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NEW METHODS FOR SOLVING LINEAR FUNCTIONAL EQUATIONS WITH BOUNDED OPERATORS

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1. In this note some new methods concerning the solution of the functional equations with linear bounded operators in Hilbert spaces are given.

Let X be real or complex Hilbert space, $A : X \rightarrow X$ a linear continuous operator from X into X . We shall solve the equation

$$(1) \quad Ax = f$$

by the successive approximations

$$(2) \quad x_{n+1} = Pf + \beta_n(I - PA)x_n, \quad (n = 0, 1, 2, \dots),$$

where P is a linear continuous mapping of X into X having a bounded inverse P^{-1} . The parameters β_n ($n = 0, 1, 2, \dots$) are to be determined either from

$$(3) \quad \|f - \beta_n Ax_n\|^2 = \text{Min}$$

or from the conditions

$$(4) \quad \|f - Ax_{n+1}\|^2 = \text{Min}, \quad (n = 0, 1, 2, \dots).$$

From (3), (4) we obtain that

$$(5) \quad \beta_n = \frac{\text{Re}(f, Ax_n)}{\|Ax_n\|^2},$$

$$(6) \quad \tilde{\beta}_n = \frac{\text{Re}(Lf, LAx_n)}{\|LAx_n\|^2},$$

where $L = I - PA$. The following theorem is valid:

Theorem 1. ([1], [2].) Let A, P be linear bounded commutative operators in X such that P^{-1} exists and is bounded, and $\|I - PA\| = q < 1$. Then the equation (1) has a unique solution x^* in X . The sequences $\{x_n\}, \{\tilde{x}_n\}$ defined by (2), (5); (2), (6) converge in the norm topology of X to the solution x^* of (1) and their errors are bounded by

$$\|x^* - x_n\| \leq kq\|f - Ax_{n-1}\|, \quad \|x^* - \tilde{x}_n\| \leq kq\|f - A\tilde{x}_{n-1}\|,$$

where $k = \|A^{-1}\| \leq \|P\|/(1 - q)$.

Let us set $A = I - \lambda K$, where K is a linear bounded operator from X into X , λ is a complex parameter.

Theorem 2. Let one of the following conditions be fulfilled:

- 1) $P = I, \|\lambda K\| < 1$.
- 2) $P = \vartheta I, A$ is a self-adjoint operator in $X, mI \leq A \leq MI, 0 < m \leq M, \vartheta = 2/(M + m)$, where $m = \inf_{\|x\|=1} (Ax, x), M = \sup_{\|x\|=1} (Ax, x)$.
- 3) $P = \vartheta I, \operatorname{Re} (Ax, x) \geq m\|x\|^2$ for every $x \in X, (m > 0)$ and $0 < \vartheta < 2m/\|A\|^2$.
- 4) $P = \vartheta^2(I - \bar{\lambda}K^*)$, where $\bar{\lambda}$ is the complex conjugate number to λ, K^* is an adjoint operator to K, K is normal, $\|Ax\| \geq k\|x\|$ holds for every $x \in X, (k > 0)$ and $0 < \vartheta < k\sqrt{2}/(1 + \|\lambda K\|)^2$.
- 5) $P = I + J$, where J is a linear bounded operator in X , commutative with K and such that $\|\lambda G - J\| < 1/(1 + \|\lambda K\|)$, where G is the resolvent operator of K .

Then the equation (1) has a unique solution x^* in X . The sequences $\{x_n\}, \{\tilde{x}_n\}$ converge in the norm topology of X to x^* at least with speed of a geometric sequence.

Proof. 1) It follows immediately from Theorem 1.

2) Assuming that $P = \vartheta I, \vartheta = 2/(M + m)$, we shall prove $\|I - PA\| = q < 1$. The operator $I - PA$ is self-adjoint, its upper bound is $1 - \vartheta m$ lower one $1 - \vartheta M$. The number ϑ is to be determined from the condition that the norm $\|I - \vartheta A\|$ is to assume its minimum value, i.e. from the equality $1 - \vartheta m = -(1 - \vartheta M)$.

Then

$$\|I - \vartheta A\| = 1 - \vartheta m = \vartheta M - 1 = (M - m)/(M + m) < 1.$$

3) We have

$$\begin{aligned} \|I - \vartheta A\|^2 &= \sup_{\|x\|=1} (x - \vartheta Ax, x - \vartheta Ax) \leq \\ &\leq \sup_{\|x\|=1} (1 - 2\vartheta \operatorname{Re} (Ax, x) + \vartheta^2\|A\|^2) \leq 1 - 2m\vartheta + \vartheta^2\|A\|^2. \end{aligned}$$

Let us put $f(\vartheta) = 1 - 2m\vartheta + \vartheta^2\|A\|^2$. Then $f(0) = 1, f(2m/\|A\|^2) = 1$. The function $f(\vartheta)$ assumes for $\vartheta_0 = m/\|A\|^2$ the minimum value $f(\vartheta_0) = 1 - m^2/\|A\|^2$. It is evident that $f(\vartheta) < 1$ for every $\vartheta \in (0, 2m/\|A\|^2)$. Hence $\|I - \vartheta A\| < 1$ for every $\vartheta \in (0, 2m/\|A\|^2)$.

4) We have $\|Ax\| \geq k\|x\|$ for every $x \in X, (k > 0)$. Hence there exist the bounded operators $A^{-1}, (A^{-1})^*, (A^*)^{-1}$ and $(A^{-1})^* = (A^*)^{-1}$. Therefore P has the bounded inverse P^{-1} . Because K is normal, then A is normal and hence A is commutative with P . From the definition of the norm in X it follows that we have to prove

$$\sup_{\|x\|=1} (\vartheta^2\|A^*A\|^2 - 2\|Ax\|^2) < 0.$$

Because $\|A^*A\| = \|AA^*\| = \|A\|^2$, the above inequality will be satisfied if $\|A\|^4\vartheta^2 - 2k^2 < 0$. But this inequality is fulfilled, since

$$0 < \vartheta < k\sqrt{2}/(1 + \|\lambda K\|)^2 \leq k\sqrt{2}/\|A\|^2.$$

5) We shall prove that $q = \|I - (I + J)A\| < 1$. According to the definition of the resolvent operator G we conclude that

$$PA = [(I + \lambda G) - (\lambda G - J)](I - \lambda K) = I - (\lambda G - J)A.$$

Hence

$$\begin{aligned} q &= \|I - (I + J)A\| = \|(\lambda G - J)A\| \leq \|\lambda G - J\| \|A\| \leq \\ &\leq \|\lambda G - J\| (1 + \|\lambda K\|) < 1. \end{aligned}$$

To show that $P^{-1} = (I + J)^{-1}$ exists and is bounded we use the following lemma: Let T_0, T_1 be linear continuous mappings of X into X such that T_0^{-1} exists and $\|T_1\| < 1/\|T_0^{-1}\|$. Then $T = T_0 + T_1$ has an inverse T^{-1} and

$$\|T^{-1}\| \leq \|T_0^{-1}\| (1 - \|T_0^{-1}\| \|T_1\|)^{-1}, \quad (\|T_0^{-1}\| < +\infty).$$

That P^{-1} exists is now clear. It is sufficient to set $T_0 = I + \lambda G, T_1 = P - (I + \lambda G) = J - \lambda G$. This concludes the proof.

Remark. It is useful to estimate the number q . The theorems 1,2 are valid if the number q is replaced by q' such that $q \leq q' < 1$. We may set in the theorem 2: 1) $q' = \|\lambda K\|, 2) q' = (M - m)/(M + m), 3) \text{ if } \vartheta = m/\|A\|^2, \text{ then } q' = (1 - (m/\|A\|)^2)^{\frac{1}{2}}, 4) q' = (1 - 2\vartheta^2k^2 + \vartheta^4\|A\|^4)^{\frac{1}{2}}, 5) q' = \|\lambda G - J\| \|I - \lambda K\|.$

2. Under the assumptions of Theorem 1, let A be a self-adjoint operator in X . Let us denote by x^* the unique solution of (1) in X . We shall solve the equation (1) by an iterative process (2). The real parameters $\beta_n (n = 0, 1, 2, \dots)$ are to be determined now either from

$$(7) \quad \|x^* - \beta_n x_n\|^2 = \text{Min},$$

or from

$$(8) \quad \|x^* - x_{n+1}\|^2 = \text{Min}, \quad (n = 0, 1, 2, \dots).$$

First of all we shall determine β_n ($n = 0, 1, 2, \dots$) from the conditions (7). According to (2)

$$\begin{aligned} & \|x^* - \beta_n x_n\|^2 = \|x^* - x_{n+1} + Pf - \beta_n PAX_n\|^2 = \\ & = \|x^* - x_{n+1} + Pf\|^2 - 2\beta_n \text{Re}(x^* - x_{n+1} + Pf, PAX_n) + \beta_n^2 \|PAX_n\|^2. \end{aligned}$$

From (7) it follows that

$$\beta_n = \frac{\text{Re}(x^* - x_{n+1} + Pf, PAX_n)}{\|PAX_n\|^2}.$$

Hence from the above equality and according to (2) we have

$$\begin{aligned} \beta_n \|PAX_n\|^2 & = \text{Re}(x^* - Pf - \beta_n x_n + \beta_n PAX_n + Pf, PAX_n) = \\ & = \text{Re}(x^*, PAX_n) - \beta_n \text{Re}(x_n, PAX_n) + \beta_n \|PAX_n\|^2. \end{aligned}$$

Therefore

$$\beta_n = \frac{\text{Re}(x^*, PAX_n)}{\text{Re}(x_n, PAX_n)}.$$

The operators A, P are commutative and A is self-adjoint. Hence

$$(9) \quad \beta_n = \frac{\text{Re}(f, Px_n)}{\text{Re}(x_n, PAX_n)}.$$

Now we shall determinate β_n ($n = 0, 1, 2, \dots$) from the conditions (8). From (2) we obtain

$$\begin{aligned} \|x^* - x_{n+1}\|^2 & = \|L(x^* - \beta_n x_n)\|^2 = \|L(x^* - x_{n+1} + Pf - \beta_n PAX_n)\|^2 = \\ & = \|L(x^* - x_{n+1} + Pf) - \beta_n LPAX_n\|^2 = \\ & = \|L(x^* - x_{n+1} + Pf)\|^2 - 2\beta_n \text{Re}(L(x^* - x_{n+1} + Pf), LPAX_n) + \\ & \quad + \beta_n^2 \|LPAX_n\|^2, \end{aligned}$$

where $L = I - PA$. From (8) we get

$$\text{Re}(L(x^* - x_{n+1} + Pf), LPAX_n) = \beta_n \|LPAX_n\|^2.$$

From the above equality and from (2) we see that

$$(10) \quad \beta_n = \frac{\text{Re}(Lf, PLX_n)}{\text{Re}(Lx_n, PLAX_n)}.$$

The formulae (2), (9); (2), (10) give new methods for solving linear functional equations in Hilbert space X . The methods (2), (9); (2), (10) are more simple for

computation in comparison with (2), (5); (2), (6). In a real Hilbert space X formulae (9), (10) are as follows:

$$\beta_n = (f, P_n)(x_n, P_n A x_n)^{-1}, \quad \beta_n = (L f, P L x_n)(L x_n, P L A x_n)^{-1}.$$

Taking $P = \vartheta I$, where $\vartheta > 0$, then formulae (9), (10) have the simple form:

$$\beta_n = \frac{(f, x_n)}{(x_n, A x_n)}, \quad \beta_n = \frac{(L f, L x_n)}{(L x_n, L A x_n)}.$$

Theorem 3. *Under the assumptions of Theorem 1, let A be a self-adjoint operator in X . Then the equation (1) has a unique solution x^* in X . The sequences $\{x_n\}$, $\{\tilde{x}_n\}$ defined by (2), (9); (2), (10) are convergent in the norm topology of X to the solution x^* at least with the speed of a geometric sequence.*

Proof. Because

$$A^{-1} = \sum_{j=0}^{\infty} (I - PA)^j P,$$

then $\|A^{-1}\| \leq \|P\|/(1 - q)$. Hence the equation (1) has a unique solution x^* in X . We shall prove that $\|x^* - x_n\| \rightarrow 0$ whenever $n \rightarrow \infty$. According to (7) we conclude that $\|\beta_n x_n - x^*\| \leq \|x^* - x_n\|$ for every n ($n = 0, 1, 2, \dots$). Further

$$\begin{aligned} \|x^* - x_{n+1}\| &= \|P f + \beta_n x_n - \beta_n P A x_n - x^*\| = \\ &= \|P A x^* - x^* + \beta_n x_n - \beta_n P A x_n\| = \\ &= \|(I - PA)(x^* - \beta_n x_n)\| \leq q \|x^* - \beta_n x_n\| \leq q \|x^* - x_n\|. \end{aligned}$$

Now it is evident that $\|x^* - x_n\| \rightarrow 0$ whenever $n \rightarrow \infty$. We shall show that $\|x^* - \tilde{x}_n\| \rightarrow 0$ when $n \rightarrow \infty$. Since the following equalities

$$F(\tilde{x}_{n+1}) = \|x^* - \tilde{x}_{n+1}\|^2 = \|L(x^* - \beta_n \tilde{x}_n)\|^2$$

hold (β_n is defined by the equality (10)), the element \tilde{x}_{n+1} defined by (2), (10) minimizes the functional $F(x) = \|x^* - x\|^2$ on the set of all the elements of the form $P f + \alpha L \tilde{x}_n$, $\alpha \in \mathfrak{R}$ (where \mathfrak{R} is the set of all real numbers), then the element $\beta_n \tilde{x}_n$ gives the minimum value of $\tilde{F}(x) = \|L(x^* - x)\|^2$ on the set of all points $\alpha \tilde{x}_n$, where $\alpha \in \mathfrak{R}$. Hence

$$F(\tilde{x}_{n+1}) = \tilde{F}(\beta_n \tilde{x}_n) = \min_{\alpha \in \mathfrak{R}} \tilde{F}(\alpha \tilde{x}_n) \leq \tilde{F}(\tilde{x}_n).$$

Observe that

$$\tilde{F}(\tilde{x}_n) = \|L(x^* - \tilde{x}_n)\|^2 \leq q^2 \|x^* - \tilde{x}_n\|^2.$$

Then, clearly, $\|x^* - \tilde{x}_{n+1}\| \leq q \|x^* - \tilde{x}_n\|$. Thus $\|x^* - \tilde{x}_n\| \rightarrow 0$ whenever $n \rightarrow \infty$. The proof is complete.

The analogical theorem to theorem 2 is valid. Let us set again $A = I - \lambda K$, where K is a linear bounded operator from X into X , λ is a complex parameter generally.

Theorem 4. *Let one of the following conditions be fulfilled:*

- 1) $P = I$, $\|\lambda K\| < 1$, K is a self-adjoint operator.
- 2) $P = \vartheta I$, A is self-adjoint, $mI \leq A \leq MI$, $0 < m \leq M$, where $m = \inf_{\|x\|=1} (Ax, x)$, $M = \sup_{\|x\|=1} (Ax, x)$, $\vartheta = 2/(M + m)$.
- 3) $P = \vartheta I$, A is self-adjoint, $(Ax, x) \geq m\|x\|^2$ holds for every $x \in X$, ($m > 0$) and $0 < \vartheta < 2m/\|A\|^2$.
- 4) $P = \vartheta^2(I - \bar{\lambda}K^*)$, where $\bar{\lambda}$ is a complex conjugate number to λ , K^* is an adjoint operator to normal mapping K , $\|Ax\| \geq k\|x\|$ holds for every $x \in X$, ($k > 0$) and $0 < \vartheta < k\sqrt{2}/(1 + \|\lambda K\|)^2$.
- 5) $P = I + J$, where J is a linear bounded operator in X , commutative with K and such that $\|\lambda G - J\| < 1/(1 + \|\lambda K\|)$ holds, where G is the resolvent operator of the self-adjoint mapping K .

Then the equation (1) has a unique solution x^* in X . The sequences $\{x_n\}$, $\{\tilde{x}_n\}$ defined by (2), (9); (2), (10) respectively, converge in the norm topology of X to x^* at least with the speed of the geometric sequence.

Remark. If $x_n \rightarrow x^*$ (or $\tilde{x}_n \rightarrow x^*$) in the norm topology of X , then the sequences $\{\beta_n\}$ defined by (9), (10) respectively converge to one, i.e. $\lim_{n \rightarrow \infty} \beta_n = 1$. It follows immediately from the continuity of the inner product of X . Under the assumptions of theorem 3, $F(\beta_n x_n) \searrow 0$, $F(\tilde{x}_n \beta_n) \searrow 0$ (here β_n is defined by (10)). In fact

$$F(x_{n+1}) \leq q^2 F(\beta_n x_n) \leq q^2 F(x_n) < F(x_n),$$

$$F(\beta_n x_n) \leq q^2 F(\beta_{n-1} x_{n-1}) < F(\beta_{n-1} x_{n-1}), \quad (n = 0, 1, 2, \dots).$$

3. The purpose of this paragraph is to remove the condition that A be a self-adjoint operator in X . We shall solve again the equation (1) with linear bounded operator A . Let us suppose that A maps X into itself. The equation (1) is equivalent to the equation

$$(11) \quad B_1 x = f_1,$$

where $B_1 = A^*A$, $f_1 = A^*f$. But the equation (11) is equivalent to

$$(12) \quad x = (I - \vartheta B_1) x + \vartheta f_1,$$

where ϑ is an arbitrary positive number. Suppose that A^{-1} exists and is bounded ope-

rator in X . Let us denote by x^* the unique solution of (1). Hence x^* is the unique solution of (12) in X . We solve (12) by an iterative process

$$(13) \quad x_{n+1} = \vartheta f_1 + \alpha_n(I - \vartheta B_1) x_n,$$

where the parameters α_n ($n = 0, 1, 2, \dots$) are to be determined from the conditions that $\|x^* - \alpha_n x_n\|^2 = \text{Min}$. Since B_1 is self-adjoint, then according to (9)

$$(14) \quad \alpha_n = \frac{\text{Re}(f_1, x_n)}{(x_n, B_1 x_n)}.$$

Finally we have

$$(15) \quad \alpha_n = \frac{\text{Re}(f, Ax_n)}{\|Ax_n\|^2}.$$

It is easy to show now that the following theorem is valid.

Theorem 5. *Let A be a linear bounded operator which maps X into itself such that A^{-1} exists and is bounded in X . Furthermore let the inequality $0 < \vartheta < 1/\|A\|^2$ be fulfilled. Define the sequence $\{x_n\}$ as follows:*

$$x_{n+1} = \vartheta A^* f + \alpha_n(I - \vartheta A^* A) x_n, \\ \alpha_n = \text{Re}(f, Ax_n) / \|Ax_n\|^2, \quad (n = 0, 1, 2, \dots).$$

Then $\|x^* - x_n\| \rightarrow 0$ whenever $n \rightarrow \infty$, $\|x^* - x_n\| \leq q^n(1 - q)^{-1} \|x_1 - x_0\|$, where $q = \|I - \vartheta A^* A\|$. Moreover $\|x_{n+1} - x^*\| < \|x_n - x^*\|$ for every n ($n = 0, 1, 2, \dots$).

Proof. Since A has the bounded A^{-1} , then there exists a positive constant k such that $\|Ax\| \geq k\|x\|$ holds for every $x \in X$. Because $I - \vartheta A^* A$ is self-adjoint, then

$$\|I - \vartheta A^* A\| = \sup_{\|x\|=1} |(I - \vartheta A^* A)x, x| = \sup_{\|x\|=1} |\|x\|^2 - \vartheta \|Ax\|^2| = \\ = \sup_{\|x\|=1} (1 - \vartheta \|Ax\|^2),$$

since

$$1 - \vartheta \|Ax\|^2 \geq 1 - \vartheta \|A\|^2 \|x\|^2 = 1 - \vartheta \|A\|^2 > 0$$

for every $x \in X$ with $\|x\| = 1$. Further

$$q = \|I - \vartheta A^* A\| \leq \sup_{\|x\|=1} (1 - k^2 \vartheta \|x\|^2) = 1 - \vartheta k^2 < 1,$$

since $0 < \vartheta k^2 < 1$. Hence, we infer that conditions of theorem 3 are satisfied and the theorem is proved.

Remark. Let us solve the equation (1) again by the successive approximations

$$(16) \quad x_{n+1} = \vartheta f_1 + \gamma_n(I - \vartheta B_1) x_n, \quad (\vartheta > 0, P = \vartheta I),$$

where $f_1 = A^*f$, $B_1 = A^*A$. If the constants γ_n ($n = 0, 1, 2, \dots$) are to be determined from the conditions that $\|x^* - x_{n+1}\|^2 = \text{Min}$ ($n = 0, 1, 2, \dots$) then

$$(17) \quad \gamma_n = \frac{\text{Re}(L_1^2 f_1, x_n)}{\text{Re}(B_1 L_1 x_n, L_1 x_n)},$$

where $L_1 = I - \vartheta B_1$. Under the assumptions of theorem 5, the sequence $\{x_n\}$ defined by (16), (17) is convergent in the norm topology of X to x^* and $\|x^* - x_n\| = O(q^n)$, where $q = \|I - \vartheta A^*A\|$. But formulae (16), (17) are rather complicated for practical computation.

It was shown that the iterative processes (2), (9); (2), (10) are convergent at least with the rate of the geometric sequence with the quotient $q = \|I - PA\|$. Hence it is natural to choose the operator P such that the norm $\|I - PA\|$ assumes its minimum value.

Let $P = \vartheta I$, A is a self-adjoint operator in X , $mI \leq A \leq MI$, $m > 0$. The constant ϑ (Theorem 4) is chosen such that $\|I - \vartheta A\|$ catches its minimum value. The lower bound of $I - \vartheta A$ is $1 - \vartheta M$, the upper one is $1 - \vartheta m$. The spectrum of $I - \vartheta A$ lies on the segment $\langle 1 - \vartheta M, 1 - \vartheta m \rangle$. Suppose that B is a linear self-adjoint operator in X with the lower and the upper bounds b, N respectively. Then $\|B\| = \max(|b|, |N|)$. The norm $\|I - \vartheta A\|$ will be smallest, if $1 - \vartheta M = -(1 - \vartheta m)$, i.e. if $\vartheta = 2/(M + m)$. Then $\|I - \vartheta A\| = (M - m)/(M + m)$. Hence the processes (2), (9); (2), (10) are convergent (under the best choice of ϑ) at least with the speed of the geometric sequence with the quotient $q = (M - m)/(M + m)$. The constant ϑ satisfies in 3) of theorem 4 the inequality $0 < \vartheta < 2m/\|A\|^2$. The estimate of the speed of the convergence is best when $\vartheta = \vartheta_0 = m/\|A\|^2$. Then $\|I - \vartheta A\| \leq [1 - (m/\|A\|^2)^2]^{\frac{1}{2}}$. Taking in 4) of theorem 4 $\vartheta = \vartheta_1 = k\sqrt{2}/(2\|A\|^2)$, then $\|I - \vartheta A\| \leq [1 - 1/2(k/\|A\|)^4]^{\frac{1}{2}}$. The parameter ϑ of theorem 5 satisfies the inequality $0 < \vartheta < 1/\|A\|^2$. Then the norm $\|I - \vartheta A^*A\|$ will be smaller when ϑ is nearer to $1/\|A\|^2$. The conclusions of theorem 5 remain valid if ϑ satisfies $0 < \vartheta < 2k/\|A\|^4$, where $k > 0$ is the number from the inequality $\|Ax\| \geq k\|x\|^2$ (the existence of a bounded A^{-1}). The best choice of ϑ is $\vartheta = k/\|A\|^4$. Then $\|I - \vartheta A^*A\| \leq [1 - (k/\|A\|)^2]^{\frac{1}{2}}$.

Remark. Some results of this paper were published without proofs in [3], [4].

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Резюме

НОВЫЕ МЕТОДЫ РЕШЕНИЯ ЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ С ОГРАНИЧЕННЫМИ ОПЕРАТОРАМИ

ЙОСЕФ КОЛОМЫ (Josef Kolomý), Прага

Пусть дано уравнение (1) в действительном или комплексном гильбертовом пространстве X . Пусть A, P — линейные ограниченные перестановочные операторы в X , и P таков, что существует ограниченный P^{-1} в X и выполнено условие $\|I - PA\| = q < 1$. Пусть, далее, A — самосопряженный оператор в X . Обозначим через x^* единственное в X решение уравнения (1). Последовательные приближения вычисляются по формуле (2), где действительные коэффициенты β_n ($n = 0, 1, 2, \dots$) определим так, чтобы выполнялись условия (7), (8). Формулы (2), (9); (2), (10) определяют новые итерационные процессы, обеспечивающие на каждом шаге наибольшее уменьшение норм ошибок $\|x^* - \beta_n x_n\|$, $\|x^* - x_{n+1}\|$. Если выбрать P подходящим образом (напр., $P = \vartheta I$, где $\vartheta > 0$), то формулы (9), (10) более просты, чем (5), (6), соответственно.

Теорема. Пусть выполнены выше сформулированные предположения. Тогда последовательности $\{x_n\}$, $\{\tilde{x}_n\}$, определенные равенствами (2), (9); (2), (10) сходятся по норме X к единственному решению x^* уравнения (1) с быстротой геометрической прогрессии $\{q^n\}$.

В случае несамосопряженного оператора A имеет место следующая теорема.

Теорема. Пусть A — линейный ограниченный оператор в X такой, что существует ограниченный A^{-1} . Пусть выполнено неравенство $0 < \vartheta < 1/\|A\|^2$. Определим последовательность $\{x_n\}$ следующим образом:

$$x_{n+1} = \vartheta A^* f + \alpha_n (I - \vartheta A^* A) x_n, \quad \alpha_n = \operatorname{Re}(f, Ax_n) / \|Ax_n\|^2, \quad (n = 0, 1, 2, \dots).$$

Тогда $\|x^* - x_n\| \rightarrow 0$ ($n \rightarrow \infty$) и имеет место оценка $\|x^* - x_n\| \leq q^n (1 - q)^{-1} \cdot \|x_1 - x_0\|$, где $q = \|I - \vartheta A^* A\|$.