(k(A), k(B))$.

Proof. Denote $k_0 = \max\,(k(A), k(B))$, $d_0 = (d(A) \,.\, d(B))/(d(A), d(B))$. Then

$$(C_A C_B)^{k_0 + d_0} = C_A^{k_0 + d(A)(d_0/d(A))}\, C_B^{k_0 + d(B)(d_0/d(B))} = C_A^{k_0}\, C_B^{k_0} = (C_{AB})^{k_0}\,.$$

Hence $(C_{AB})^{k_0} \in \mathfrak{G}(AB)$. Therefore $k(AB) \leqq k_0$ and $d(AB)/d_0$. This proves our Theorem.

Remark. If $A = B$, we have $d(A^2) \leqq d(A)$ and even in this case simple examples show that the sign of equality need not hold.

Theorem 5. *If A is irreducible, B is primitive and C_A, C_B commute, then AB is primitive and $k(AB) \leqq k(B)$.*

Proof. By supposition there is an integer $k(B)$ such that $C_B^{k(B)} = S$. We have $(C_{AB})^{k(B)} = C_B^{k(B)} \,.\, C_A^{k(B)} = S \,.\, C_A^{k(B)} = (S \,.\, C_A)\, C_A^{k(B)-1} = S C_A^{k(B)-1} = \ldots = S C_A = S$, which proves our statement.

Theorem 5 immediately implies:

Theorem 6. *If A, B are primitive and C_A, C_B commute, then AB is primitive and $k(AB) \leqq \min\,(k(A), k(B))$.*

If there is no assumption concerning the irreducibility or primitivity of A and B we have the following

Theorem 7. *Suppose that C_A and C_B commute. Then AB is primitive if and only if $C_A^{k(A)} \,.\, C_B^{k(B)} = S$.*

Proof. a) Suppose that AB is primitive. Then A and B cannot have a zero row or a zero column so that $C_A S = S C_B = S$. By supposition there is an integer $w_0 \geqq 1$ such that for all $w \geqq w_0$ we have

$$(9) \qquad (C_A C_B)^w = C_A^w C_B^w = S .$$

For a sufficiently large w we have $C_A^w \in \mathfrak{G}(A)$, $C_B^w \in \mathfrak{G}(B)$. Since we also have $C_A^{k(A)} \in \in \mathfrak{G}(A)$, there exists an integer $\alpha \geqq 1$ such that $C_A^w C_A^\alpha = C_A^{k(A)}$. Analogously $C_B^w C_B^\beta = C_B^{k(B)}$ for a suitably chosen $\beta \geqq 1$. Multiplying (9) by C_A^α and C_B^β we have $C_A^w C_A^\alpha C_B^w C_B^\beta = = C_A^\alpha S C_B^\beta = S$, so that $C_A^{k(A)} C_B^{k(B)} = S$.

b) Suppose conversely that

$$(10) \qquad C_A^{k(A)} C_B^{k(B)} = S .$$

Then neither A nor B can contain a zero row or a zero column. If $k(A) = k(B)$, we have $(C_{AB})^{k(A)} = S$, hence AB is primitive. If (without loss of generality) $k(A) > k(B)$, multiply (10) by $C_B^{k(A)-k(B)}$. We have $C_A^{k(A)} . C_B^{k(B)} . C_B^{k(A)-k(B)} = S C_B^{k(A)-k(B)} = S$, i.e. $(C_{AB})^{k(A)} = S$. Hence again AB is primitive.

Remark. Theorem 7 does not necessarily hold if C_A, C_B do not commute. This can be shown on the example at the beginning of this section. Here $C_A^{k(A)} . C_B^{k(B)} = S . S = = S$, but AB is reducible (hence not primitive).

We now give some results concerning the case when C_A, C_B do not necessarily commute.

Theorem 8. *Suppose that A, B are irreducible. If AB is irreducible, then*

a) *BA is also irreducible,*

b) $d(BA) = d(AB)$,

c) $|k(AB) - k(BA)| \leqq 1$.

Proof. By supposition

$$C_{AB}^{k(AB)} \cup C_{AB}^{k(AB)+1} \cup \ldots \cup C_{AB}^{k(AB)+d(AB)-1} = S .$$

Multiply this relation by C_B to the left and by C_A to the right. Since C_A, C_B contain in each row and column at least one element different from zero, we have

$$(11) \qquad C_{BA}^{k(AB)+1} \cup C_{BA}^{k(AB)+2} \cup \ldots \cup C_{BA}^{k(AB)+d(AB)} = C_B S C_A = S .$$

Lemma 1 implies that BA is irreducible and $d(BA) \leqq d(AB)$.

Further it follows from Lemma 2 that $k(AB) + 1 \geqq k(BA)$. By symmetry we get $d(AB) \leqq d(BA)$ and $k(BA) + 1 \geqq k(AB)$. This proves Theorem 8.

Corollary 8. *Suppose that A, B are primitive. If AB is primitive, so is BA and $|k(AB) - k(BA)| \leqq 1$.*

Remark 1. In Theorem 8 the supposition that A, B are irreducible can be weakened by requiring only that A contains no zero column and B contains no zero row. (Since AB is supposed to be irreducible, A cannot contain a zero row and B cannot contain a zero column.) But such a supposition is necessary. To show this consider the matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is primitive while $BA = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

By the same argument as in the proof of Theorem 8 we can prove:

Theorem 9. *If A, B are non-negative matrices such that both AB and BA are irreducible, then $d(AB) = d(BA)$ and $|k(AB) - k(BA)| \leqq 1$.*

Rematk 2. In general it is not true that $k(AB) = k(BA)$. This is shown on the following example. Consider the primitive matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Both products

$$AB = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

are primitive matrices. Direct computation shows that $k(AB) = 4$ while $k(BA) = 3$.

Theorem 10. *If A, B are irreducible, then there is an integer $h \leqq \varrho(B)$ such that AB^h and B^hA are irreducible and $d(AB^h) = d(B^hA) \leqq d(A)$.*

Proof. Since B is irreducible there exists an integer h, $1 \leqq h \leqq \varrho(\beta)$, such that $E \subset C_B^h$. This implies $C_A = C_A E \subset C_A C_B^h$. Therefore $C_A^v \subset (C_{AB^h})^v$ for any integer $v \geqq 1$. Now

$$S = C_A^{k(A)} \cup \ldots \cup C_A^{k(A)+d(A)-1} \subset (C_{AB^h})^{k(A)} \cup \ldots \cup (C_{AB^h})^{k(A)+d(A)-1}.$$

This implies that AB^h is irreducible and by Lemma 1 $d(AB^h) \leqq d(A)$. Analogously: B^hA is irreducible and $d(AB^h) = d(B^hA)$ follows by Theorem 9.

Theorem 11. *If A is primitive and B irreducible, then there exists an integer h, $1 \leqq h \leqq \varrho(B)$ such that AB^h is primitive and $k(AB^h) \leqq k(A)$.*

Proof. Again there is an integer h, $1 \leqq h \leqq \varrho(B)$, such that $E \subset C_B^h$. Hence $C_A \subset C_A C_B^h$. Now we have $S = C_A^{k(A)} \subset (C_A C_B^h)^{k(A)}$, hence $(C_{AB^h})^{k(A)} = S$, which proves our assertion.

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Резюме

НОВЫЙ КЛАСС ТЕОРЕМ ОБ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦАХ

ШТЕФАН ШВАРЦ, (Štefan Schwarz), Братислава

Целью статьи является изучение некоторых свойств суммы $A + B$ и произведения AB двух квадратных неотрицательных матриц A, B; предпологается, что известны некоторые (в работе определенные) характеристики матриц A и B.