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A NEW APPROACH TO SOME PROBLEMS IN THE THEORY OF NON-NEGATIVE MATRICES

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In the paper [11] I developed a semigroup treatment of some theorems concerning non-negative matrices. The substance of this method is the following.

Denote $N = \{1, 2, \ldots, n\}$ and consider the set of all "$n \times n$ matrix units," i.e. the set of symbols $\{e_{ij} \mid i \in N, j \in N\}$ together with a zero 0 adjoined. Define in $S = \{0\} \cup \{e_{ij} \mid i \in N, j \in N\}$ a multiplication by

$$e_{ij}e_{ml} = \begin{cases} e_{il} & \text{for } j = m\,, \\ 0 & \text{for } j \neq m\,, \end{cases}$$

the zero element having the usual properties of a multiplicative zero. The set S with this multiplication is a 0-simple semigroup containing n non-zero idempotents $e_{11}, e_{22}, \ldots, e_{nn}$.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support C_A of A we shall mean the subset of S containing 0 and all e_{ij} for which $a_{ij} > 0$.

For any two non-negative $n \times n$ matrices A, B we have $C_{AB} = C_A C_B$, where the multiplication of subsets of S has the usual meaning used in the theory of semigroups.

Consider the sequence

$$A, A^2, A^3, \ldots$$

The sequence of the corresponding supports

$$(1) \qquad C_A, C_A^2, C_A^3, \ldots$$

has clearly only a finite number of different members.

Let $k = k(A)$ be the least positive integer such that $C_A^k = C_A^{l_1}$ for some $l_1 > k$. Let further $l = k + d$ $[d = d(A) \geqq 1]$ be the least positive integer for which $C_A^k = C_A^{k+d}$ holds. Then the sequence (1) is of the form

$$C_A, C_A^2, \ldots, C_A^{k-1} \,|\, C_A^k, \ldots, C_A^{k+d-1} \,|\, C_A^k, \ldots, C_A^{k+d-1} \,|\, \ldots$$

The system of sets $\{C_A, C_A^2, \ldots, C_A^{k+d-1}\}$ with respect to the multiplication of subsets of S forms a finite semigroup of order $k + d - 1$. It is well known from the elements of the theory of finite semigroups that $\mathfrak{G}_A = \{C_A^k, C_A^{k+1}, \ldots, C_A^{k+d-1}\}$ (with respect to the same multiplication) is a cyclic group of order d. We mention by the way (though it will not be used in this paper) that the unit element of the group $\mathfrak{G}_A$ is the set C_A^{ϱ}, where ϱ is the uniquely defined multiple τd satisfying $k \leqq \tau d = \varrho \leqq k + d - 1$.

In this manner we have associated to any non-negative matrix A three positive integers $k = k(A)$, $d = d(A)$, $\varrho = \varrho(A)$.

A non-negative $n \times n$ matrix A is called reducible if N can be decomposed in two non-empty disjoint subsets $N = I \cup J$, $I \cap J = \Phi$ such that $a_{ij} = 0$ for $i \in I$ and $j \in J$. Otherwise it is called irreducible.

In [11] we have shown: For an irreducible matrix A the number $d = d(A)$ is simply the index of imprimitivity of A and we always have $d \leqq n$. [For a characterization of $d(A)$ in the general case see [12].]

A matrix A is irreducible if and only if

$$C_A \cup C_A^2 \cup \ldots \cup C_A^n = S .$$

It turns out that this is the case if and only if

$$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S . \tag{2}$$

Note also that an irreducible matrix is primitive if and only if $d(A) = 1$.

In this paper we shall use a refinement of the argument used in [11] in order to find estimations for the number $k = k(A)$ for any irreducible matrix.

For a primitive matrix it is well known that $k(A) \leqq (n-1)^2 + 1$ and that this result is sharp. (See [1]–[4], [6], [7], [8], [10], [11], [15].)

An analogous question for irreducible (but not necessarily primitive) matrices has been recently treated in [5] and in some special cases in [10].

The refinement of our argument consists in the fact that instead of studying the global behaviour of the sequence (1) we shall first study the behaviour of a fixed "row" in the sequence (1).

To this end we introduce the following notations: We denote $\{e_{i1}, e_{i2}, \ldots, e_{in}\} \cup \cup \{0\} = S_i$, so that $S_1 \cup S_2 \cup \ldots \cup S_n = S$. If A is a given $n \times n$ matrix, we further denote $F_i = F_i(A) = S_i \cap C_A$. Hence $F_i = F_i(A)$ is the "support of the i-th row of A". For further purposes note that $F_i = e_{ii}C_A$.

For brevity we shall occasionally say that F_i is "the i-th row of C_A" by considering hereby in a natural manner the set C_A (subset of S) written in the form of a square block with the non-zero entries e_{ij} on appropriate places. For instance for the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{pmatrix}$$

we shall write $C_A = \{0, e_{11}, e_{13}, e_{22}, e_{31}, e_{32}, e_{33}\}$ in the form

$$C_A = \begin{pmatrix} e_{11}, 0, & e_{13} \\ 0, & e_{22}, 0 \\ e_{31}, e_{32}, e_{33} \end{pmatrix} \cup \{0\}\,.^1)$$

Here

$$F_1 = \{0, e_{11}, e_{13}\}\,, \quad F_2 = \{0, e_{22}\}\,, \quad F_3 = \{0, e_{31}, e_{32}, e_{33}\}\,.$$

Consider now the sequence

$$(3) \qquad F_i,\, F_iC_A,\, F_iC_A^2,\, \ldots$$

and define $F_iC_A^0 = F_i$. The members of this sequence are clearly the supports of the i-th rows in the sequence (1).

Again (3) contains only a finite numbers of different sets. Denote by $k_i = k_i(A)$ the least integer such that $F_iC_A^{k_i-1}$ occurs in (3) more then once. Let further $d_i = d_i(A)$ be the least integer $\geqq 1$ such that $F_iC_A^{k_i-1} = F_iC_A^{k_i-1+d_i}$. Then the sequence (3) is of the form

$$F_i,\, F_iC_A,\, \ldots,\, F_iC_A^{k_i-2} \mid F_iC_A^{k_i-1},\, \ldots,\, F_iC_A^{k_i-1+d_i-1} \mid F_iC_A^{k_i-1},\, \ldots$$

Clearly $k_i \leqq k$, $d_i \leqq d$ (for $i = 1, 2, \ldots, n$) so that, in particular, $\max_i k_i \leqq k$. Conversely, if $k^* = \max_i k_i$, then the term $F_iC_A^{k^*-1}$ (for any i) occurs in the sequence (3) more then once, hence $F_iC_A^{k^*-1} = F_iC_A^{k^*-1+d_i}$ (for any i). This implies that for any integer $\lambda_i \geqq 1$ we have $F_iC_A^{k^*-1} = F_iC_A^{k^*-1+\lambda_i d_i}$. Let d^* be the least common multiple of the numbers $d_1, d_2, \ldots, d_n$ and put $\lambda_i = d^*/d_i$. We then have $F_iC_A^{k^*-1} = F_iC_A^{k^*-1+d^*}$ and $\left(\bigcup_{i=1}^{n} F_i\right) C_A^{k^*-1} = \left(\bigcup_{i=1}^{n} F_i\right) C_A^{k^*-1+d^*}$, i.e. $C_A^{k^*} = C_A^{k^*+d^*}$. This shows that $C_A^{k^*}$ occurs in (1) more then once, so that $k \leqq k^*$. Hence $k = k^* = \max_i k_i$.

Remark 1. By the way: $C_A^{k^*} = C_A^{k^*+d^*}$ immediately implies that $d \leqq d^*$ and $d \mid d^*$. Since it is easy to see that $d_i \mid d$, we also have $d^* \mid d$, so that $d = d^*$. We shall not need this fact in the present paper.

Remark 2. If A is irreducible, then (2) implies that

$$F_iC_A^{k_i-1} \cup F_iC_A^{k_i} \cup \ldots \cup F_iC_A^{k_i+d_i-2} = S_i$$

for $i = 1, 2, \ldots, n$. In particular, if A is primitive, then $F_iC_A^{k_i-1} = S_i$.

Remark 3. It is easy to introduce in the sequence (3) a multiplication $\odot$ so that (3) becomes a cyclic semigroup. To this end it is sufficient to define $F_iC_A^{\alpha} \odot F_iC_A^{\beta} =$

[1]) The set $\{0\}$ can be omitted if A contains a zero entry.

$F_i C_A^{\alpha+\beta+1}$. Then the set $\{F_i C_A^{k_i-1}, \ldots, F_i C_A^{k_i+d_i-2}\}$ (with the same multiplication) is a cyclic group of order d_i.

1. THE GENERAL CASE

The goal of this section is to prove some theorems, which hold for any non-negative irreducible matrix. Some of the lemmas are of independent interest.

All matrices considered below are $n \times n$ matrices, $n > 1$.

We begin with the decisive

Lemma 1. *Suppose that A is irreducible and M any proper subset of S_i containing 0 and at least one non-zero element. Then MC_A contains at least one non-zero element $\in S_i$, which is not contained in M.*

Proof. Let $M = \{0, e_{i\alpha}, e_{i\beta}, \ldots, e_{i\nu}\}$, $\{\alpha, \beta, \ldots, \nu\} \subsetneqq N$. Suppose for an indirect proof that for all elements $e_{\varrho\sigma} \in C_A$ we have

$$\{e_{i\alpha}, e_{i\beta}, \ldots, e_{i\nu}\}\, e_{\varrho\sigma} \subset \{e_{i\alpha}, e_{i\beta}, \ldots, e_{i\nu}\} \cup \{0\}\,.$$

If $\varrho \in \{\alpha, \beta, \ldots, \nu\}$, we necessarily have $\sigma \in \{\alpha, \beta, \ldots, \nu\}$. In other words: If $\varrho \in \{\alpha, \beta, \ldots, \nu\}$ and $\sigma \in N - \{\alpha, \beta, \ldots, \nu\}$, we have $a_{\varrho\sigma} = 0$. This says that A is reducible, contrary to the assumption.

Lemma 2. *Suppose that A is irreducible.*

a) *If F_i contains $g \geqq 1$ non-zero elements $\in S_i$, we have*

$$F_i \cup F_i C_A \cup \ldots F_i C_A^{n-g} = S_i\,.$$

b) *In particular we always have*

$$F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-1} = S_i\,.$$

c) *If $i \neq j$ we always have*

$$e_{ij} \in F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-2}\,.$$

Proof. a) By Lemma 1 $F_i \cup F_i C_A$ contains at least $g + 1$ non-zero elements. Again by Lemma 1

$$(F_i \cup F_i C_A) \cup (F_i \cup F_i C_A)\, C_A = F_i \cup F_i C_A \cup F_i C_A^2$$

contains at least $g + 2$ non-zero elements. Repeating this argument we find that $F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-g}$ contains at least n non-zero elements $\in S_i$, i.e. the whole set S_i.

b) Follows from the fact that an irreducible matrix has in each row at least one element different from zero.

c) Since $e_{ii}C_A$ contains at least one non-zero element $\neq e_{ii}$, the set $e_{ii} \cup e_{ii}C_A$ contains at least two non-zero elements $\in S_i$. Analogously $(e_{ii} \cup e_{ii}C_A) \cup (e_{ii} \cup e_{ii}C_A) C_A = e_{ii} \cup e_{ii}C_A \cup e_{ii}C_A^2$ contains at least 3 non-zero elements, and so on. We finally have

$$e_{ii} \cup e_{ii}C_A \cup e_{ii}C_A^2 \cup \dots e_{ii}C_A^{n-1} = S_i \, .$$

Since $e_{ii}C_A = F_i$, the last equality can be written in the form

$$e_{ii} \cup F_i \cup F_iC_A \cup \dots \cup F_iC_A^{n-2} = S_i \, ,$$

from which our assertion immediately follows.

Lemma 3. *If A is irreducible, then there is an integer $h = h(i)$ such that $1 \leqq h \leqq n$ and $F_i \subset F_iC_A^h$. Here:*

a) *If $e_{ii} \in F_i$, we may choose $h = 1$.*

b) *If F_i contains g non-zero elements $\in S_i$, we may choose $h \leqq n - g + 1$.*

Proof. a) If $e_{ii} \in F_i$, then $F_i = e_{ii}C_A \subset F_iC_A$, and our statement is true with $h = 1$.

b) By Lemma 2b there is an integer u, $1 \leqq u \leqq n - g$ such that $e_{ii} \in F_iC_A^u$. Multiplying by C_A we get $F_i = e_{ii}C_A \subset F_iC_A^{u+1}$. Since $u + 1 \leqq n - g + 1$, our statement holds.

Remark. The example of the irreducible permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

shows that $F_i \subset F_iC_A^n$, but $F_i \not\subset F_iC_A^h$ for $h = 1, 2, \dots, n - 1$. Hence the estimation $h \leqq n$ in Lemma 3 is — in general — the best possible.

Theorem 1. *If A is irreducible, F_i contains g non-zero elements and $F_i \subset F_iC_A^h$, $h \geqq 1$, then $k_i \leqq (n - g) h + 1$.*

Proof. The supposition implies

$$(4) \qquad F_i \subset F_iC_A^h \subset F_iC_A^{2h} \subset \dots \subset F_iC_A^{(n-g)h} \subset F_iC_A^{(n-g+1)h} \subset \dots$$

Since F_i contains g non-zero elements $\in S_i$, the set $F_iC_A^h$ is either equal to F_i or contains at least $g + 1$ non-zero elements $\in S_i$. Further $F_iC_A^{2h}$ is again either equal to $F_iC_A^h$ or contains at least $g + 2$ non-zero elements $\in S_i$; and so on. The chain (4) cannot have more than $n - g + 1$ different members. There exists therefore a τ, $0 \leqq \tau \leqq n - g$, such that $F_i C_A^{\tau h} = F_i C_A^{(\tau+1)h}$. Hence $k_i - 1 \leqq \tau h \leqq (n - g) h$. This proves our Theorem.

Theorem 2. *If A is irreducible and F_i contains g non-zero elements $\in S_i$, we have $k_i \leqq (n - g)^2 + (n - g) + 1$.*

Proof. By Lemma 3b we have $h \leqq n - g + 1$, hence

$$k_i \leqq (n - g)(n - g + 1) + 1 = (n - g)^2 + (n - g) + 1 .$$

Remark. The results of Theorem 1 and Theorem 2 cannot be – in general – sharpened. To show this consider the matrix A with

$$C_A = \begin{Bmatrix} 0, & e_{12}, & 0 \\ 0, & 0, & e_{23} \\ e_{31}, & e_{32}, & 0 \end{Bmatrix}$$

and the third row $F_3 = \{0, e_{31}, e_{32}\}$. Here $n = 3$, $g = 2$. We have $F_3 C_A = \{0, e_{32}, e_{33}\}$, $F_3 C_A^2 = \{0, e_{31}, e_{32}, e_{33}\}$ so that $k_3 = 3$. On the other hand $(n - g)^2 + (n - g) + 1 = 3$.

With respect to the relation $k(A) = \max_i k_i$ we immediately get:

Corollary 1. *For any irreducible non-negative $n \times n$ matrix A we always have $k(A) \leqq n^2 - n + 1$.*

Proof. Since $g \geqq 1$, we have $k(A) \leqq (n - 1)^2 + (n - 1) + 1 = n^2 - n + 1$.

Corollary 2. *If A is irreducible and each row contains at least two non-zero elements, we have $k(A) \leqq n^2 - 3n + 3$.*

Proof. Follows from $k(A) = \max_i k_i \leqq (n - 2)^2 + (n - 2) + 1 = n^2 - 3n + 3$.

The result of Corollary 1 is not the best possible. It is intuitively clear that a possible sharpening of this estimation depends on the possibility to sharpen Theorem 1 for the rows containing a unique non-zero element.

Note first: If A is irreducible and F_i contains a unique non-zero element $\in S_i$ there cannot hold $F_i = \{0, e_{ii}\}$ since such a matrix is reducible. Therefore in the following Theorem 3 we may suppose $F_i = \{0, e_{ij}\}$ with $i \neq j$.

Theorem 3. *Suppose that A is irreducible and F_i contains exactly one non-zero element $\in S_i$. Let h_i be the least integer $\geqq 1$ such that $F_i \subset F_i C_A^{h_i}$.*

A) *If $h_i \leqq n - 1$, we have $k_i \leqq (n - 1) h_i + 1 \leqq (n - 1)^2 + 1$.*

B) *If $h_i = n$, we have $k_i \leqq n^2 - 3n + 4$.*

Proof. A) This follows from Theorem 1 by putting $g = 1$ and $h = n - 1$.

B) We first show that in this case $e_{ii} \in F_i C_A^{n-1}$ and $e_{ii} \notin F_i C_A^h$ with $h \leqq n - 2$.

By Lemma 2b we have $e_{ii} \in F_i C_A^h$ with $1 \leqq h \leqq n - 1$. If there were $h \leqq n - 2$, we would have $e_{ii} C_A \subset F_i C_A^{h+1}$, i.e. $F_i \subset F_i C_A^{h+1}$ with $h + 1 \leqq n - 1$, contrary to the assumption.

Next we show that for $t = 1, 2, \ldots, n$ the set $F_i C_A^t$ contains exactly one element $\in S_i$ which is not contained in the union $F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{t-1}$. (Hereby $F_i C_A^0 = F_i$.)

By the same argument as in the proof of Lemma 2 a it follows that $F_i \cup \ldots \cup F_i C_A^{t-1}$ contains at least t different non-zero elements $\in S_i$. Suppose for an indirect proof that $F_i C_A^t$ has at least two non-zero elements not contained in $F_i \cup \ldots \cup F_i C_A^{t-1}$. Then $F_i \cup \ldots \cup F_i C_A^t$ contains at least $t + 2$ non-zero elements $\in S_i$. By Lemma 1 $(F_i \cup \ldots \cup F_i C_A^t) \cup (F_i \cup \ldots \cup F_i C_A^t) C_A = F_i \cup \ldots \cup F_i C_A^{t+1}$ contains at least $t + 3$ non-zero elements, and repeating this process we obtain that $F_i \cup \ldots \cup F_i C_A^{n-2} = S_i$. Hence $e_{ii} \in F_i C_A^h$ with $h \leqq n - 2$, which has been shown impossible.

In particular: $F_i C_A$ contains exactly one element not contained in F_i. But since $F_i \not\subset F_i C_A$, we conclude that $F_i C_A$ contains exactly one non-zero element $\in S_i$.

Consider now the finite sequence $F_i, F_i C_A, \ldots, F_i C_A^{n-1}, F_i C_A^n$, and let l_0 be the least integer such that $F_i C_A^{l_0}$ contains more than one non-zero element $\in S_i$. We have just seen that $l_0 > 1$.

α) If $l_0 = n$, then each of the sets $F_i, \ldots, F_i C_A^{n-1}$, contains a unique element and since $e_{ii} \in F_i C_A^{n-1}$, we have $\{0, e_{ii}\} = F_i C_A^{n-1}$. Therefore $e_{ii} C_A = F_i C_A^n$, i.e. $F_i = F_i C_A^n$, so that $k_i = 1$.

β) Suppose next $l_0 \leqq n - 1$ and let $F_i = \{0, e_{i\alpha}\}$, $F_i C_A = \{0, e_{i\beta}\}, \ldots, F C_A^{l_0 - 1} = \{0, e_{i\lambda}\}$. Since $F_i C_A^{l_0}$ contains at least two non-zero elements $\in S_i$ and only one not contained in $\{e_{i\alpha}, e_{i\beta}, \ldots, e_{i\lambda}\}$, there is necessarily an index $\xi \in \{\alpha, \beta, \ldots, \lambda\}$ such that $e_{i\xi} \in F_i C_A^{l_0}$. Consequently: There is an integer τ, $1 \leqq \tau \leqq l_0$, such that

$$\{0, e_{i\xi}\} = F_i C_A^{l_0 - \tau} \subset F_i C_A^{l_0} . \tag{5}$$

Now τ cannot be l_0 since $F_i \subset F_i C_A^{l_0}$ with $l_0 \leqq n - 1$ contradicts our assumption. Therefore we have $1 \leqq \tau \leqq l_0 - 1$. The relation (5) implies

$$F_i C_A^{l_0 - \tau} \subset F_i C_A^{l_0} \subset F_i C_A^{l_0 + \tau} \subset \ldots \subset F_i C_A^{l_0 + (n-1)\tau} .$$

This chain of $n + 1$ sets cannot have all members different one from the other. There is therefore an integer u, $-1 \leqq u \leqq n - 2$, such that

$$F_i C_A^{l_0 + u\tau} = F_i C_A^{l_0 + (u+1)\tau} .$$

Hence

$$k_i - 1 \leqq l_0 + u\tau \leqq l_0 + u(l_0 - 1) \leqq n - 1 + (n - 2)(n - 2) = n^2 - 3n + 3 .$$

This proves Theorem 3.

Remark. The result $k_i \leqq n^2 - 3n + 4$ cannot be – in general – sharpened. To show this consider the matrix A with

$$C_A = \begin{Bmatrix} 0, & e_{12}, & 0 \\ 0, & 0, & e_{23} \\ e_{31}, & 0, & e_{32} \end{Bmatrix}.$$

We have

$$C_A^2 = \begin{Bmatrix} 0, & 0, & e_{13} \\ e_{21}, & 0, & e_{23} \\ e_{31}, & e_{32}, & e_{33} \end{Bmatrix}, \quad C_A^3 = \begin{Bmatrix} e_{11}, & 0, & e_{13} \\ e_{21}, & e_{22}, & e_{23} \\ e_{31}, & e_{32}, & e_{33} \end{Bmatrix}, \quad C_A^4 = \begin{Bmatrix} e_{11}, & e_{12}, & e_{13} \\ e_{21}, & e_{22}, & e_{23} \\ e_{31}, & e_{32}, & e_{33} \end{Bmatrix} \cup \{0\},$$

so that A is primitive (hence irreducible). Now

$$F_1 = \{0, e_{12}\}, \quad F_1C_A = \{0, e_{13}\}, \quad F_1C_A^2 = \{0, e_{11}, e_{13}\}, \quad F_1C_A^3 = \{0, e_{11}, e_{12}, e_{13}\}$$

so that indeed $F_1 \subset F_1C_A^3$ and $k_1 = 4$. On the other hand $n^2 - 3n + 4$ for $n = 3$ is equal to 4.

Theorems 2 and 3 allow the following conclusions. If $n \geqq 2$, we have for the rows with at least two non-zero elements

$$k_i \leqq (n-g)^2 + (n-g) + 1 \leqq (n-2)^2 + (n-2) + 1 = n^2 - 3n + 3.$$

For the rows with a unique non-zero element we have (with h_i defined above)

$$\begin{aligned} &\text{either} \quad k_i \leqq n^2 - 3n + 4 && \text{if} \quad h_i = n, \\ &\text{or} \quad k_i \leqq (n-1)h_i + 1 \leqq (n-1)^2 + 1 && \text{if} \quad h_i \leqq n - 1. \end{aligned}$$

Since (for $n \geqq 2$) we have

$$(n-1)(n-2) + 1 = (n-2)^2 + (n-2) + 1 = n^2 - 3n + 3 < n^2 - 3n + 4 \leqq$$
$$\leqq (n-1)^2 + 1,$$

we get with respect to $k(A) = \max_i k_i$:

Theorem 4. *For any non-negative irreducible matrix A we always have $k(A) \leqq \leqq (n-1)^2 + 1$.*

Theorem 5. *Let A be irreducible. Denote h_i the least positive integer for which $F_i \subset F_iC_A^{h_i}$. If for every row F_i containing a unique non-zero element we have $h_i \neq n - 1$ (i.e. either $h_i = n$ or $h_i \leqq n - 2$), then $k(A) \leqq n^2 - 3n + 4$.*

Remark 1. The result of Theorem 4 is the best possible for it is known that to every $n \geqq 2$ there is a primitive matrix A with $k(A) = (n-1)^2 + 1$. This property has the "Wielandt matrix", which is a matrix with $C_A = \{0, e_{12}, e_{23}, e_{34}, \ldots, \ldots, e_{n-1,n}, e_{n1}, e_{n2}\}$.

Remark 2. Also the result of Theorem 5 cannot be — in general — sharpened. This shows the example in the Remark after Theorem 3. Here $F_1 = \{0, e_{12}\}$ and $h_1 = 3$, $F_2 = \{0, e_{23}\}$ and $h_2 = 1$ so that the suppositions of Theorem 5 are satisfied. On the other hand $k(A) = 4 = n^2 - 3n + 4$.

2. THE CASE OF A PRIMITIVE MATRIX

We shall now apply our results to the case of a primitive matrix. For a primitive matrix A the set $F_i C_A^{k-1}$ is the whole set S_i.

Theorem 6. *If A is primitive, then $k(A) \leqq n - 1 + \min_i k_i$.*

Proof. Let $e_{i\alpha}$ be any element $\in S_i$. Take $j \neq i$ and write $e_{i\alpha} = e_{ij}e_{j\alpha}$. By Lemma 2 $e_{ij} \in F_i C_A^t$, where $t = t(i, j)$ satisfies $0 \leqq t \leqq n - 2$. By definition of the number k_j we have (for any α) $e_{j\alpha} \in S_j = F_j C_A^{k_j - 1}$. Hence

$$S_i = \{0, e_{i1}, e_{i2}, \ldots, e_{in}\} \subset F_i C_A^t F_j C_A^{k_j - 1} \subset F_i C_A^{t+k_j}.$$

Therefore $k_i - 1 \leqq t + k_j$, i.e. $k_i \leqq t + 1 + k_j$. (This is, of course, trivially true also for $i = j$.) Since j is arbitrary, we have $k_i \leqq (n - 2) + 1 + \min_j k_j = n - 1 + \min_j k_j$. Taking account of $k(A) = \max_i k_i$, we finally get $k(A) \leqq n - 1 + \min_j k_j$.

By the way we have also proved[2]):

Theorem 7. *For any primitive $n \times n$ matrix A we always have*

$$\max_i k_i - \min_i k_i \leqq n - 1.$$

Remark. The result of Theorem 6 is sharp in the following sense. In any primitive matrix there is at least one row, say j-th row, containing at least $g = 2$ non-zero elements. By Theorem 2 $k_j \leqq n^2 - 3n + 3$. Hence by Theorem 6 $k(A) \leqq (n - 1) + (n^2 - 3n + 3) = n^2 - 2n + 2$ and the "Wielandt matrix" attains this upper bound.

Also simple examples show that the result of Theorem 7 is the best possible.

The following result described in Theorem 8 is known. (See [1], [4], [11].)

Lemma 4. *If A is irreducible and $e_{jj} \in F_j$, then $k_j \leqq n - 1$.*

Remark. It is well known that in this case irreducibility implies primitivity.

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[2]) (Added in proofs, May 1966.) In a forthcomming paper ([16]) we shall show that Theorem 7 holds for any non-negative irreducible matrix A and we use it to obtain estimates for $k(A)$ in the case of imprimitive matrices.

Proof. By supposition $e_{jj} \in F_j$, hence $F_j = e_{jj}C_A \subset F_jC_A$. This implies $F_j \subset \subset F_jC_A \subset F_jC_A^2 \subset \ldots \subset F_jC_A^{n-2}$. By Lemma 2c we have for $j \neq \alpha$

$$e_{j\alpha} \in F_j \cup F_jC_A \cup \ldots \cup F_jC_A^{n-2} = F_jC_A^{n-2}, \quad \text{i.e.} \quad S_j = F_jC_A^{n-2}.$$

Hence there is a τ, $0 \leqq \tau \leqq n-2$, such that $F_jC_A^\tau = F_jC_A^{\tau+1}$. Therefore $k_j - 1 \leqq \tau$, i.e. $k_j \leqq \tau + 1 \leqq (n-2) + 1 = n - 1$.

Remark. The result of Lemma 4 is sharp, since e.g. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is primitive and direct computation shows that $k_2 = k_3 = 2(=n-1)$.

Under the suppositions of Lemma 4 we have $\min_i k_i \leqq n - 1$. This combined with Theorem 6 gives the following

Corollary. *If A is irreducible and contains a non-zero element in the main diagonal, then $k(A) \leqq 2n - 2$.*

In the proof of the next Theorem 8 we shall again use the inequality $k_i \leqq t(i, j) + 1 + k_j$ (proved in the proof of Theorem 6).

Theorem 8. *If A is primitive and contains $r \geqq 1$ non-zero elements in the main diagonal, we have $k(A) \leqq 2n - r - 1$.*

Proof. Suppose that $\{e_{j_1j_1}, e_{j_2j_2}, \ldots, e_{j_rj_r}\} \subset C_A$. Then $k_{j_1} \leqq n-1, \ldots, k_{j_r} \leqq \leqq n - 1$.

If $r = n$, then $k(A) = \max_j k_j \leqq n - 1$, and our statement holds,.

Suppose $r < n$ and choose an index $i \notin \{j_1, j_2, \ldots, j_r\}$. Since

$$e_{ii} \cup e_{ii}C_A \cup \ldots \cup e_{ii}C_A^{n-r} = e_{ii} \cup F_i \cup F_iC_A \cup \ldots \cup F_iC_A^{n-r-1}$$

contains at least $n - r + 1$ non-zero elements $\in S_i$ and $\{e_{ij_1}, e_{ij_2}, \ldots, e_{ij_r}\}$ contains exactly r elements, these sets intersect and there is a j, say j_1, such that $e_{ij_1} \in F_iC_A^t$ with $0 \leqq t(i, j_1) \leqq n - r - 1$. Now $k_i \leqq t(i, j_1) + 1 + k_{j_1}$ implies $k_i \leqq (n - r - 1) + + 1 + (n - 1) = 2n - r - 1$. Hence $k(A) = \max k_i \leqq 2n - r - 1$, q.e.d.

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Резюме

НОВЫЙ МЕТОД РЕШЕНИЯ НЕКОТОРЫХ ВОПРОСОВ ТЕОРИИ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦ

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава

Пусть A — квадратная неотрицательная матрица. Распределение нулевых и ненулевых элементов в последовательности $A, A^2, A^3, \ldots$, начиная с некоторой степени $k(A)$, периодически повторяется. Цель статьи — получить оценки для числа $k(A)$ в случае неразложимых матриц. При этом используется новый метод, являющийся уточнением метода, использованного автором в работе [11].