Czechoslovak Mathematical Journal

Ivo Vrkoč

On homogeneous linear differential equations with random perturbations

Czechoslovak Mathematical Journal, Vol. 16 (1966), No. 2, 199–230

Persistent URL: http://dml.cz/dmlcz/100724

Чехословацкий математический журнал, т. 16 (91) 1966, Прага

ON HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH RANDOM PERTURBATIONS

Ivo Vrkoč, Praha

(Received April 1, 1965)

Let there be given a sequence of differential equations

$$\dot{x} = -\lambda_n x + S_n(t, x, \omega_n), \tag{1}$$

a sequence of initial conditions $x_n(0, \omega_n)$, a sequence of numbers v_n, a sequence of intervals $\langle 0, T_n \rangle$ and a sequence of $(\Omega_n, \mathscr{F}_n, P_n)$. The S_n is a stochastic process defined for $[t, x, \omega_n] \in \langle 0, \infty) \times (-\infty, \infty) \times \Omega_n$, and describes random perturbations. Finally x is a scalar variable, λ_n are positive numbers. We shall investigate two different kinds of assumptions concerning S_n. Firstly, we may assume that the perturbations act only at discrete instants $t_n^{(k)}$, causing discontinuities of the solution; secondly, they may act on the entire intervals $\langle t^{(k)}, t^{(k+1)})$ being bounded there however a mixed case may also be considered. For the sake of simplicity we shall assume that these instants $t_n^{(k)}$ are regularly distributed, i.e. there exist numbers $d_n > 0$ such that $t_n^{(k)} = kd_n$ (k are nonnegative integers). The assumptions on S_n are given more precisely in § 2: in the discrete case in (16) to (18), and in the continuous case in (19) or (20) together with a further assumption. However, these assumptions do not ensure the uniqueness of the process S_n nor of the solutions x.

We shall deal with the expressions $\sup_{x_n(t,\omega_n)} P(\sup_{\tau\in\langle 0,T_n\rangle} x_n(\tau, \omega_n) \geqq v_n)$ and $\sup_{x_n(t,\omega_n)} P(\sup_{\tau\in\langle 0,T_n\rangle} |x_n(\tau, \omega_n)| \geqq v_n)$; here $x_n(t, \omega_n)$ is a solution of the n-th differential equation of (1) with the given initial condition $x_n(0, \omega_n)$. These expressions mean the maximal probability that the solutions $x_n(t, \omega_n)$ (not uniquely determined by its initial condition) exceed at least once the bounds $x = v_n$ in the interval $\langle 0, T_n \rangle$. If the assumptions (7) to (10) are satisfied, the limits of these expressions exist for $n \to \infty$ and it is possible to determine them in terms of solutions of a parabolic differential equation (see Theorem 1 or 2). The assumptions (7) to (10) imply that the density of the points $t_n^{(k)}$ on the considered intervals $\langle 0, T_n \rangle$ is sufficiently large for large n.

In the first part of this paper, in lemmas 1 to 11 as long as we look for the process $y_n(t, \omega_n)$, which gives the maximum at

$$\sup_{x_n(t,\omega_n)} P(\sup_{\tau\in\langle 0,T_n\rangle} x_n(\tau, \omega_n) \geqq v_n), \quad \sup_{x_n(t,\omega_n)} P(\sup_{\tau\in\langle 0,T_n\rangle} |x_n(\tau, \omega_n)| \geqq v_n)$$

we proceed similarly as in [1] and we do not use the subscript n. We shall not give proofs of Lemmas 1–9 because they are similar to those in [1]. Lemmas 10 and 11 will be proved because the difference in method is significant. The remaining portions that is the determination of the limit of the expression $P(\sup_{\tau\in\langle 0,T_n\rangle} y_n(\tau, \omega_n) \geqq v_n)$ is quite different from that in [1].

We shall employ the following notation: Let $x_1(\omega), \ldots, x_n(\omega)$ be random variables on a common space $(\Omega, \mathscr{F}, P)$ then conditional distributions [2] exist

$$F(\lambda_1 \mid \lambda_2, \ldots, \lambda_n) = P(x_1(\omega) \leqq \lambda_1 \mid x_2(\omega) = \lambda_2, \ldots, x_n(\omega) = \lambda_n).$$

We shall use the conditional expectation

$$E(\varphi(x_1)) \mid x_2 = \lambda_2, \ldots, x_n = \lambda_n) = \int \varphi(\lambda_1) \, \mathrm{d}_{\lambda_1} F(\lambda_1 \mid \lambda_2, \ldots, \lambda_n)$$

The problem is formulated and solved as in [1], first without using the concept of a differential equation, and only in a conclusion – see § 2 – the results are applied to the differential equation (1).

The conditions (2) to (4) are so chosen that solutions of (1) fulfil (2) to (4) if S fulfils the assumptions of § 2 (for example (16) to (18) or (19) or (20)).

Definition 1. Let numbers $d > 0$, $\delta > 0$, $K > 0$, $\lambda > 0$ and a distribution $F_0(\theta)$ be given. The class of processes $x(t, \omega)$ which are defined for all $t \geqq 0$ (but not necessarily on the same Ω) and which fulfil the following conditions $P(x(0, \omega) \leqq \theta) = F_0(\theta)$,

$$\begin{aligned} E[x(t^{(k+1)}, \omega) - \mu e^{-\lambda d} \mid x(t^{(k)}, \omega) = \mu, \\ x(t^{(k-1)}, \omega) = \alpha_{k-1}, \ldots, x(0, \omega) = \alpha_0] = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} E[|x(t^{(k+1)}, \omega) - \mu e^{-\lambda d}| \mid x(t^{(k)}, \omega) = \mu, \\ x(t^{(k-1)}, \omega) = \alpha_{k-1}, \ldots, x(0, \omega) = \alpha_0] \leqq \delta d, \end{aligned} \tag{3}$$

$$\begin{aligned} P[|x(t, \omega) - \mu e^{-\lambda(t-t^{(k)})}| > K(t - t^{(k)}) \mid x(t^{(k)}, \omega) = \mu, \\ x(t^{(k-1)}, \omega) = \alpha_{k-1}, \ldots, x(0, \omega) = \alpha_0] = 0 \end{aligned} \tag{4}$$

for all $t : t^{(k)} < t \leqq t^{(k+1)}$, $(t^{(k)} = kd)$ will be denoted by $X(F_0, \delta, K, d, \lambda)$ (or briefly X).

As mentioned above, we shall be concerned with estimating the expressions

$$P_1(\delta, K, v, d, t, \lambda) = \sup_{x(t,\omega)} \bar{P}(\sup_{\tau\in\langle 0,t\rangle} x(\tau, \omega) \geqq v) \tag{5}$$

$$P_2(\delta, K, v, d, t, \lambda) = \sup_{x(t,\omega)} \bar{P}(\sup_{\tau\in\langle 0,t\rangle} |x(\tau, \omega)| \geqq v) \tag{6}$$

where $x(t, \omega)$ is arbitrary process belonging to $X(F_0, \delta, K, d, \lambda)$ ($\bar{P}(Q) = \inf_A P(A)$, $A \supset Q$, A are mesurable).

The meaning of (2), (3), (4), and similarly of (16), (17), (18) or (19) or (20) of § 2, is as follows: (2) – random perturbantions do not cause any "systematic error", (3) – random perturbations are small in a certain sense, (3) – this condition has the same meaning as that in [1]. If condition (3) were not satisfied, a "systematic error" would arise and the results would be weaker. The following definition of bounded processes differs only slight by from that in [1].

Definition 2. Let positive numbers δ, λ, K, d, v and a process $x(t, \omega)$ be given. Assume that $x(t, \omega)$ is defined for all $t \geqq 0$. Define $t(\omega) = \min t^{(l)}(\omega)$ if $x(t^{(l)}(\omega), \omega) \geqq$ $\geqq (v - Kd)\, e^{\lambda d}$ (values of $t^{(l)}(\omega)$ are only $t^{(l)}$). Or if such $t^{(l)}(\omega)$ do not exist set $t(\omega) =$ $= \infty$. *The process $x(t, \omega)$ is bounded at v* if $x(t, \omega) = x(t(\omega), \omega)$ for $t \geqq t(\omega)$. The process $x_v(t, \omega)$ will be termed *the process bounded at v corresponding to $x(t, \omega)$* if $x_v(t, \omega) = x(t(\omega), \omega)$ for $t \geqq t(\omega)$, $x_v(t, \omega) = x(t, \omega)$ for $t \leqq t(\omega)$.

Remark 1. The inequality $(v_n - K_n d_n) \exp\{\lambda_n d_n\} < v_n$ must be fulfilled for large n according to (7) to (10), and thus we can assume that $(v - Kd)\, e^{\lambda d} < v$. By (3), (4) the inequality $\delta > K$ cannot be satisfied. The meaning of the expression $(v - Kd)\, e^{\lambda d}$ in Definition 2 is obvious from the last inequality of Lemma 2.

The following lemma differs slightly from [1, Lemma 1] but the method of the proof is similar.

Lemma 1. *Let $x(t, \omega) \in X(F_0, \delta, K, d, \lambda)$; then $x_v(t, \omega)$ satisfies (2) to (4) for $\mu < (v - Kd)\, e^{\lambda d}$.*

The change of the definition of a bounded process influences Lemma 2, too.

Lemma 2. *Let $x \in X(F_0, \delta, K, d, \lambda)$; then*

$$P(x_{ve^{-\lambda d}+Kd}(t, \omega) \geqq v) \leqq P^-(\sup_{0 \leqq \tau \leqq t} x(\tau, \omega) \geqq v) \leqq \bar{P}(\sup_{0 \leqq \tau \leqq t} x(\tau, \omega) \geqq v) \leqq$$
$$\leqq P(x_v(\bar{t}^{(k)}, \omega) \geqq (v - Kd)\, e^{\lambda d})$$

where $\bar{t}^{(k)}$ is the greatest $t^{(k)}$ such that $t^{(k)} \leqq t$ ($P^-(Q) = \sup_A P(A)$, $A \subset Q$, A are measurable).

The following three Lemmas remain unchanged.

Lemma 3. *The expression $P(x_v(t, \omega) \geqq (v - Kd)\, e^{\lambda d})$ is a non-increasing function of v for fixed t and for fixed $x(t, \omega) \in X(F_0, \delta, K, d, \lambda)$.*

Lemma 4. *The expression $P(x_v(t^{(k)}, \omega) \geqq (v - Kd)\, e^{\lambda d})$ is a non-decreasing function of k for fixed v and for fixed $x(t, \omega) \in X(F_0, \delta, K, d, \lambda)$.*

Let the numbers K, d, λ and the distribution F_0 be fixed. Denote $\zeta(\theta) =$ $= \sup P(x_v(t, \omega) \geqq (v - Kd)\, e^{\lambda d})$ for $x(t, \omega) \in X(F_0, \theta, K, d, \lambda)$.

Lemma 5. *The function $\zeta(\theta)$ is non-decreasing.*

As Lemma 1 holds in form weaker than in [1] we cannot use the principle of reflection, and we must introduce the following definition which slightly differs from that in [1].

Definition 3. $X^\nu(F_0, \delta, K, d, \lambda)$ is the class of processes with initial distribution F_0 [1]) which are bounded at ν, satisfy (2) to (4) for $\mu < (\nu - Kd)\, e^{\lambda d}$, and also

$$P(x(t) \in A \,|\, x(t^{(k)}) = \mu,\, x(t^{(k-1)}) = \alpha_{k-1}, \ldots, x(0) = \alpha_0) = \\ = P(x(t) \in A \,|\, x(t^{(k)}) = \mu)$$

for all $t \in \langle t^{(k)}, t^{(k+1)}\rangle$ and for all sets A which are B-mesurable on $(-\infty, \infty)$.

The following Lemma 6 is quite similar to that in [1], and now the Lemmas 7, 8 hold for the class X^ν.

Lemma 6. *Let $x(t, \omega) \in X(F_0, \delta, K, d, \lambda)$; then there exists a process $x^*(t, \varrho)$ such that $x^*(t, \varrho) \in X^\nu(F_0, \delta, K, d, \lambda)$ (x^* is not necesserily defined on the same Ω), and*

$$(6,1) \qquad P_\varrho(x^*(t^{(k+1)}) \in A \,|\, x^*(t^{(k)}) = \mu) = P_\omega(x(t^{(k+1)}) \in A \,|\, x(t^{(k)}) = \mu)$$

holds for all $t \in \langle t^{(k)}, t^{(k+1)}\rangle$ and for all sets A which are B-mesurable on $(-\infty, \infty)$.

Remark 2. If $x \in X$, we can construct the bounded process x_ν at ν corresponding to x. According to Lemma 6 we may construct the process x^* which corresponds to x_ν. (The condition $x \in X$ is necessery only for the existence of conditional distributions.) According to (6,1) $x^* \in X^\nu$ is satisfied, and for every nonnegative integer $P(x^*(t^{(k)}) \leqq \theta) = P(x(t^{(k)}) \leqq \theta)$.

Definition 4. Let a, b be nonnegative integers, $a \leqq b$. Denote

$$\varphi_a^b(\theta) = \sup_x E(\varphi(x(t^{(b)})) \,|\, x(t^{(a)}) = \theta)\,,$$

where x belongs to $X^\nu(F_0, \delta, K, d, \lambda)$. The expression $\varphi_a^b(\theta)$ also depends on ν, but this will not be emphasised because it is not important.

Lemma 7. *Let a, b be nonnegative integers, $a \leqq b$, and let $\varphi(\theta), \psi(\theta)$ be B-mesurable functions; if $\varphi(\theta) \leqq \psi(\theta)$ then $\varphi_a^b(\theta) \leqq \psi_a^b(\theta)$.*

Lemma 8. *Let a, b, c be nonnegative integers, $a \leqq b \leqq c$, and let $\varphi(\theta), \psi(\theta)$ be B-mesurable functions; if $\varphi_b^c(\theta) \leqq \psi(\theta)$ then $\varphi_a^c(\theta) \leqq \psi_a^b(\theta)$.*

The following Lemma 9 is a modification of that in [1].

---

[1]) i.e. $P(x(0) \leqq \Theta) = F_0(\Theta)$.

Lemma 9. *Let $\varphi(\theta)$ be a convex function; then each $\varphi_k^{k+1}(\theta)$ is convex for $\theta < (v - Kd)\, e^{\lambda d}$, and the relation*

$$(9,1) \qquad \varphi_k^{k+1}(\theta) = E(\varphi(x^*(t^{(k+1)}))\,|\, x^*(t^{(k)}) = \theta) \quad \text{for} \quad \theta < (v - Kd)\, e^{\lambda d}$$

is satisfied, where x^* is the process belonging to $X^v(F_0, \delta, K, d, \lambda)$ for which $F(\theta_{k+1}\,|\,\theta_k) = F(\theta_{k+1} - \theta_k e^{-\lambda d})$. The function $F(u)$ is defined by

$$(9,2) \quad F(u) = 0 \quad \text{for} \quad u < -Kd\,, \qquad F(u) = 1 \quad \text{for} \quad u \geqq Kd\,,$$
$$F(u) = \delta/(2K) \text{ for } -Kd \leqq u < 0\,, \qquad F(u) = 1 - \delta/(2K) \text{ for } 0 \leqq u < Kd\,.$$

The necessary modifications of Lemmas 10 and 11 are more essential and therefore they will be proved in detail.

Lemma 10. *Let $\varphi(\theta)$ satisfy: $0 \leqq \varphi(\theta) \leqq 1$, $\varphi(\theta) = 1$ for $\theta \geqq v - Kd$, and be $\varphi(\theta)$ convex in the region $\theta \leqq v - Kd$ (v is a positive number). Let $\varphi^*(\theta)$ be defined by $\varphi^*(\theta) = \varphi(\theta)$ for $\theta \notin (v - 3Kd - \varepsilon, v - Kd)$ and*

$$(10,1) \qquad \varphi^*(\theta) = 1 \quad \text{for} \quad \theta \geqq v - 3Kd\,.$$

$\varphi^*(\theta)$ is linear in $\langle v - 3Kd - \varepsilon, v - 3Kd\rangle$. The process y is bounded at $(v - 3Kd)\, e^{-\lambda d} + Kd$ corresponding to $z \in X(F_0, \delta, K, d, \lambda)$, $P(z(t^{(k)} + 0) = \mu e^{-\lambda d} + uKd\,|\, z(t^{(k)}) = \mu) = 1 - 2p + |u|\,(3p - 1)$ where $p = \min(\delta/K, 1)/2$, the parameter u may equal only $-1, 0, 1$, and $z(t)$ is a solution of $\dot{z} = -\lambda z$ for all other t.

Then the inequality

$$(10,2) \qquad E(\varphi(x(t^{(n)}))) \leqq E(\varphi^*(y(t^{(n)})))$$

holds for each $x \in X^v(F_0, \delta, K, d, \lambda)$ arbitrary $\varepsilon > 0$ and arbitrary nonnegative integer n.

Proof. According to the definition of z the equations

$$(10,3) \qquad \begin{aligned} P(z(t^{(k+1)}) &= \mu e^{-\lambda d} - Kd\,|\, z(t^{(k)}) = \mu) = p \\ P(z(t^{(k+1)}) &= \mu e^{-\lambda d} + Kd\,|\, z(t^{(k)}) = \mu) = p \\ P(z(t^{(k+1)}) &= \mu e^{-\lambda d}\,|\, z(t^{(k)}) = \mu) = 1 - 2p \\ P(z(0) \leqq \theta) &= P(x(0) \leqq \theta)\,. \end{aligned}$$

hold. Obviously

$$(10,4) \qquad 0 \leqq \int \varphi(\mu)\, \mathrm{d}_\mu F(\mu \mid \theta) \leqq 1\,.$$

Let $V(\theta) = a\theta + b$ and $x \in X(F_0, \delta, K, d, \lambda)$; then

$$(10,5) \qquad E(V(x(t^{(k+1)}))\,|\, x(t^{(k)}) = \theta) = a\theta e^{-\lambda d} + b\,.$$

Equation (10,5) also holds for the processes $x \in X^v$, for $\theta < (v - Kd)\,e^{\lambda d}$, whose distributions satisfy

$$\begin{aligned} (10,6) \qquad & F(\mu \mid \theta) = 0 && \text{for} \quad \mu < \theta e^{-\lambda d} - Kd \\ & F(\mu \mid \theta) = 1 && \text{for} \quad \mu \geqq \theta e^{-\lambda d} + Kd \\ & F(\mu \mid \theta) = \delta/(2K) && \text{for} \quad \theta e^{-\lambda d} - Kd \leqq \mu < \theta e^{-\lambda d} \\ & F(\mu \mid \theta) = 1 - \delta/(2K) && \text{for} \quad \theta e^{-\lambda d} \leqq \mu < \theta e^{-\lambda d} + Kd\,. \end{aligned}$$

Next we shall prove the following statement:

Let $\varphi(\theta)$ be convex for $\theta \leqq v - Kd$, linear in $\langle v - 3Kd, v - Kd \rangle$ and $\varphi(\theta) = 1$ for $\theta \geqq v - Kd$, then every process for which

$$\varphi_k^{k+1}(\theta) = E(\varphi(y(t^{(k+1)})) \mid y(t^{(k)}) = \theta)$$

is bounded at v. The distributions of these processes satisfy (10,6) for $\theta < (v - 2Kd)\,e^{\lambda d}$,

$$\begin{aligned} (10,7) \quad & F(\mu \mid \theta) = 0 \quad \text{for} \quad \mu < \theta e^{-\lambda d},\ \theta \in \langle (v - 2Kd)\,e^{\lambda d}, (v - Kd)\,e^{\lambda d} \rangle \\ & F(\mu \mid \theta) = 1 \quad \text{for} \quad \mu \geqq \theta e^{-\lambda d},\ \theta \in \langle (v - 2Kd)\,e^{\lambda d}, (v - Kd)\,e^{\lambda d} \rangle \end{aligned}$$

$\varphi_k^{k+1}(\theta)$ is convex for $\theta \leqq (v - Kd)\,e^{\lambda d}$.

Let $\chi(\theta) = \varphi(\theta)$ for $\theta \leqq v - Kd$, $\chi(\theta)$ linear for $\theta \geqq v - 3Kd$. Obviously $\varphi(\theta) \leqq \chi(\theta)$. By Lemma 7 $\varphi_k^{k+1}(\theta) \leqq \chi_k^{k+1}(\theta)$ and by Lemma 8 (since $\chi(\theta)$ is convex) $\chi_k^{k+1}(\theta) = \int \chi(\mu)\, \mathrm{d}F(\mu \mid \theta)$, where $F(\mu \mid \theta)$ is a conditional distribution fulfilling (10,6) for $\theta < (v - Kd)\,e^{\lambda d}$ of a process which is bounded at v. We have obtained that $\chi_k^{k+1}(\theta)$ is also convex in $\theta < (v - Kd)\,e^{\lambda d}$. By (10,5) the equation $\chi_k^{k+1}(\theta) = \int \chi(\mu)\, d\tilde{F}(\mu \mid \theta)$ holds, where $\tilde{F}(\mu \mid \theta)$ satisfies (10,6) or (10,7) respectively and moreover $\tilde{F}(\mu \mid \theta)$ is a conditional distribution of a process bounded at v. The process x which is defined by (10,7) and for which $x(t^{(k)}) = \theta < (v - Kd)\,e^{\lambda d}$ cannot exceed the bounds $x = v - Kd$ at $t^{(k+1)}$ and hence $\varphi_k^{k+1}(\theta) = \chi_k^{k+1}(\theta)$ for $\theta < (v - Kd)\,e^{\lambda d}$. (Note that $\varphi \neq \chi$ may hold for $\theta > v - Kd$.) As $\varphi(\theta) \leqq 1$, $\varphi(\theta) = 1$ for $\theta \geqq v - Kd$, the equation $\varphi_k^{k+1}(\theta) = \int \varphi(\mu)\, d\tilde{F}(\mu \mid \theta)$ also holds for $\theta \geqq (v - Kd)\,e^{\lambda d}$. Now start with the proof of Lemma 10. We define the function ${}^n\psi$ by ${}^n\psi(\theta) = \varphi(\theta)$ for $\theta \notin (v - 3Kd, v - Kd)$, ${}^n\psi(\theta)$ is linear in $\langle v - 3Kd, v - Kd \rangle$. Since $\varphi(\theta)$ is convex in $\theta \leqq v - Kd$, we have

$$(10,8) \qquad \varphi(\theta) \leqq {}^n\psi(\theta)$$

Since $\varphi^*(\theta) = 1$ for $\theta \geqq v - 3Kd$, we obtain

$$(10,9) \qquad {}^n\psi(\theta) \leqq \varphi^*(\theta)$$

By Lemma 7 we have

$$(10,10) \qquad \varphi_{n-1}^n(\theta) \leqq {}^n\psi_{n-1}^n(\theta)\,.$$

According to the statement proved above and (10,9),

$$(10,11) \qquad {}^n\psi^n_{n-1}(\theta) = \int {}^n\psi(\mu)\,\mathrm{d}\tilde{F}(\mu \mid \theta) \leqq \int \varphi^*(\mu)\,\mathrm{d}\tilde{F}(\mu \mid \theta),$$

where $\tilde{F}(\mu \mid \theta)$ is conditional distribution fulfilling (10,6) and (10,7) of a process bounded at v. Set $\varphi^*_{n-1}(\theta) = \int \varphi^*(\mu)\,\mathrm{d}_\mu \hat{F}(\mu \mid \theta)$, where $\hat{F}(\mu \mid \theta)$ is a conditional distribution fulfilling (10,6) of a process bounded at $(v - 3Kd)\,e^{\lambda d} + Kd$. According to (10,1) we have

$$(10,12) \qquad \varphi^*_{n-1}(\theta) = 1 \quad \text{for} \quad \theta \geqq v - 3Kd\,.$$

According to (10,4), (10,1) and (10,11) the inequality

$$(10,13) \qquad {}^n\psi^n_{n-1}(\theta) \leqq \varphi^*_{n-1}(\theta)$$

holds. By the statement proved above, ${}^n\psi^n_{n-1}(\theta)$ is convex for $\theta \leqq (v - Kd)\,e^{\lambda d}$, and by (10,4) ${}^n\psi^n_{n-1}(\theta) \leqq 1$. We can now define the function ${}^{n-1}\psi(\theta)$ thus:

$$\begin{aligned} {}^{n-1}\psi(\theta) &= {}^n\psi^n_{n-1}(\theta) \quad &&\text{for} \quad \theta \notin (v - 3Kd,\ (v - Kd)\,e^{\lambda d}),\\ {}^{n-1}\psi(\theta) &= 1 &&\text{for} \quad \theta \in \langle v - Kd,\ (v - Kd)\,e^{\lambda d}\rangle, \end{aligned}$$

${}^{n-1}\psi(\theta)$ is linear in $\langle v - 3Kd,\ v - Kd\rangle$. Obviously ${}^{n-1}\psi(\theta) \geqq {}^n\psi^n_{n-1}(\theta)$, and by (10,10) the inequality

$$(10,14) \qquad \varphi^n_{n-1}(\theta) \leqq {}^{n-1}\psi(\theta)$$

holds. According to (10,12), (10,13) and (10,4) we have ${}^{n-1}\psi(\theta) \leqq \varphi^*_{n-1}(\theta)$.

Now we proceed by induction. Let there exist functions ${}^k\psi$, φ^*_k such that

$$(10,15) \qquad \varphi^n_k(\theta) \leqq {}^k\psi(\theta),$$

$$(10,16) \qquad {}^k\psi(\theta) \leqq \varphi^*_k(\theta), \ \varphi^*_k(\theta) = 1 \quad \text{for} \quad \theta \geqq v - 3Kd\,,$$

$$(10,17) \qquad {}^k\psi(\theta) \text{ is convex for } \theta \leqq v - Kd\,,$$

$$(10,18) \qquad {}^k\psi(\theta) \text{ is linear in } \langle v - 3Kd,\ v - Kd\rangle\,,$$

$$(10,19) \qquad {}^k\psi(\theta) \leqq 1, \ {}^k\psi(\theta) = 1 \quad \text{for} \quad \theta \geqq v - Kd\,.$$

By Lemma 8 we have

$$(10,20) \qquad \varphi^k_{k-1}(\theta) \leqq {}^k\psi^k_{k-1}(\theta)\,.$$

By the statement, (10,17), (10,18) and (10,19), ${}^k\psi^k_{k-1}(\theta)$ is convex for $\theta \leqq (v - Kd)\,e^{\lambda d}$ and the inequality ${}^k\psi^k_{k-1}(\theta) = \int {}^k\psi(\mu)\,d\tilde{F}(\mu \mid \theta) \leqq \int \varphi^*_k(\mu)\,\mathrm{d}\tilde{F}(\mu \mid \theta)$ holds, where $\tilde{F}(\mu$ [illegible] s a conditional distribution fulfilling (10,6) and (10,7) of a process bounded at v. The last inequality holds according to (10,16). By (10,4) and (10,16) we obtain $\int \varphi_k^*(\mu)\,\mathrm{d}\tilde{F}(\mu \mid \theta) \leqq \int \varphi_k^*(\mu)\,\mathrm{d}\hat{F}(\mu \mid \theta) = \varphi_{k-1}^*(\mu)$ and

$$(10,21) \qquad {}^k\psi_{k-1}^k(\theta) \leqq \varphi_{k-1}^*(\theta)\,.$$

Set ${}^{k-1}\psi(\theta) = {}^k\psi_{k-1}^k(\theta)$ for $\theta \notin (v - 3Kd,\ (v - Kd)\,e^{\lambda d})$, ${}^{k-1}\psi(\theta) = 1$ for $\theta \in \langle v - Kd,\ (v - Kd)\,e^{\lambda d}\rangle$, ${}^{k-1}\psi(\theta)$ is linear in $\langle v - 3Kd,\ v - Kd\rangle$. Since ${}^k\psi_{k-1}^k(\theta)$ is convex for $\theta \leqq (v - Kd)\,e^{\lambda d}$, we obtain ${}^k\psi_{k-1}^k(\theta) \leqq {}^{k-1}\psi(\theta)$. By (10,20) the inequality (10,15) holds also for $k - 1$. By (10,21) and according to $\varphi_{k-1}^*(\theta) = 1$ for $\theta \geqq v - 3Kd$, the relations (10,16) hold also for $k - 1$. By the statement proved above, (10,17) is true. According to definition of the function ${}^{k-1}\psi(\theta)$ we obtain (10,18). According to (10,4) and the definition of $\tilde{F}(\mu \mid \theta)$ we conclude (10,19).

From (10,15) and (10,16) we have

$$(10,22) \qquad \varphi_k^n(\theta) \leqq \varphi_k^*(\theta) \quad \text{for every } k\,.$$

The function $\varphi_k^*(\theta)$ is obtained from $\varphi_{k+1}^*(\theta)$ by means of

$$(10,23) \qquad \varphi_k^*(\theta) = \int \varphi_{k+1}^*(\mu)\,\mathrm{d}\hat{F}(\mu \mid \theta) = E(\varphi_{k+1}^*(y(t^{(k+1)})) \mid y(t^{(k)}) = \theta)\,,$$

where y is the process described in Lemma 10. We shall prove that

$$(10,24) \qquad \varphi_k^*(\theta) = E(\varphi^*(y(t^{(n)})) \mid y(t^{(k)}) = \theta)\,.$$

Obviously the relation

$$(10,25) \qquad E(\varphi^*(y(t^{(n)})) \mid y(t^{(k-1)}) = \theta) = E[E(\varphi^*(y(t^{(n)})) \mid y(t^{(k)}),\ y(t^{(k-1)})) \mid y(t^{(k-1)}) = \theta] = E[E(\varphi^*(y(t^{(n)})) \mid y(t^{(k)})) \mid y(t^{(k-1)}) = \theta]$$

holds since $y \in X^v$. By (10,24) we obtain the relation

$$(10,26) \qquad E(\varphi^*(y(t^{(n)}, \omega)) \mid y(t^{(k)}, \omega)) = \varphi_k^*(y(t^{(k)}, \omega))$$

almost everywhere. If apply (10,26) in (10,25) we obtain $E(\varphi^*(y(t^{(n)})) \mid y(t^{(k-1)}) = \theta) = \varphi_{k-1}^*(\theta)$ by means of (10,23). We have proved (10,24) for all k. According to the definition of $\varphi_0^n(\theta)$ and (10,22) and (10,24), we obtain $E(\varphi(x(t^{(n)})) \mid x(0) = \theta) \leqq \varphi_0^{(n)}(\theta) \leqq \varphi_0^*(\theta) \leqq E(\varphi^*(y(t^{(n)})) \mid y(0) = \theta)$. Then (10,2) is easily proved.

Previous lemmas yield

Lemma 11. *Let* $x \in X(F_0, \delta, K, d, \lambda)$ *and* y *be the process bounded at* $(v - 3Kd)\,e^{\lambda d} + Kd$ *corresponding to the process* z *which is described in Lemma* 10. *The inequality*

$$(11,1) \qquad \bar{P}(\sup_{0 \leqq \tau \leqq t} x(\tau, \omega) \geqq v) \leqq P(y(\bar{t}^{(k)}, \omega) \geqq v - 3Kd)$$

holds, where $\bar{t}^{(k)} \leqq t$, $t - \bar{t}^{(k)} < d$.

Proof. Choose a sequence of positive numbers $\varepsilon_m \to 0$ (for $m \to \infty$) and a sequence of functions $\varphi_m(\theta)$ such that $\varphi_m(\theta)$ is convex for $\theta \leqq v - Kd$, $\varphi_m(\theta) = 1$ for $\theta \geqq v - Kd$, $\varphi_m(\theta) \to \varphi(\theta)$ which is defined by $\varphi(\theta) = 0$ for $\theta < v - Kd$, $\varphi(\theta) = 1$ for $\theta \geqq v - Kd$. The functions $\varphi_m^*(\theta)$ (see Lemma 10, $\varepsilon_m \to 0$) converge to a function $\varphi^*(\theta)$. The function φ^* fulfils $\varphi^*(\theta) = 0$ for $\theta < v - 3Kd$, $\varphi^*(\theta) = 1$ for $\theta \geqq v - 3Kd$. For arbitrary $x \in X$ choose x_v — see Definition 2. According to Lemma 2 we have

$$(11,2) \qquad \bar{P}(\sup_{0 \leqq \tau \leqq t} x(\tau, \omega) \geqq v) \leqq P(x_v(\bar{t}^{(k)}) \geqq (v - Kd)\, e^{\lambda d})$$

According to Remark 2 we construct $x^* \in X^v$ for the process x_v so that the inequality

$$(11,3) \qquad P(x_v(\bar{t}^{(k)}) \geqq (v - Kd)\, e^{\lambda d}) \leqq P(x^*(\bar{t}^{(k)}) \geqq v - Kd)$$

holds. As the functions $\varphi_m(\theta)$ coverge to $\varphi(\theta)$, we have

$$(11,4) \qquad E(\varphi_m(x^*(\bar{t}^{(k)}))) \to E(\varphi(x^*(\bar{t}^{(k)}))) = P(x^*(\bar{t}^{(k)}) \geqq v - Kd)$$

Since the functions $\varphi_m^*(\theta)$ converge to $\varphi^*(\theta)$ we have, similarly,

$$(11,5) \qquad E(\varphi_m^*(y(\bar{t}^{(k)}))) \to E(\varphi^*(y(\bar{t}^{(k)}))) = P(y(\bar{t}^{(k)}) \geqq v - 3Kd)$$

By Lemma 10

$$(11,6) \qquad E(\varphi_m(x^*(\bar{t}^{(k)}))) \leqq E(\varphi_m^*(y(\bar{t}^{(k)}))) \,.$$

The inequalities (11,2)–(11,6) imply (11,1).

We have obtaine the process y, and by means of this process we are able to estimate (5). In the case that $y(0, \omega) = y_0$ is a number (and not a random variable) we can conceive this process in the following way. Consider a point A. For $t \leqq 0$, A lies at y_0. For $0 < t < d$ the point A moves in accordance to the differential equation $\dot{x} = -\lambda x$. At the instant $t = d$ the point A moves to a distance Kd to the right or left with the same probability $\delta/(2K)$ or it remains at the same position with probability $1 - \delta/K$. In the time interval $d < t < 2d$ the movement of A is governed by the differential equation $\dot{x} = -\lambda x$. At the instant $t = 2d$ the whole situation is repeated. We still have to determine the value of $\lim_{n \to \infty} P(\max_{0 \leqq l \leqq T_n/d_n} y(t_n^{(l)}, \omega) \geqq v_n)$. In [1], the modified Lagrange integral theorem was sufficient for this. But now we must find the distribution as the solution of certain parabolic equation. We shall introduce three transformations which will transfer the process y to a process v more convenient for our considerations. Before formulating Lemma 12 we must modify the definition of processes bounded at v.

Definition 5. Let there be given a function $f(\xi)$, a sequence $\xi^{(0)} < \xi^{(1)} < \ldots$ and a process $x(\xi, \omega)$. We assume that the process $x(\xi, \omega)$ has values from $\bar{E}$ [2]).

---

[2]) $\bar{E}$ arises from $(-\infty, \infty)$ by adding $\pm\infty$.

The process $x(\xi, \omega)$ is bounded by $f(\xi)$ at $\xi^{(i)}$ if $x(\xi, \omega) = \infty$ for $\xi \geqq \xi(\omega)$, where $\xi(\omega)$ is the minimum of $\xi^{(i)}(\omega) : x(\xi^{(i)}(\omega), \omega) \geqq f(\xi^{(i)}(\omega))$ and the value of $\xi^{(i)}(\omega)$ are $\xi^{(j)}$ only. If no such $\xi^{(i)}(\omega)$ exists we put $\xi(\omega) = \infty$. The process $z(\xi, \omega)$ is bounded by $f(\xi)$ at $\xi^{(i)}$ and corresponds to $x(\xi, \omega)$ (for an arbitrary process x) if $z(\xi, \omega) = x(\xi, \omega)$ for $\xi \leqq \xi(\omega)$, $z(\xi, \omega) = \infty$ for $\xi > \xi(\omega)$, where $\xi(\omega)$ is defined above.

Lemma 12. *The transformation* $y = e^{-\lambda t}\gamma$ *transfers the process* y *to a process* γ $P(\gamma(t^{(i+1)}) = \gamma(t^{(i)}) + uKde^{\lambda(t^{(i)}+d)} \mid \gamma(t^{(i)}), \ldots, \gamma(0)) = 1 - \delta/K + |u|\,(3\delta/(2K) - 1)$, *where the parameter* u *may equal only* $-1, 0, 1$. *The absorbing barrier is described by* $\gamma = (v - 3Kd)\,e^{\lambda t}$. *The transformation* $\xi = e^{2\lambda t}$, $\xi^{(i)} = e^{2\lambda t^{(i)}}$, $\tilde{\gamma}(\xi^{(i)}) = \gamma(t^{(i)})$ *transfers the process* $\gamma(t)$ *to a process* $\tilde{\gamma}(\xi)$

$$P(\tilde{\gamma}(\xi^{(i+1)}) = \tilde{\gamma}(\xi^{(i)}) + uKde^{\lambda d}\sqrt{(\xi^{(i)})} \mid \tilde{\gamma}(\xi^{(i)}), \ldots, \tilde{\gamma}(\xi^{(0)})) = \\ = 1 - \delta/K + |u|\,(3\delta/(2K) - 1),$$

where again $u = -1, 0, 1$. *The absorbing barrier is described by* $\tilde{\gamma} = (v - 3Kd)\sqrt{\xi}$. *The transformation* $\tilde{\gamma} = vv$ *transfers the process* $\tilde{\gamma}$ *to a process* v

$$(12,1) \qquad P(v(\xi^{(i+1)}) = v(\xi^{(i)}) + u\,\frac{Kd}{v}\,e^{\lambda d}\sqrt{(\xi^{(i)})} \mid v(\xi^{(i)}), \ldots, v(\xi^{(0)}) = \\ = 1 - \delta/K + |u|\,(3\delta/(2K - 1),$$

where the parameter may equal only $-1, 0, 1$. *The absorbing barrier is described by*

$$(12,2) \qquad v = \left(1 - \frac{3Kd}{v}\right)\sqrt{\xi}\,.$$

The initial condition is

$$(12,3) \qquad P(v(1) \leqq \theta) = P(y(0) \leqq v\theta)\,.$$

Consider now the processes $v_n(\xi, \omega)$; these processes are defined by (12,1), with $\delta, K, d, v, \lambda, \xi^{(i)}$ replaced by $\delta_n, K_n, d_n, v_n, \lambda_n, \xi_n^{(i)}$. If the processes v_n are not bounded by any f we shall prove that under certain conditions the distributions $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$ converge to a solution of a certain parabolic equation.

Hypothesis. Let sequences of real positive numbers $\delta_1, \ldots, \delta_n, \ldots, K_1, \ldots, K_n, \ldots;$ $d_1, \ldots, d_n, \ldots;$ $\lambda_1, \ldots, \lambda_n \ldots;$ $T_1, \ldots, T_n, \ldots, v_1, \ldots, v_n, \ldots$ satisfy

$$(7) \qquad \frac{K_n \delta_n d_n}{\lambda_n v_n^2} \to A^2\,, \quad \delta_n \leqq K_n$$

$$(8) \qquad \lambda_n T_n \to T > 0$$

$$(9) \qquad \lambda_n d_n \to 0$$

$$(10) \qquad \frac{K_n d_n}{v_n} \to 0\,.$$

Remark 3. By (8), (9) we have $T_n/d_n \to \infty$. If we assume that δ and K are independent of n (as in [1]) then we the assumptions (7)–(10) reduce to $\delta \leqq K$, $d_n/(\lambda_n v_n^2) \to A^2/(\delta K)$, $\lambda_n T_n \to T$, $\lambda_n d_n \to 0$ (or $T_n/d_n \to \infty$). By Lemma 12 we have $\xi_n^{(i)} \in \langle 1, e^{2T} \rangle$.

Lemma 13. *Let v_n be defined by (12,1) (where instead of $\delta, K, v, d, \xi^{(i)}, \lambda$ we put $\delta_n, K_n, v_n, d_n, \xi_n^{(i)}, \lambda_n$). Let the initial distributions $F_n(1, \theta)$ converge uniformly to a distribution $F(1, \theta)$. If (7)–(10) are satisfied then the distributions $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$ converge to the bounded solution of the parabolic equation*

$$\frac{\partial F}{\partial \xi} = \frac{A^2}{4} \frac{\partial^2 F}{\partial \theta^2} \tag{13,1}$$

which has initial condition $F(1, \theta)$ and is given by Poisson formula [3]

$$F(\xi, \theta) = \frac{1}{A \sqrt{[\pi(\xi - 1)]}} \int_{-\infty}^{\infty} \exp\left\{- \frac{(\theta - x)^2}{A^2(\xi - 1)}\right\} F(1, x) \, \mathrm{d}x .$$

This covergence is uniform in each region $\xi \in \langle 1 + h, H \rangle, \theta$; $h > 0$, $H > 1 + h$. If $F(1, \theta)$ is uniformly continuous then $F_n(\xi, \theta)$ converge uniformly to $F(\xi, \theta)$ in the region $\xi \in \langle 1, H \rangle, \theta$.

Proof. Choose a positive number ε. Observe that each bounded solution of (13,1) has uniformly bounded partial derivatives $\partial^2 F/\partial \xi^2$, $\partial^2 F/\partial \theta^2$, $\partial^2 F/\partial \theta^3$ in each region $\xi \in \langle 1 + h, H \rangle, \theta$. (A simple consequence of the Poisson formula.) If $F(1, \theta)$ is uniformly continuous then $F(1 + \varepsilon, \theta)$ converges uniformly to $F(1, \theta)$ for $\varepsilon \to 0+$, this follows from the Poisson formula.

First we shall assume that the function $F(1, \theta)$ is uniformly continuous. Let s be the least superscript satisfying $\xi_n^{(s)} \geqq 1 + \varepsilon$ (s depends on n but we do not indicate this). We shall prove that the value

$$\beta_n(\xi_n^{(i)}, \theta) = F(\xi_n^{(i)}, \theta) - F_n(\xi_n^{(i-s)}, \theta) \tag{13,2}$$

converges to zero. To obtain this we shall estimate the expression (in the following two formulae we shall not indicate the dependence on n)

$$\begin{aligned} \alpha(\xi^{(i)}, \theta) &= F(\xi^{(i+1)}, \theta) - F\left(\xi^{(i)}, \theta + \frac{Kd}{v} e^{\lambda d} \sqrt{(\xi^{(i)})}\right) \frac{\delta}{2K} - \\ &- F(\xi^{(i)}, \theta)\left(1 - \frac{\delta}{K}\right) - F\left(\xi^{(i)}, \theta - \frac{Kd}{v} e^{\lambda d} \sqrt{(\xi^{(i)})}\right) \frac{\delta}{2K} = \\ &= \frac{\partial F}{\partial \xi}(\xi^{(i)}, \theta)(\xi^{(i+1)} - \xi^{(i)}) + \frac{1}{2} \frac{\partial^2 F}{\partial \xi^2}(\xi^*, \theta)(\xi^{(i+1)} - \xi^{(i)})^2 - \\ &- \frac{1}{2} \frac{\partial^2 F}{\partial \theta^2}(\xi^{(i)}, \theta) \frac{Kd\delta}{v^2} e^{2\lambda d} \xi^{(i)} - \frac{1}{12} \frac{\partial^3 F}{\partial \theta^3}(\xi^{(i)}, \theta^*) \frac{K^2 d^3 \delta}{v^3} e^{3\lambda d} (\xi^{(i)})^{3/2} + \\ &+ \frac{1}{12} \frac{\partial^3 F}{\partial \theta^3}(\xi^{(i)}, \theta^{**}) \frac{K^2 d^3 \delta}{v^3} e^{3\lambda d} (\xi^{(i)})^{3/2} , \end{aligned}$$

where

$$\xi^{(i)} < \xi^* < \xi^{(i+1)},\ \theta < \theta^* < \theta + \frac{Kd}{v} e^{\lambda d} \sqrt{(\xi^{(i)})},\ \theta - \frac{Kd}{v} e^{\lambda d} \sqrt{(\xi^{(i)})} < \theta^{**} < \theta\,.$$

Dividing the equation by $\xi^{(i+1)} - \xi^{(i)}$ and considering that F is a solution of (13,1), we obtain

$$\frac{\alpha(\xi^{(i)}, \theta)}{\xi^{(i+1)} - \xi^{(i)}} = \frac{1}{4} \frac{\partial^2 F}{\partial \theta^2} (\xi^{(i)}, \theta) \left(A^2 - \frac{Kd\delta}{v^2 \lambda} e^{2\lambda d} \right) + \frac{\partial^2 F}{\partial \xi^2} (\xi^*, \theta)\, \xi^{(i)} \lambda d -$$

$$- \frac{1}{24} \left(\frac{\partial^3 F}{\partial \theta^3} (\xi^{(i)}, \theta^*) - \frac{\partial^3 F}{\partial \theta^3} (\xi^{(i)}, \theta^{**}) \right) \frac{K^2 d^2 \delta}{v^3 \lambda} e^{3\lambda d} \sqrt{(\xi^{(i)})} + 0(\lambda d)\,.$$

By (7)–(10), $\xi_n^{(i)} \leqq e^{2\lambda_n T_n} \to e^{2T}$ for $n \to \infty$ and since the partial derivatives are bounded, we have

$$\text{(13,3)} \qquad \left| \frac{\alpha_n(\xi_n^{(i)}, \theta)}{4\xi_n^{(i)} \lambda_n d_n} \right| \leqq \left| \frac{\alpha_n(\xi_n^{(i)}, \theta)}{\xi_n^{(i+1)} - \xi_n^{(i)}} \right| \to 0 \quad \text{for} \quad n \to \infty\,.$$

Let us denote $\beta_i^{(n)} = \max_\theta |\beta^{(n)}(\xi_n^{(i)}, \theta)|$, $\alpha_i^{(n)} = \max_\theta |\alpha^{(n)}(\xi_n^{(i)}, \theta)|$. By (13,2) we obtain $\beta_n^{(j+1)} \leqq \alpha_n^{(j)} + \beta_n^{(j)}$ and furthermore $\beta_n^{(j+1)} \leqq \beta_n^{(s)} + \sum_{i=s}^{j} \alpha_n^{(i)}$. T_n/d_n is the greatest value of j (that is the number of instants at which perturbations act). Hence

$$\text{(13,4)} \qquad \beta_n^{(j)} \leqq \beta_n^{(s)} + \frac{T_n}{d_n} \max_{i \geqq s} \alpha_n^{(i)} \quad \text{for} \quad s \leqq j \leqq \frac{T_n}{d_n}\,.$$

We may estimate the last term of (13,4) $T_n/d_n \max_{i \geqq s} \alpha_n^{(i)} \leqq 2Te^{2T} \max_{i \geqq s} \alpha_n^{(i)}/(\xi_n^{(i)} \lambda_n d_n)$, and by (13,3) this term converges uniformly to 0 for $n \to \infty$. Since $\beta_n(\xi_n^{(s)}, \theta) = F(\xi_n^{(s)}, \theta) - F_n(\xi_n^{(0)}, \theta) = F(\xi_n^{(s)}, \theta) - F_n(1, \theta)$ and since $F(\xi_n^{(s)}, \theta)$ converge uniformly to $F(1, \theta)$ for $\varepsilon \to 0$, we obtain that $\beta_n^{(s)}$ converges uniformly to 0 for $n \to \infty$, $\varepsilon \to 0$. By (13,2)

$$\text{(13,5)} \qquad F(\xi_n^{(i)}, \theta) - F_n(\xi_n^{(i)}, \theta) = \beta_n(\xi_n^{(i+s)}, \theta) - [F(\xi_n^{(i+s)}, \theta) - F(\xi_n^{(i)}, \theta)]\,.$$

Since $\xi_n^{(i+s)} - \xi_n^{(i)} \leqq 2\varepsilon e^{2T}$ (as for large n there is $\xi_n^{(s)} < 1 + 2\varepsilon$), we have $\xi_n^{(i+s)} - \xi_n^{(i)} \to 0$ for $\varepsilon \to 0$, $n \to \infty$, and the difference $F(\xi_n^{(i+s)}, \theta) - F(\xi_n^{(i)}, \theta)$ converges uniformly to 0. Let $\xi > 1$ be given. We shall choose least $\xi_n^{(i)}$ with $\xi_n^{(i)} > \xi$. Then $\xi_n^{(i)} \to \xi$ holds, and by (12,1) we have $|F_n(\xi_n^{(i)}, \theta) - F_n(\xi, \theta)| \to 0$ uniformly with respect to θ. By (13,5) $F_n(\xi, \theta)$ converges uniformly to the solution of (13,1) described in Lemma 13.

We have assumed that the function $F(1, \theta)$ is uniformly continuous. For arbitrary $\varepsilon_1 > 0$ and for an arbitrary distribution $F(1, \theta)$ we can find distributions $F_1(1, \theta)$ and $F_2(1, \theta)$ such that

$$\text{(13,6)} \qquad F_2(1, \theta) - F_1(1, \theta) < \varepsilon_1$$

except of finite number of regions (the number of these regions is at most $[2/\varepsilon_1]$, $[x]$ stands for the greatest integer in x) and the Lebesgue measure of each is less than ε_1^2. We can assume that

$$(13,7) \qquad \begin{aligned} &F_2(1, \theta) - F(1, \theta) > \varepsilon_1/4 \quad \text{or} \quad F_2(1, \theta) = 1 \\ &F(1, \theta) - F_1(1, \theta) > \varepsilon_1/4 \quad \text{or} \quad F_1(1, \theta) = 0 . \end{aligned}$$

Denote by $v_n^{(1)}$ the process defined by (12,1) ($\delta = \delta_n, \ldots$) with initial condition $F_1(1, \theta)$; $v_n^{(2)}$ is defined in the same manner as $v_n^{(1)}$ but with initial condition $F_2(1, \theta)$. Put

$$F_1^{(n)}(\xi, \theta) = P(v_n^{(1)}(\xi) \leqq \theta), \quad F_2^{(n)}(\xi, \theta) = P(v_n^{(2)}(\xi) \leqq \theta) .$$

By (13,7), the inequality $F_1^{(n)}(\xi, \theta) \leqq F_n(\xi, \theta) \leqq F_2^{(n)}(\xi, \theta)$ holds for large n. Passing to the limit for $n \to \infty$ we obtain

$$(13,8) \qquad \begin{aligned} F_1(\xi, \theta) = \lim_{n\to\infty} F_1^{(n)}(\xi, \theta) &\leqq F^*(\xi, \theta) \leqq F^{**}(\xi, \theta) \leqq \\ &\leqq \lim_{n\to\infty} F_2^{(n)}(\xi, \theta) = F_2(\xi, \theta) , \end{aligned}$$

where

$$F^*(\xi, \theta) = \liminf_{n\to\infty} F_n(\xi, \theta) , \quad F^{**}(\xi, \theta) = \limsup_{n\to\infty} F_n(\xi, \theta) ;$$

these $F_1(\xi, \theta)$, $F_2(\xi, \theta)$ exist because $F_1(1, \theta)$, $F_2(1, \theta)$ are uniformly continuous. By (13,7) and the Poisson formula the inequality

$$(13,9) \qquad F_1(\xi, \theta) \leqq F(\xi, \theta) \leqq F_2(\xi, \theta)$$

holds, and by (13,6) and the Poisson formula again,

$$(13,10) \qquad F_2(\xi, \theta) - F_1(\xi, \theta) \leqq \varepsilon_1(1 + \psi(\xi)) .$$

The function $\psi(\xi)$ is monotone decreasing. By (13,8), (13,9) and (13,10) we have $F^*(\xi, \theta) = F^{**}(\xi, \theta) = F(\xi, \theta)$.

Now consider the case when v is a bounded process (by certain f at certain $\xi^{(i)}$). We choose the absorbing barrier in a simpler way than in (12,2) in order to be able to apply the principle of reflection. Assume we have a finite set of numbers $\zeta^{(\alpha)} : 1 = \zeta^{(0)} < \zeta^{(1)} < \ldots < \zeta^{(q)} = e^{2T}$ and $g^{(\alpha)}$: $0 < g^{(0)} < g^{(1)} < \ldots < g^{(q)}$. Now define the function f – see Definition 5 – by $f(\theta) = g^{(i)}$ for $(\zeta^{(i)}, \zeta^{(i+1)}\rangle$.

Lemma 14. *Let v_n be the process defined by (12,1), bounded by f at $\zeta^{(i)}$. If the assumptions of Lemma 13 are satisfied, then $F_n(\xi, \theta)$ converge to $F(\xi, \theta)$. The function F is the bounded solution of (13,1) in the region $\xi \in \langle 1, \zeta^{(1)}\rangle$, $\theta \leqq g^{(0)}$ with initial condition $F(1, \theta)$ for $\theta \leqq g^{(0)}$ and with boundary condition $(\partial F/\partial\theta)(\xi, g^{(0)}) = 0$ for $\xi \in \langle 1, \zeta^{(1)}\rangle$. In the region $\mathscr{H}_i : \xi \in (\zeta^{(i)}, \zeta^{(i+1)}\rangle$, $\theta \leqq g^{(i)}$, F is the bounded solution of (13,1) with initial condition $F(\zeta^{(i)}, \theta)$ for $\theta \leqq g^{(i-1)}$, $F(\zeta^{(i)}, \theta)$ is constant* *for* $\theta \in \langle g^{(i-1)}, g^{(i)}\rangle$, *and with boundary condition* $(\partial F/\partial\theta)(\xi, g^{(i)}) = 0$ *for* $\xi \in \in \langle\zeta^{(i)}, \zeta^{(i+1)}\rangle$. *The convergence of the* F_n *is uniform in each region* $\mathscr{H} \cap \bigcup \mathscr{H}_i$ *where* $\mathscr{H}$ *is described by* $\xi \in \langle 1 + h, H\rangle, \theta$. *If* $F(1, \theta)$ *is uniformly continuous then the* $F_n(\xi, \theta)$ *converge uniformly in the region* $\bigcup \mathscr{H}_i$.

Proof. Without loss of generality we may restrict ourselves to the first region $\mathscr{H}_0$. Let $\tilde{v}_n$ be the process fulfilling (12,1) (the numbers δ, λ, v, K, d depend on n). Let v_n be the process bounded by $f(\theta) = g^{(0)}$ at $\zeta^{(i)}$ corresponding to $\tilde{v}_n$. As the initial distribution of v_n we can take $F_n(1, \theta)$ defined thus: $F_n(1, \theta) = \tilde{F}_n(1, \theta)$ for $\theta \leqq g^{(0)}$, $F_n(1, \theta) = l_n$ for $\theta \geqq g^{(0)}$, where $\tilde{F}_n(1, \theta)$ is initial distribution of $\tilde{v}_n$ and $l_n = = \lim_{\theta \to g^{(0)}-} \tilde{F}_n(1, \theta)$. Set $\tilde{F}_n(\xi, \theta) = P(\tilde{v}_n(\xi) \leqq \theta)$, $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$. It is easily proved that

$$(14,1) \qquad \tilde{F}_n(\xi, \theta) + \tilde{F}_n(\xi, 2g^{(0)} - \theta + \frac{K_n d_n}{v_n} e^{\lambda_n(d_n + t_n)}) - l_n \geqq F_n(\xi, \theta) \geqq$$

$$\geqq \tilde{F}_n(\xi, \theta) + \tilde{F}_n(\xi, 2g^{(0)} - \theta - \frac{K_n d_n}{v_n} e^{\lambda_n(d_n + t_n)}) - l_n \quad \text{for} \quad \theta \leqq g^{(0)}.$$

Since both the left and right sides of (14,1) converge to the solution of (13,1) (taking $(1/l_n)\tilde{F}_n$ instead of $\tilde{F}_n$), we obtain that $F_n(\xi, \theta)$ converge to the solution of (13,1).

The processes v_n determined by Lemma 12 are bounded by functions which converge to $\sqrt{\xi}$ for $n \to \infty$ (see (12,2)). Before considering the function (12,2) we shall formulate certain results of the theory of differential equations which we shall need later.

Lemma 15. *Let* $F(\xi, \theta)$ *be the bounded solution of* (13,1) *in the region* $\xi \geqq 1$, $\theta \leqq \sqrt{\xi}$, *with initial condition* $F(1, \theta)$ *and with boundary condition* $(\partial F/\partial\theta)(\xi, \sqrt{\xi}) = = 0$ *for* $\xi \geqq 1$. *Assume that* $(\partial F/\partial\theta)(\xi, \theta)$ *is continuous at* $[\xi, \sqrt{\xi}]$ *for* $\xi \geqq 1$, *the initial function has continuous derivative at* $\theta = 1$ *and* $(\mathrm{d}F/\mathrm{d}\theta)(1, 1) = 0$, $F(1, \theta)$ *is nondecreasing,* $\lim_{\theta \to -\infty} F(1, \theta) = 0$. *Then the following statements hold.*

1) *Solutions are uniquely determined in the class of bounded solutions.*

2) *The solutions with the properties mentioned above exist and*

$$F(\xi, \theta) = F^*(\xi, \theta) - \int_{-\infty}^{\Theta} \int_1^{\xi} \Gamma_\Theta(\xi, \gamma; y, \sqrt{y})\, F^*_\Theta(y, \sqrt{y})\, \mathrm{d}y\, \mathrm{d}\gamma$$

holds, where $F^*(\xi, \theta)$ *is determined by the Poisson formula and* $\Gamma(\xi, \theta; x, y)$ *is Green's function of* (13,1) *for our region.*

$$F^*_\theta(\xi, \theta) = \frac{\partial F^*}{\partial\theta}, \quad \Gamma_\theta = \frac{\partial \Gamma}{\partial\theta}.$$

3) *Let a, ε be positive $0 < 2\varepsilon < a$. Define $F_\varepsilon(\xi, \theta)$ thus $F_\varepsilon(\xi, \theta)$ is the bounded solution of (13,1) for $\xi \in \langle 1, 1 + a \rangle$, $\theta \leqq \sqrt{(1 + a)}$, with initial condition $F_\varepsilon(1, \theta) = F(1, \theta)$ for $\theta \leqq 1$, $F_\varepsilon(1, \theta) = F(1, 1)$ for $\theta \geqq 1$, and with boundary condition $(\partial F_\varepsilon/\partial\theta)\,[\xi, \sqrt{(1 + a)}] = 0$. In the region $\xi \geqq 1 + a$, $\theta \leqq \sqrt{(\xi + \varepsilon)}$, F_ε is the bounded solution of (13,1) with initial condition $F_\varepsilon(1 + a, \theta) = F_\varepsilon(1 + a, \sqrt{(1 + a)})$ for $\theta \geqq 1 + a$, and with boundary condition $(\partial F_\varepsilon/\partial\theta)\,(\xi, \sqrt{(\xi + \varepsilon)}) = 0$ for $\xi \geqq 1 + a$. Let $F_a^+(\xi, \theta)$ be the continuous function defined by $F_a^+(\xi, \theta) = F_\varepsilon(\xi, \theta)$ in the region $\xi \in \langle 1, 1 + a \rangle$, $\theta \leqq \sqrt{(1 + a)}$. In the region $\xi \geqq 1 + a$, $\theta \leqq \sqrt{\xi}$, F_a^+ is the bounded solution of (13,1) with the same initial conditions as F_ε but with boundary condition $(\partial F_a^+/\partial\theta)\,(\xi, \sqrt{\xi}) = 0$. We shall also need a function $F_{-\varepsilon}(\xi, \theta)$ defined as the solution of (13,1) in the region $\xi \in \langle 1, 1 + a \rangle$, $\theta \leqq 1$, with initial condition $F(1, \theta)$ and with boundary condition $(\partial F_{-\varepsilon}/\partial\theta)\,(\xi, 1) = 0$. Further $F_{-\varepsilon}(\xi, \theta)$ is bounded solution of (13,1) in the region $\xi \geqq 1 + a$, $\theta \leqq \sqrt{(\xi - \varepsilon)}$ with initial condition $F_{-\varepsilon}(1 + a, \theta) = F_{-\varepsilon}(1 + a, 1)$ for $\theta \geqq 1$ and with boundary condition $(\partial F_{-\varepsilon}/\partial\theta)\,(\xi, \sqrt{(\xi - \varepsilon)}) = 0$ for $\xi \geqq 1 + a$. $F_{-a}^+(\xi, \theta)$ is defined in the same manner as $F_{-\varepsilon}$ with only the difference that for $\xi \geqq 1 + a$ the region of definition is described by $\theta \leqq \sqrt{\xi}$ and the boundary condition is $(\partial F_{-a}^+/\partial\theta)\,(\xi, \sqrt{\xi}) = 0$. Let $F(\xi, \theta)$ be the bounded solution of (13,1) with initial condition $F(1, \theta)$ and boundary condition $(\partial F/\partial\theta)\,(\xi, \sqrt{\xi}) = 0$. The region of definition is described by $\xi \geqq 1$, $\theta \leqq \sqrt{\xi}$. Then the inequalities*

$$F_{-\varepsilon}(\xi, \theta) \leqq F_{-a}^+(\xi, \theta)\,, \quad F_a^+(\xi, \theta) \leqq F_\varepsilon(\xi, \theta)$$

hold for $\xi \geqq 1$.

4) *There is $\lim_{\varepsilon \to 0+} F_\varepsilon(\xi, \theta) = F_a^+(\xi, \theta)$, $\lim_{\varepsilon \to 0+} F_{-\varepsilon}(\xi, \theta) = F_{-a}^+(\xi, \theta)$. The convergence is uniform with respect to ξ, θ in the region described by $H \geqq \xi \geqq 1 + a$, $\theta \leqq \sqrt{\xi}$.*

5) *There is $\lim_{a \to 0+} [F_a^+(\xi, \theta) - F_{-a}^+(\xi, \theta)] = 0$. The convergence is uniform in any region described by $\xi \in \langle 1 + h, H \rangle$, $\theta \leqq \sqrt{\xi}$ with arbitrary $h > 0$, $1 + h < H$.*

6) *If $F(1, \theta)$ is continuous then $F_a^+ - F_{-a}^+$ converges uniformly to 0 in the region $H \geqq \xi \geqq 1$, $\theta \leqq \sqrt{\xi}$.*

7) *There is $F_{-a}^+(\xi, \theta) \leqq F(\xi, \theta) \leqq F_a^+(\xi, \theta)$.*
If $F(1, \theta)$ has a continuous derivation, then

8) *$F(\xi, \theta)$ is non-decreasing with respect to θ for fixed ξ,*

9) $\lim_{\theta \to -\infty} F(\xi, \theta) = 0$ *uniformly with respect to all ξ,*

10) *$F(\xi, \theta) = \int_{-\infty}^{\theta} G(\xi, \gamma)\,d\gamma$, where $G(\xi, \theta)$ is the solution of (13,1) with initial condition $G(1, \theta) = (\partial F/\partial\theta)\,(1, \theta)$ and with boundary condition $G(\xi, \sqrt{\xi}) = 0$ for $\xi \geqq 1$.*

11) *$F(\xi, \sqrt{\xi})$ is a non-increasing function of ξ.*

Proof. If two solutions F_1, F_2 exist, then for almost all $\theta \leqq 1 - \tau$, $F(1, \theta) = 0$ with $F = F_1 - F_2$ (F_1, F_2 are continuous at $\theta = 1, \tau > 0$). Since these solutions are bounded we obtain $F(1, \theta) = 0$ for all $\theta \leqq 1$. For F the boundary condition $(\partial F/\partial\theta)(\xi, \sqrt{\xi}) = 0$ also holds. By the Theorem 3 in [4] we obtain $F = 0$.

8) If we realize that $(\partial F/\partial\theta)(\xi, \theta)$ is also the solution of (13,1) we can easily prove item 8. On comparing the solution F with the solution defined on the region described by $\xi \geqq 1$, $\theta \leqq 1$, we obtain item 9; and also that there exists a constant K such that $\lim_{\theta \to -\infty} e^{K\theta^2} F(\xi, \theta) = 0$ uniformly with respect to all ξ provided that $F(1, \theta) = 0$ for sufficiently small θ. Using this remark we can easily prove items 10; and also 2) because $\int_0^\xi \Gamma_\theta(\xi, \theta; y, \sqrt{y}) F_a^*(y, \sqrt{y})\, dy$ is a solution of (13,1) which is equal to zero for $\xi = 1$.

Proof of 11. Take $1 \leqq \xi_1 < \xi_2$, denote by $\hat{F}(\xi, \theta)$ the solution of (13,1) which satisfies $\hat{F}(\xi_1, \theta) = F(\xi_1, \theta)$ for $\theta \leqq \sqrt{\xi}$, $\hat{F}(\xi_1, \theta) = F(\xi_1, \sqrt{\xi_1})$ for $\theta \in (\sqrt{\xi_1}, \sqrt{\xi_2})$, $(\partial\hat{F}/\partial\theta)(\xi, \sqrt{\xi_2}) = 0$. By Lemma 14, $\hat{F}$ is the limit of distributions of processes bounded by 1 (at $\zeta^{(i)}$). Then the inequality $\hat{F}(\xi_2, \sqrt{\xi_2}) \leqq \hat{F}(\xi_1, \sqrt{\xi_2})$ must hold and also $F(\xi_2, \sqrt{\xi_2}) \leqq \hat{F}(\xi_2, \sqrt{\xi_2}) \leqq \hat{F}(\xi_1, \sqrt{\xi_2}) = F(\xi_1, \sqrt{\xi_1})$. To prove Item 3 we consider the partial derivative $\partial F/\partial\theta$ again. Proof of 4. Set $G_\varepsilon(\xi, \theta) = (\partial F_\varepsilon/\partial\theta)(\xi, \theta)$, $G_a^+(\xi, \theta) = (\partial F_a^+/\partial\theta)(\xi, \theta)$. First we shall show that the G_ε converge to G_a^+. Consider the auxiliary function $\hat{G}_\varepsilon(\xi, \theta) = G_\varepsilon(\xi - \varepsilon, \theta)$. Then $\hat{G}_\varepsilon$ is a solution of (13,1) with the condition $\hat{G}_\varepsilon(\xi, \sqrt{\xi}) = 0$ for $\xi \geqq 1 + a + \varepsilon$. By the maximum principle,

$$
\begin{aligned}
&|G_a^+(\xi, \theta) - G_\varepsilon(\xi, \theta)| = |G_a^+(\xi, \theta) - \hat{G}_\varepsilon(\xi + \varepsilon, \theta)| \leqq \\
&\leqq |G_a^+(\xi, \theta) - G_a^+(\xi + \varepsilon, \theta)| + |G_a^+(\xi + \varepsilon, \theta) - \hat{G}_\varepsilon(\xi + \varepsilon, \theta)| \leqq \\
&\leqq |G_a^+(\xi, \theta) - G_a^+(\xi + \varepsilon, \theta)| + \sup_\theta |G_a^+(1 + a + \varepsilon, \theta) - G_a^+(1 + a, \theta)| \,.
\end{aligned}
$$

The last term can be estimate by $2M\varepsilon$, where $M = \sup_\theta (\partial^3 F_a^+/\partial\theta^3)(1 + a, \theta)$. The term $|G_a^+(\xi, \theta) - G_a^+(\xi + \varepsilon, \theta)|$ converges to 0 (with $\varepsilon \to 0$) because the function G_a^+ is continuous. As $G_\varepsilon(\xi, \theta)$ converges to $G_a^+(\xi, \theta)$ monotonously, we obtain that $\lim_{\varepsilon \to 0+} F_\varepsilon(\xi, \theta) = F_a^+(\xi, \theta)$, $\lim_{\varepsilon \to 0+} F_\varepsilon(\xi, \sqrt{(\xi + \varepsilon)}) = F_a^+(\xi, \sqrt{\xi})$ for fixed ξ, θ (see 10). By 11, the function $F_\varepsilon(\xi, \sqrt{\xi})$ is non-increasing. Using the continuity of $F_a^+(\xi, \sqrt{\xi})$, the statement to follow and the maximum principle, we see that the functions $F_\varepsilon(\xi, \theta)$ converge uniformly to $F_a^+(\xi, \theta)$ in the region mentioned above.

Statement. Let there be given sequence of non-increasing functions $f_n(x)$ with $f_n(x) \geqq f_0(x)$ or $(f_n(x) \leqq f_0(x))$ for $n > 0$, $\lim_{n \to \infty} f_n(x) = f_0(x)$ for each $x \in \langle a, b\rangle$, $-\infty < a < b < \infty$ and let the function $f_0(x)$ be continuous. Then the functions $f_n(x)$ converge uniformly to $f_0(x)$ on $\langle a, b\rangle$.

Proof of 5. First we shall prove $\lim_{a\to 0+}[F_a^+(1+a,\theta) - F_{-a}^+(1+a,\theta)] = 0$ uniformly with respect to $\theta \leqq 1$. Setting $a^* = aA^2$, we have

$$F_a^+(1+a,\theta) - F_{-a}^+(1+a,\theta) =$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} F(1,\gamma) \frac{1}{\sqrt{a^*}} \left[\exp\left\{-\frac{(2+2v(a)-\theta-\gamma)^2}{a^*}\right\} - \exp\left\{-\frac{(2-\theta-\gamma)^2}{a^*}\right\}\right] d\gamma =$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} (F(1, x+2+2v(a)-\theta) - F(1, x+2-\theta)) \frac{1}{\sqrt{a^*}} \exp\left\{-\frac{x^2}{a^*}\right\} dx,$$

for $\theta \leqq 1$, where $v(a) = \sqrt{(1+a)} - 1$. We write the function $F(1,\theta)$ as the sum of the continuous part $\hat{F}$ and of functions of discontinuities $F_i(1,\theta)$: Let the i-th discontinuity be at the point h_i (there are only countably many such points) and $w_i = F(1,h_i) - F(1,h_i - 0)$. Then $F_i(1,\theta) = 0$ for $\theta < h_i$, $F_i(1,\theta) = w_i$ for $\theta \geqq h_i$. We obtain $F_a(1+a,\theta) - F_{-a}(1+a,\theta) = (1/2\sqrt{\pi}) \int_{-\infty}^{\infty} [\hat{F}(1, x+2+2v(a)-\theta) - \hat{F}(1, x+2-\theta)]\,(1/\sqrt{a^*}) \exp\,(-x^2/a^*)\}\,dx + \sum_{i=1}^{\infty} (1/2\sqrt{\pi})\,.$ $.\int_{-\infty}^{\infty} [F_i(1, x+2+2v(a)-\theta) - F_i(1, x+2-\theta)]\,(1/\sqrt{a^*}) \exp\,\{-(x^2/a^*)\}\,dx.$ The first expression obviously converges to 0 for $a \to 0$, uniformly with respect to θ. For the remaining expressions we obtain the estimate

$$\frac{1}{2\sqrt{\pi}} \sum \int_{h_i - 2v(a)+\theta-2}^{h_i+\theta-2} w_i \frac{1}{\sqrt{a^*}} \exp\left\{-\frac{x^2}{a^*}\right\} dx =$$

$$= \frac{1}{2\sqrt{\pi}} \sum \int_{[(h_i+\theta-2)/\sqrt{a^*}] - 2(v(a)/\sqrt{a^*})}^{(h_i+\theta-2)/\sqrt{a^*}} w_i \exp\{-y^2\}\,dy \leqq$$

$$\leqq \frac{1}{2\sqrt{\pi}} \sum \int_{n-(v(a)/\sqrt{a^*})}^{n+(v(a)/\sqrt{a^*})} w_i^*(a) \exp\{-y^2\}\,dy,$$

where $w_n^*(a)$ is the sum of those w_i for which

$$\left(\frac{h_i+\theta-2}{\sqrt{a^*}} - 2\frac{v(a)}{\sqrt{a^*}}, \frac{h_i+\theta-2}{\sqrt{a^*}}\right) \cap (n, n+1) \neq 0 \quad \text{for} \quad n > 0,$$

$$\left(\frac{h_i+\theta-2}{\sqrt{a^*}} - 2\frac{v(a)}{\sqrt{a^*}}, \frac{h_i+\theta-2}{\sqrt{a^*}}\right) \cap (n-1, n) \neq 0 \quad \text{for} \quad n < 0,$$

$w_0^*(a)$ is the sum of the remaining w_i. Since $\sum w_i \leqq 1$ we obtain that $\sum w_n^*(a) \leqq 1$, and thus estimate the above expression by

$$\frac{1}{2\sqrt{\pi}} \sum \int_{n-(v(a)/\sqrt{a^*})}^{n+(v(a)/\sqrt{a^*})} \exp(-y^2)\,dy.$$

This last expression converges to 0 for $a \to 0$ because $\nu(a) \sim \frac{1}{2}a = a^*/2A^2$. Further, for θ in $(1, 1 + \nu(a))$, the function $F^+_{-a}(1 + a, \theta)$ is constant, and evidently $|F^+_{-a}(1 + a, 1) - F^+_{-a}(1, 1)| \to 0$ for $a \to 0$. As $F(1, \theta)$ is continuous at $\theta = 1$ we obtain that $|F^+_a(1 + a, \sqrt{(1 + a)}) - F(1, 1)| \to 0$, for $a \to 0$. Evidently the inequalities $F^+_a(1 + a, 1) \leqq F^+_a(1 + a, \theta) \leqq F^+_a(1 + a, \sqrt{(1 + a)})$ hold for $1 \leqq \theta \leqq \sqrt{(1 + a)}$. We have proved that $\lim_{a \to 0} |F^+_a(1 + a, \theta) - F^+_{-a}(1 + a, \theta)| = 0$. Item 5 follows from Theorem 3 in [4].

6) If $F(1, \theta)$ is continuous, the functions F_i in the estimates introduced above are all zero. The expression for the remainder converges uniformly to zero.

7) If $(\partial F/\partial \theta)(1, \theta)$ is continuous for all θ we shall prove 7 by means of 10. If not, we can approximate all F^* by functions which have continuous derivatives. Substitute F^* into the formula in 2; the statement of 7 is then obtained by passing to the limit.

Now, we shall consider the case when the absorbing barrier is given by (12,2).

Lemma 16. *Let v_n be given by (12,1) (where d, $\xi^{(i)}$, ν, δ, K depend on n) and v_n is bounded by (12,2) at $\xi^{(i)}_n$. Assume that $F_n(1, \theta)$ converge uniformly to $F(1, \theta)$, $(\mathrm{d}F/\mathrm{d}\theta)(1, \theta)$ is continuous at $\theta = 1$ and $(\mathrm{d}F/\mathrm{d}\theta)(1, 1) = 0$. Then $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$ converge to the bounded solution $F(\xi, \theta)$ of (13,1) which is defined in the region described by $\xi \geqq 1$, $\theta \leqq \sqrt{\xi}$, satisfies the intial condition $F(1, \theta)$ and the boundary condition $(\partial F/\partial \theta)(\xi, \sqrt{\xi}) = 0$ for $\xi \geqq 1$. This convergence is uniform in any region described by $\xi \in \langle 1 + h, H\rangle$, $\theta \leqq \sqrt{\xi}$ with $h > 0$, $1 + h < H$. If $F(1, \theta)$ is uniformly continuous then the convergence is uniform in the region described by $H \geqq \xi \geqq 1$, $\theta \leqq \sqrt{\xi}$.*

Proof. Take any $\eta > 0$, $h > 0$. By Lemma 15 (5), we can choose a number $a > 0$ such that

$$|F^+_a(\xi, \theta) - F^+_{-a}(\xi, \theta)| < \eta \quad \text{for} \quad \xi \in \langle 1 + h, H\rangle, \tag{16,1}$$

where F^+_a, F^+_{-a} were defined in Lemma 15 (3). By Lemma 15 (3), $F^+_a \leqq F_\varepsilon$, $F_{-\varepsilon} \leqq F^+_{-a}$. The proof of the inequality $F_{-\varepsilon} \leqq F^+_{-a}$ is the same as that of the previous inequality. Obviously $F_{-\varepsilon}(1 + a, \theta) = F^+_{-a}(1 + a, \theta)$; $F_\varepsilon(1 + a, \theta) = F^+_a(1 + a, \theta)$. By Lemma 15 (4), we can choose $\varepsilon > 0$ such that

$$F_\varepsilon < F^+_a + \eta\,, \quad F^+_{-a} < F_{-\varepsilon} + \eta\,. \tag{16,2}$$

Let us define the following regions: Choose $\zeta_k : \zeta_0 = 1$, $\zeta_k = 1 + a + (k - 1)\,\varepsilon/2$ for $k \geqq 1$ and $g^{(i)}_k$, $i = 1, 2 : g^{(1)}_0 = \sqrt{(1 + a)}$, $g^{(1)}_k = \sqrt{(\zeta_k + \varepsilon)}$; $g^{(2)}_0 = 1$, $g^{(2)}_k = \sqrt{(\zeta_k - \varepsilon/2)}$ for $k \geqq 1$. Denote by Q the set of points $[\xi, \theta]$ with $\xi \in \langle \zeta_k, \zeta_{k+1})$, $\xi \leqq H$, $\theta \leqq g^{(1)}_k$. Let F_Q be the continuous function which is the solution of (13,1) in any region described by $\xi \in \langle \zeta_k, \zeta_{k+1})$, $\theta \leqq g^{(1)}_k$, with initial conditions $F_Q(\zeta_k, \theta)$ for $\theta \leqq g^{(1)}_{k-1}$, $F_Q(\zeta_k, \theta) = F_Q(\zeta_k, g^{(1)}_{k-1})$ for $\theta \in (g^{(1)}_{k-1}, g^{(1)}_k)$ with boundary condition $(\partial F_Q/\partial \theta)(\xi, g^{(1)}_k) = 0$ for $\xi \in \langle \zeta_k, \zeta_{k+1})$ and with initial condition $F_Q(1, \theta) = F(1, \theta)$ for $\theta \leqq 1$, $F_Q(1, \theta) = F(1, 1)$ for $\theta \in (1, g^{(1)}_0)$ for $k = 0$. Denote by I

the set of points $[\xi, \theta]$ with $\xi \in \langle\zeta_k, \zeta_{k+1}), \xi \leqq H, \theta \leqq g_k^{(2)}$. We shall define a function $F_I(\xi, \theta)$ in I similarly as F_Q in Q. In any region described by $\xi \in \langle\zeta_k, \zeta_{k+1})$, $\theta \leqq g_k^{(2)}$, $F_I(\xi, \theta)$ is the solution of (13,1) with initial conditions $F_I(\zeta_k, \theta)$ for $\theta \leqq g_{k-1}^{(2)}$, $F_I(\zeta_k, \theta) = F_I(\zeta_k, g_{k-1}^{(2)})$ for $\theta \in (g_{k-1}^{(2)}, g_k^{(2)})$, with boundary condition $(\partial F_I/\partial\theta)(\zeta, g_k^{(2)}) = 0$ and with initial condition $F_I(1, \theta) = F(1, \theta)$ for $\theta \leqq 1$, $k = 0$. By Lemma 15 (11,7), the inequality

$$(16,3) \qquad F_{-\varepsilon}(\xi, \theta) \leqq F_I(\xi, \theta) \leqq F_{-a}^{+}(\xi, \theta) \leqq F(\xi, \theta) \leqq F_a^{+}(\xi, \theta) \leqq \\ \leqq F_Q(\xi, \theta) \leqq F_\varepsilon(\xi, \theta)$$

holds, where $F(\xi, \theta)$ is defined in Lemma 15 (3). Define $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$, where v_n is the process defined by (12,1), bounded by (12,2) at $\xi_n^{(i)}$. $F_n^Q(\xi, \theta) = P(v_n^Q(\xi) \leqq \theta)$, where v_n^Q is defined by (12,1) and it is the process bounded by $g^{(1)}(\xi)$ at $\zeta^{(i)}$ and $g^{(1)}(\xi) = g_k^{(1)}$ for $\xi \in \langle\zeta_k, \zeta_{k+1})$. Define $F_n^I(\xi, \theta) = P(v_n^I(\xi) \leqq \theta)$ similarly, where v_n^I is defined by (12,1) and it is the process bounded by $g^{(2)}(\xi)$ at $\zeta^{(i)}$ and $g^{(2)}(\xi) = g_k^{(2)}$ for $\xi \in \langle\zeta_k, \zeta_{k+1})$. For large n, the inequalities

$$(16,4) \qquad F_n^I(\xi, \theta) \leqq F_n(\xi, \theta) \leqq F_n^Q(\xi, \theta)$$

hold (n must be large than $K_n d_n / v_n < \varepsilon/4H$). We can choose n_0 for $\eta > 0$ and $h > 0$ by Lemma 14 such that

$$(16,5) \qquad |F_n^I(\xi, \theta) - F_I(\xi, \theta)| < \eta, \quad |F_n^Q(\xi, \theta) - F_Q(\xi, \theta)| < \eta$$

hold for $n \geqq n_0$ in the regions described by $\xi \in \langle 1 + h, H\rangle$, $[\xi, \theta] \in I$ or by $\xi \in \langle 1 + h, H\rangle$, $[\xi, \theta] \in Q$ respectively. We shall conclude Lemma 16 from the inequalities (16,1)–(16,5). If $F(1, \theta)$ is uniformly continuous, then by Lemma 15 the inequalities (16,1)–(16,5) hold uniformly with respect to all $H \geqq \xi \geqq 1$.

Now we have everything prepared for the formulation of the following theorem.

Theorem 1. *Let there be given sequences of numbers $\delta_n, K_n, v_n, d_n, T_n, \lambda_n$ and a sequence of distributions $F_n(1, \theta)$. Assume that the number sequences satisfy (7)–(10), and that the functions $F_n(1, \theta v_n)$ converge uniformly to $F(\theta)$. Let the function $F(\theta)$ have a continuous derivative at $\theta = 1$ and $(dF/d\theta)(1, 1) = 0$. Set $P_1^{(n)} = P_1(\delta_n, K_n, v_n, d_n, T_n, \lambda_n)$ (see (5) and note that we consider processes belonging to $X(F_n(1, \theta), \delta_n, K_n, d_n, \lambda_n)$.*

Then the limit $\lim_{n\to\infty} P_1^{(n)}$ exists and

$$\lim_{n\to\infty} P_1^{(n)} = 1 - F(e^{2T}, e^T),$$

where $F(\xi, \theta)$ is the bounded solution of the equation

$$\frac{\partial F}{\partial \xi} = \frac{A^2}{4} \frac{\partial^2 F}{\partial \theta^2}$$

in the region described by $\xi > 1$, $\theta < \sqrt{\xi}$ with the initial condition $F(\theta)$ and with the boundary condition $(\partial F/\partial\theta)(\xi, \sqrt{\xi}) = 0$.

Proof. By Lemma 11 we obtain

$$\bar{P}(\sup_{0 \leqq \tau \leqq T_n} x_n(\tau, \omega) \geqq v_n) \leqq P(y_n(\bar{t}_n^{(k)}) \geqq v_n - 3K_n d_n) . \tag{I,1}$$

Let the process v_n corresponds to the process y_n (see 12,1). Then the process v_n is bounded by (12,2) at $\xi_n^{(i)}$ and its initial distribution fulfils (12,3). Evidently

$$P(y_n(\bar{t}_n^{(k)}) \leqq v_n - 3K_n d_n) = P(v_n(\exp\{2\lambda_n \bar{t}_n^{(k)}\}) \leqq \left(1 - \frac{3K_n d_n}{v_n}\right) \exp\{\lambda_n \bar{t}_n^{(k)}\} . \tag{I,2}$$

Put $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$. By Lemma 16, the functions $F_n(\xi, \theta)$ converge to the function $F(\xi, \theta)$ described in Theorem 1. By (I,1), (I,2) and (7)–(10) (the assumptions (7)–(10) remain valid if we substitude $\bar{t}_n^{(k)}$ for T_n because $0 \leqq T_n - \bar{t}_n^{(k)} < d_n$) we obtain that

$$\limsup_{n \to \infty} P_1^{(n)} \leqq 1 - F(e^{2T}, e^T) . \tag{I,3}$$

Conversely, by Lemma 2 we have

$$\sup P^-(\sup_{0 \leqq \tau \leqq T_n} x_n(\tau, \omega) \geqq v_n) \geqq P(y_n^*(T_n) \geqq v_n) , \tag{I,4}$$

where the processes x_n belong to $X(F_n(1, \theta), \delta_n, K_n, d_n, \lambda_n)$ and y_n^* is the process bounded at $v_n e^{-\lambda_n d_n} + K_n d_n$ corresponding to z_n, and z_n is defined in Lemma 10 with initial distribution $F_n(1, \theta)$. (Now $\delta, K, v, d, \xi^{(i)}, \lambda$ depend on n). Let v_n^* be the process which corresponds to y_n^* according to Lemma 12; it is the process bounded by (12,2) at $\xi_n^{(i)}$. Then the following equation

$$P(y_n^*(T_n) \leqq v_n) = P(v_n^*(\exp\{2\lambda_n T_n\}) \leqq \exp\{\lambda_n t_n\}) \tag{I,5}$$

holds. Set $F_n^*(\xi, \theta) = P(v_n^*(\xi) \leqq \theta)$. By Lemma 16, the functions F_n^* converge to the function $F(\xi, \theta)$. By means of (I,4), (I,5) and (7)–(10) we obtain

$$\liminf_{n \to \infty} P_1^{(n)} \geqq 1 - F(e^{2T}, e^T) . \tag{I,6}$$

The theorem follows from the inequalities (I,3) and (I,6).

By means of the transformation $\tau = \lg \xi/2T$, $\mu = \theta/\sqrt{\xi}$, $G(\tau, \mu) = F(\xi, \theta)$ it is easily seen that Theorem 1 is equivalent to the following theorem.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied; then* $\lim_{n \to \infty} P_1^{(n)} = 1 - G(1, 1)$, *where $G(\tau, \mu)$ is the bounded solution of the equation*

$$\frac{\partial G}{\partial \tau} = \mu T \frac{\partial G}{\partial \mu} + \frac{TA^2}{2} \frac{\partial^2 G}{\partial \mu^2}$$

in the region described by $\tau > 0$, $\mu < 1$ *with the initial condition* $G(0, \mu) = F(\mu)$ *and boundary condition* $(\partial G/\partial \mu)(\tau, 1) = 0$.

Remark 4. Evidently TA^2 is the limit of $(\delta_n K_n d_n T_n)/(v_n^2)$. Theorem 2 also applies in the case with $\lambda_n = 0$. In this case set $\lim (\delta_n K_n d_n T_n)/(v_n^2) = a^2$ and then our parabolic equation is $(\partial G/\partial \tau) = \frac{1}{2}a^2(\partial^2 G/\partial \mu^2)$. The region of definition of G and the initial boundary conditions will be the same as above.

Next we shall deal with the expression (6), but we shall formulate our problem more generally. Set

(6′) $$P_3(\delta, K, v^{(1)}, v^{(2)}, d, T, \lambda) = \sup_x \bar{P}\Big(\sup_{0 \leqq \tau \leqq T} x(\tau) \geqq v^{(2)} \,;\ \inf_{0 \leqq \tau \leqq T} x(\tau) \leqq -\, v^{(1)}\Big)\ ^{3)}$$

where $x(t)$ belong to $X(F_0, \delta, K, d, \lambda)$ and $v^{(1)}, v^{(2)}$ are given positive numbers. The following definition is analogous to Definition 2.

Definition 6. Let there be given positive numbers $\delta, \lambda, K, d, v^{(1)}, v^{(2)}$ and a process $x(t, \omega)$. Assume that $x(t, \omega)$ is defined for all $t \geqq 0$. Define $t(\omega) = \min t^{(k)}(\omega)$ where $t^{(k)}(\omega)$ assume only the values $t^{(l)}$ and $x(t^{(k)}(\omega), \omega) \geqq (v^{(2)} - Kd)\, e^{\lambda d}$ or $x(t^{(k)}(\omega), \omega) \leqq \leqq -(v^{(1)} - Kd)\, e^{\lambda d}$; if such $t^{(k)}(\omega)$ do not exist put $t(\omega) = \infty$. *The process x is bounded at* $-v^{(1)}, v^{(2)}$ if $x(t, \omega) = x(t(\omega), \omega)$ for $t \geqq t(\omega)$. *The process* $x_{v^{(1)}v^{(2)}}$ *is the process bounded at* $-v^{(1)}, v^{(2)}$ *corresponding to* x if $x_{v^{(1)}v^{(2)}}(t, \omega) = x(t(\omega), \omega)$ for $t \geqq t(\omega)$, $x_{v^{(1)}v^{(2)}}(t, \omega) = x(t, \omega)$ for $t \leqq t(\omega)$.

The following modification of Lemma 1 is evident:

Lemma 17. *Let* $x \in X(F_0, \delta, K, d, \lambda)$; *then the process* $x_{v^{(1)}v^{(2)}}$ *satisfies the relations* (2)–(4) *for* $-(v^{(1)} - Kd)\, e^{\lambda d} < \mu < (v^{(2)} - Kd)\, e^{\lambda d}$.

The following lemma differs from Lemma 2 only slightly

Lemma 18. *Let* $x \in X(F_0, \delta, K, d, \lambda)$; *then*

$$P(x_{a,b}(t) \geqq v^{(2)};\ x_{a,b}(t) \leqq -v^{(1)}) \leqq P^-\Big(\sup_{0 \leqq \tau \leqq t} x(\tau) \geqq v^{(2)};\ \inf_{0 \leqq \tau \leqq t} x(\tau) \leqq -v^{(1)}\Big) \leqq$$
$$\leqq \bar{P}\Big(\sup_{0 \leqq \tau \leqq t} x(\tau) \geqq v^{(2)};\ \inf_{0 \leqq \tau \leqq t} x(\tau) \leqq -v^{(1)}\Big) \leqq$$
$$\leqq P\big(x_{v^{(1)}v^{(2)}}(\bar{t}^{(k)}) \geqq (v^{(2)} - Kd)\, e^{\lambda d};\ x_{v^{(1)}v^{(2)}}(\bar{t}^{(k)}) \leqq -\,(v^{(1)} - Kd)\, e^{\lambda d}\big),$$

where $a = v^{(1)}e^{-\lambda d} + Kd$, $b = v^{(2)}e^{-\lambda d} + Kd$, *and* $\bar{t}^{(k)}$ *is the maximal* $t^{(k)}$ *such that* $t^{(k)} \leqq t$.

The lemmas analogous to Lemmas 3–5 will not be formulated explicitly because they are very similar to the latter.

---

3) $P(A; B) = P(A \cup B)$.

Definition 7. $X^{\nu^{(1)}\nu^{(2)}}(F_0, \delta, K, d, \lambda)$ is the class of processes with initial distribution F_0 which are bounded at $-\nu^{(1)}, \nu^{(2)}$, satisfy (2)–(4) for $-(\nu^{(1)} - Kd)\, e^{\lambda d} < \mu < (\nu^{(2)} - Kd)\, e^{\lambda d}$ and satisfy $P(x(t) \in A \mid x(t^{(k)}) = \mu,\ x(t^{(k-1)}) = \alpha_{k-1}, \ldots, x(0) = \alpha_0) = P(x(t) \in A \mid x(t^{(k)}) = \mu)$ for all $t \in \langle t^{(k)}, t^{(k+1)} \rangle$ and all B-mesurable $A \subset (-\infty, \infty)$.

Lemmas 6–9 need not be modified, and the necessary changes in Lemma 10 are obvious. The analogue of Lemma 11 shall be formulated only.

Lemma 19. *Let $x \in X(F_0, \delta, K, d, \lambda)$ and let y be the process bounded at $-(\nu^{(1)} - 3Kd)\, e^{\lambda d} + Kd$, $(\nu^{(2)} - 3Kd)\, e^{\lambda d} + Kd$, corresponding to z. The process z is described in Lemma 10. The inequality*

$$\bar{P}\big(\sup_{0 \leqq \tau \leqq t} x(\tau) \geqq \nu^{(2)};\ \inf_{0 \leqq \tau \leqq t} x(\tau) \leqq -\nu^{(1)}\big) \leqq$$
$$\leqq P\big(y(\bar{t}^{(k)}) \geqq \nu^{(2)} - 3Kd;\ y(\bar{t}^{(k)}) \leqq -(\nu^{(1)} - 3Kd)\big)$$

holds, where $\bar{t}^{(k)} \leqq t$, $t - \bar{t}^{(k)} < d$.

Further, we shall introduce the analogue of Definition 5.

Definition 8. Let there be given positve functions f_1, f_2, a sequence $\xi^{(0)} < \xi^{(1)} < \ldots$ and a process $x(\xi, \omega)$. Assume that the process x has values from $\bar{E}$. Denote by $\xi(\omega)$ the minimum of $\xi^{(k)}(\omega)$ where $\xi^k(\omega)$ assume values $\xi^{(l)}$ only and satisfy $x(\xi^{(k)}(\omega), \omega) \geqq f_2(\xi^{(k)}(\omega))$ or $x(\xi^{(k)}(\omega), \omega) \leqq -f_1(\xi^{(k)}(\omega))$; if no such $\xi^k(\omega)$ exists put $\xi(\omega) = \infty$. The process x is bounded by $-f_1, f_2$ at $\xi^{(k)}$ if

$$x(\xi, \omega) = +\infty \quad \text{for} \quad \xi > \xi(\omega) \quad \text{provided that} \quad x(\xi(\omega), \omega) \geqq f_2(\xi(\omega))\,,$$
$$x(\xi, \omega) = -\infty \quad \text{for} \quad \xi > \xi(\omega) \quad \text{provided that} \quad x(\xi(\omega), \omega) \leqq -f_1(\xi(\omega))\,.$$

The process $z(\xi, \omega)$ is bounded by $-f_1, f_2$ at $\xi^{(i)}$ and corresponds to $x(\xi, \omega)$ (for an arbitrary process x) if $z(\xi, \omega) = x(\xi, \omega)$ for $\xi \leqq \xi(\omega)$,

$$z(\xi, \omega) = +\infty \quad \text{for} \quad \xi > \xi(\omega) \quad \text{provided that} \quad x(\xi(\omega), \omega) \geqq f_2(\xi(\omega))\,,$$
$$z(\xi, \omega) = -\infty \quad \text{for} \quad \xi > \xi(\omega) \quad \text{provided that} \quad x(\xi(\omega), \omega) \leqq -f_1(\xi(\omega))\,,$$

where $\xi(\omega)$ is defined above.

It is easy to find the functions which describe the absorbing barriers.

Lemma 20. *Let z be the process defined in Lemma 10. Let y be the process bounded at $-\nu^{(1)}, \nu^{(2)}$ corresponding to z. Denote by v the process which corresponds to y according to Lemma 12. Then v is the process bounded by $f_1(\xi), f_2(\xi)$ at $\xi^{(i)}$, where*

$$f_1(\xi) = \left(\frac{\nu^{(1)}}{\nu^{(2)}} - 3\,\frac{Kd}{\nu^{(2)}}\right)\sqrt{\xi}\,, \quad f_2(\xi) = \left(1 - 3\,\frac{Kd}{\nu^{(2)}}\right)\sqrt{\xi}\,. \tag{20,1}$$

Next we must complete the assumptions (7)–(10).

Hypothesis. Let there be given sequences of positive numbers $\delta_1, \ldots, \delta_n, \ldots, K_1, \ldots$ $\ldots, K_n, \ldots, d_1, \ldots, d_n, \ldots, \lambda_1, \ldots, \lambda_n, \ldots, T_1, \ldots, T_n, \ldots, v_1^{(1)}, \ldots, v_n^{(1)}, \ldots, v_1^{(2)}, \ldots, v_n^{(2)} \ldots$ such that the following conditions are satisfied:

$$\frac{v_1^{(n)}}{v_2^{(n)}} \to \varrho \neq 0\,, \tag{11}$$

$$\frac{K_n \delta_n d_n}{\lambda_n (v_n^{(2)})^2} \to A^2 > 0\,, \quad \delta_n \leqq K_n\,, \tag{12}$$

$$\lambda_n T_n \to T > 0\,, \tag{13}$$

$$\lambda_n d_n \to 0\,, \tag{14}$$

$$\frac{K_n d_n}{v_n^{(2)}} \to 0\,. \tag{15}$$

Lemma 13 remains without change but the proof of Lemma 14 is changed considerably.

Assume we have a finite set of numbers $\zeta^{(\alpha)} : \zeta^{(0)} = 1 < \zeta^{(1)} < \ldots < \zeta^{(q)} = e^{2T}$ and $g_i^{(\alpha)} : 0 < g_i^{(0)} < g_i^{(1)} < \ldots g_i^{(q)}$ for $i = 1, 2$. The functions $f_1(\xi), f_2(\xi)$ – see Definition 8 – we now define by $f_1(\xi) = -\, g_1^{(i)}$ for $\xi \in (\zeta^{(i)}, \zeta^{(i+1)}\rangle$, $f_2(\xi) = g_2^{(i)}$ for $\xi \in (\zeta^{(i)}, \zeta^{(i+1)}\rangle$.

Lemma 21. *Let v_n be the process defined by (12,1) bounded by $-f_1, f_2$ at $\xi_n^{(i)}$. If the assumptions of Lemma 13 are satisfied, $F(1, \theta)$ is continuous from the left at $\theta = g_2^{(0)}$, and the assumptions (7)–(10) are replaced by (11)–(15), then $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$ converge to $F(\xi, \theta)$. The function $F(\xi, \theta)$ is the bounded solution of (13,1) in every region $\mathcal{H}_j$ described by $\xi \in (\zeta^{(j)}, \zeta^{(j+1)}\rangle$, $-g_1^{(j)} \leqq \theta \leqq g_2^{(j)}$, with initial condition $F(\zeta^{(j)}, \theta)$ for $-g_1^{(j-1)} \leqq \theta \leqq g_2^{(j-1)}$ and $F(\zeta^{(j)}, \theta) = F(\zeta^{(j)}, (-1)^i \cdot g_i^{(j-1)})$ for $\theta \in \langle (-1)^i g_i^{(j-1)}, (-1)^i g_i^{(j)} \rangle$, $i = 1, 2$, and with boundary conditions*

$$\frac{\partial F}{\partial \theta}(\xi, -g_1^{(j)}) = \frac{\partial F}{\partial \theta}(\xi, g_2^{(j)}) = 0\,.$$

For $j = 0$ the initial condition is $F(1, \theta)$. The convergence of F_n is uniform in any region $\mathcal{H} \cap \sum_i \mathcal{H}_i$ where the region $\mathcal{H}$ is described by $1 + h \leqq \xi \leqq H$. If $F(1, \theta)$ is uniformly continuous, then the F_n converge uniformly in the region $\sum_i \mathcal{H}_i$.

Proof. We can again restrict ourselves to the first region $\mathcal{H}_0$. Let $\tilde{v}_n$ be the process fulfilling (12,1). Let v_n be the process bounded by $-f_1 = -g_1^{(0)}$, $f_2 = g_2^{(0)}$ at $\zeta^{(i)}$, corresponding to $\tilde{v}_n$. As the initial distribution of $\tilde{v}_n$ we take $F_n(1, \theta)$. We can certainly assume that $F_n(1, -v_n^{(1)}) = 0$ and $\lim_{\theta \to v_n^{(2)}} F_n(1, \theta) = 1$ (since (13,1) is linear we could consider $\tilde{F}_n(1, \theta) = (F_n(1, \theta) - F_n(1, -g_1^{(0)}))/F_n(1, 1)$ instead of F_n). Denote $\eta_n = (K_n d_n / v_n^{(2)})\, e^{\lambda n d v} \sqrt{H}$. As in [1, Lemma 10] one can prove

$$
\begin{aligned}
(21{,}1) \qquad s_{k,n}^{(1)}(\xi, \theta) &= \sum_{j=-k}^{k} [\tilde{F}_n(\xi, 2jg_2^{(0)} + 2jg_1^{(0)} + 2j\eta_n + \theta) + \\
&+ \tilde{F}_n(\xi, 2jg_2^{(0)} + 2(j-1)\, g_1^{(0)} + 2j\eta_n - 3\eta_n - \theta) - 2\tilde{F}_n(\xi, 2jg_2^{(0)} + \\
&+ (2j-1)\, g_1^{(0)} + 2j\eta_n)] - \tilde{F}_n(\xi, 2(k+1)\, g_2^{(0)} + (2k+1)\, g_1^{(0)} + 2(k+1)\, \eta_n) + \\
&+ \tilde{F}_n(\xi, 2(k+1)\, g_2^{(0)} + 2kg_1^{(0)} + 2(k+1)\, \eta_n - 3\eta_n - \theta) \leqq P(-g_1^{(0)} < v_n(\xi) \leqq \theta) = \\
&= P(-g_1^{(0)} < \inf_{1 \leqq \tau \leqq \xi} \tilde{v}_n(\tau),\ \tilde{v}_n(\xi) \leqq \theta\,,\ \sup_{1 \leqq \tau \leqq \xi} \tilde{v}_n(\tau) < g_2^{(0)}) \leqq {}^{4)} \\
&\leqq \sum_{j=-k}^{k} [\tilde{F}_n(\xi, 2jg_2^{(0)} + 2jg_1^{(0)} + 2j\eta_n + \theta) + \tilde{F}_n(\xi, 2jg_2^{(0)} + 2(j-1)\, g_1^{(0)} + \\
&+ 2j\eta_n - \theta) - \tilde{F}_n(\xi, 2jg_2^{(0)} + (2j-1)\, g_1^{(0)} + 2j\eta_n) - \tilde{F}_n(\xi, 2jg_2^{(0)} + \\
&+ (2j-1)\, g_1^{(0)} + 2j\eta_n - 3\eta_n)] + \tilde{F}_n(\xi, 2(k+1)\, g_2^{(0)} + 2(k+1)\, g_1^{(0)} + \\
&+ 2(k+1)\, \eta_n + \theta) - \tilde{F}_n(\xi, 2(k+1)\, g_2^{(0)} + (2k+1)\, g_1^{(0)} + 2(k+1)\, \eta_n) = \\
&= s_{k,n}^{(2)}(\xi, \theta)\,,
\end{aligned}
$$

where $\tilde{F}_n(\xi, \theta) = P(\tilde{v}_n(\xi) \leqq \theta)$. Similarly one obtains

$$
\begin{aligned}
(21{,}2) \qquad s_{k,n}^{(3)}(\xi) &= \sum_{j=-k}^{k} [\tilde{F}_n(\xi, 2jg_2^{(0)} + (2j-1)\, g_1^{(0)} + 2j\eta_n) + \\
&+ \tilde{F}_n(\xi, 2jg_2^{(0)} + (2j-1)\, g_1^{(0)} + 2j\eta_n - 3\eta_n)] + \tilde{F}_n(\xi, 2(k+1)\, g_2^{(0)} + \\
&+ (2k+1)\, g_1^{(0)} + 2(k+1)\, \eta_n - 3\eta_n) + \tilde{F}_n(\xi, 2(k+1)\, g_2^{(0)} + (2k+1)\, g_1^{(0)} + \\
&+ 2(k+1)\, \eta_n) - 2(k+1) \leqq P(v_n(\xi) \leqq -g_1^{(0)}) \leqq 2 \sum_{j=-k}^{k} \tilde{F}_n(\xi, 2jg_2^{(0)} + \\
&+ (2j-1)\, g_1^{(0)} + 2j\eta_n) - 2k = s_{k,n}^{(4)}(\xi)\,.
\end{aligned}
$$

From (21,1) and (21,2) we have

$$
s_{k,n}^{(1)}(\xi, \theta) + s_{k,n}^{(3)}(\xi) \leqq P(v_n(\xi) \leqq \theta) \leqq s_{k,n}^{(2)}(\xi, \theta) + s_{k,n}^{(4)}(\xi)\,.
$$

By Lemma 13, the expressions $s_{k,n}^{(i)}(\xi, \theta)$ and $s_{k,n}^{(i)}(\xi)$ converge to $s_k^{(i)}(\xi, \theta)$ and $s_k^{(i)}(\xi)$ uniformly in any region described by $1 + h \leqq \xi \leqq H$ $(\lim_{n\to\infty} \eta_n = 0)$

$$
\begin{aligned}
s_k^{(1)}(\xi, \theta) + s_k^{(3)}(\xi) &\leqq \liminf_{n\to\infty} P(v_n(\xi) \leqq \theta) \leqq \limsup_{n\to\infty} P(v_n(\xi) \leqq \theta) \leqq \\
&\leqq s_k^{(2)}(\xi, \theta) + s_k^{(4)}(\xi)\,.
\end{aligned}
$$

---

[4]) $P(A, B) = P(A \cap B)$.

Since

$$\lim_{k\to\infty}\left[s_k^{(2)}(\xi,\theta) + s_k^{(4)}(\xi) - s_k^{(1)}(\xi,\theta) - s_k^{(3)}(\xi)\right] = 0$$

uniformly with respect to $1 \leqq \xi \leqq H$, θ, the limit $\lim_{n\to\infty} P(v_n(\xi) \leqq \theta)$ exists and we can write

$$(21,3) \qquad s_k^{**}(\xi,\theta) = \sum_{j=-k}^{k}\left[F(\xi, 2jg_2^{(0)} + 2jg_1^{(0)} + \theta) + F(\xi, 2jg_2^{(0)} + \right.$$
$$\left. + 2(j-1)\,g_1^{(0)} - \theta)\right] + F(\xi, 2(k+1)\,g_2^{(0)} + 2kg_1^{(0)} - \theta) - (2k+1) - \alpha_k \leqq$$
$$\leqq \lim_{n\to\infty} P(v_n(\xi) \leqq \theta) \leqq$$
$$\leqq \sum_{j=-k}^{k}\left[F(\xi, 2jg_2^{(0)} + 2jg_1^{(0)} + \theta) + F(\xi, 2jg_2^{(0)} + 2(j-1)\,g_1^{(0)} - \theta)\right] +$$
$$+ F(\xi, 2(k+1)\,g_2^{(0)} + 2(k+1)\,g_1^{(0)} + \theta) - (2k+1) + \alpha_k = s_k^*(\xi,\theta),$$

where α_k are nonnegative numbers which converge monotonously to zero, and s_k^{**}, s_k^* are solutions of (13,1). If we take $\alpha_k^* = \alpha_k + 2\beta_k$, $\beta_k = \sum_{l=k}^{\infty} \sup_{\langle 1,H\rangle} F(\xi, -2l(g_1^{(0)} + g_2^{(0)}))$ instead of α_k, then the sequence s_k^* is monotone decreasing and the limit $s^*(\xi,\theta) = \lim_{k\to\infty} s_k^*(\xi,\theta)$ exists. By (21,3), $s_k^{**} \leqq s^* \leqq s_k^*$. Since $s_k^*(\xi,\theta) - s_k^{**}(\xi,\theta)$ converge to 0 uniformly with respect to ξ, θ, the function $s^*(\xi,\theta)$ is a bounded solution of (13,1) with initial condition $F(1,\theta)$. There is $(\partial s^*/\partial\theta)(\xi,\theta) = \sum_{j=-\infty}^{\infty}[(\partial s_k^*/\partial\theta)(\xi, 2jg_2^{(0)} + 2jg_1^{(0)} + \theta) - (\partial s_k^*/\partial\theta)(\xi, 2jg_2^{(0)} + 2(j-1)\,g_1^{(0)} - \theta)]$, because the series converges uniformly with respect to all $\xi \geqq 1 + h$. On setting $\theta = -g_1^{(0)}$ or $\theta = g_2^{(0)}$ we obtain the boundary conditions which enter in Lemma 21. From (21,3) we obtain $s^*(\xi,\theta) = \lim_{n\to\infty} P(v_n(\xi) \leqq \theta)$, and the convergence is uniform with regard to ξ, θ, $\xi \geqq 1 + h$, $-g_1^{(0)} \leqq \theta \leqq g_2^{(0)}$. If $F(1,\theta)$ is continuous then it is uniformly continuous and the sequence s_k^* converges uniformly with regard to ξ, θ in the region described by $\xi \geqq 1$, $-g_1^{(0)} \leqq \theta \leqq g_2^{(0)}$.

We shall also need an auxiliery lemma.

Lemma 22. *Let $F(\xi,\theta)$ be the bounded solution of (13,1) in the region described by $\xi \geqq 1$, $-\varrho\sqrt{\xi} \leqq \theta \leqq \sqrt{\xi}$, with initial condition $F(1,\theta)$ and boundary conditions $(\partial F/\partial\theta)(\xi, -\varrho\sqrt{\xi}) = (\partial F/\partial\theta)(\xi, \sqrt{\xi}) = 0$. Assume that $(\partial F/\partial\theta)(\xi,\theta)$ is continuous at $[\xi, -\varrho\sqrt{\xi})$ and at $[\xi, \sqrt{\xi}]$ for $\xi \geqq 1$. The function $F(1,\theta)$ is non-decreasing, non-negative, has a continuous derivative at $\theta = -\varrho$ and at $\theta = 1$ with $(\mathrm{d}F/\mathrm{d}\theta)(1, -\varrho) = (\mathrm{d}F/\mathrm{d}\theta)(1, 1) = 0$. The following statements then hold.*

1) *Solutions are uniquely determined in the class of bounded solutions.*

2) *Solutions with the properties mentioned above exist and*

$$F(\xi, \theta) - F(\xi, -\varrho\sqrt{\xi}) = F^*(\xi, \theta) - F^*(\xi, -\varrho\sqrt{\xi}) + \int_{-\varrho\sqrt{\xi}}^{\theta}\int_{1}^{\xi} \Gamma_y(\xi, \gamma; \lambda, -\varrho\sqrt{\lambda}) \times$$

$$\times F_\theta^*(\lambda, -\varrho\sqrt{\lambda})\, d\lambda\, d\gamma - \int_{-\varrho\sqrt{\xi}}^{\theta}\int_{1}^{\xi} \Gamma_y(\xi, \gamma; \lambda, \sqrt{\lambda})\, F_\theta^*(\lambda, \sqrt{\lambda})\, d\lambda\, d\gamma$$

holds, where $F^*(\xi, \theta)$ is determined by the Poisson formula and $\Gamma(\xi, \gamma; x, y)$ is a Green's function of (13,1) *for our region.*

3) *Let functions $F_\varepsilon(\xi, \theta)$, $F_{-\varepsilon}(\xi, \theta)$, $F_a^+(\xi, \theta)$, $F_{-a}^+(\xi, \theta)$ be defined as in Theorem* 15 *with the following exception. If the function $F_\cdot(\xi, \theta)$ was defined in a region of the type $\theta \leqq \chi(\xi)$, $\chi(\xi) > 0$ then now we shall consider the function $F_\cdot(\xi, \theta)$ which is defined in the region $-\varrho\chi(\xi) \leqq \theta \leqq \chi(\xi)$ (ϱ is defined in* (11)), *and on the new boundary $\theta = -\varrho\chi(\xi)$ the function $F_\cdot(\xi, \theta)$ fulfils a new boundary condition $(\partial F/\partial\theta)(\xi, -\varrho\chi(\xi)) = 0$. Then the following inequalities hold:*

$$F_{-\varepsilon}(\xi, \theta) - F_{-\varepsilon}(\xi, -\varrho\sqrt{(\xi - \varepsilon)}) \leqq F_{-a}^+(\xi, \theta) - F_{-a}^+(\xi, -\varrho\sqrt{\xi}),$$

$$F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho\sqrt{\xi}) \leqq F_\varepsilon(\xi, \theta) - F_\varepsilon(\xi, -\varrho\sqrt{(\xi + \varepsilon)}) \quad \textit{for} \quad \xi \geqq 1 + a,$$

$$F_{-\varepsilon}(\xi, \theta) - F_{-\varepsilon}(\xi, -\varrho) = F_{-a}^+(\xi, \theta) - F_{-a}^+(\xi, -\varrho),$$

$$F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho) = F_\varepsilon(\xi, \theta) - F_\varepsilon(\xi, -\varrho) \quad \textit{for} \quad 1 \leqq \xi \leqq 1 + a.$$

4) *There is*

$$\lim_{\varepsilon \to 0+} [F_\varepsilon(\xi, \theta) - F_\varepsilon(\xi, -\varrho\sqrt{(\xi + \varepsilon)})] = F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho\sqrt{\xi}),$$

$$\lim_{\varepsilon \to 0+} [F_{-\varepsilon}(\xi, \theta) - F_{-\varepsilon}(\xi, -\varrho\sqrt{(\xi - \varepsilon)})] = F_{-a}^+(\xi, \theta) - F_{-a}^+(\xi, -\varrho\sqrt{\xi})$$

uniformly with respect to ξ, θ in the region described by $H \geqq \xi \geqq 1 + a$, $-\varrho\sqrt{\xi} \leqq \leqq \theta \leqq \sqrt{\xi}$.

5) *If we assume only that $F(1, \theta)$ is non-decreasing and continuous at $\theta = -\varrho$ and $\theta = 1$, then*

$$F_{-a}^+(\xi, \theta) - F_{-a}^+(\xi, -\varrho\sqrt{\xi}) \leqq F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho\sqrt{\xi}),$$

$$\lim_{a \to 0+} [F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho\sqrt{\xi}) - F_{-a}^+(\xi, \theta) + F_{-a}^+(\xi, -\varrho\sqrt{\xi})] = 0$$

uniformly in the region described by $\xi \in \langle 1 + h, H \rangle$, $-\varrho\sqrt{\xi} \leqq \theta \leqq \sqrt{\xi}$.

6) *If $F(1, \theta)$ is continuous then the expression from* 5 *converges uniformly to* 0 *in the region described by $\xi \in \langle 1, H \rangle$, $-\varrho\sqrt{\xi} \leqq \theta \leqq \sqrt{\xi}$.*

7) *The inequalities*

$$F^+_{-a}(\xi, \theta) - F^+_{-a}(\xi, -\varrho) \leqq F(\xi, \theta) - F(\xi, -\varrho\sqrt{\xi}) \leqq$$
$$\leqq F^+_a(\xi, \theta) - F^+_a(\xi, -\varrho\sqrt{(1 + a)}) \quad \textit{for} \quad \xi \in \langle 1, 1 + a \rangle\,,$$

$$F^+_{-a}(\xi, \theta) - F^+_{-a}(\xi, -\varrho\sqrt{\xi}) \leqq F(\xi, \theta) - F(\xi, -\varrho\sqrt{\xi}) \leqq$$
$$\leqq F^+_a(\xi, \theta) - F^+_a(\xi, -\varrho\sqrt{\xi}) \quad \textit{for} \quad \xi \geqq 1 + a$$

hold.

8) $F(\xi, \theta)$ *is a non-decreasing function of* θ *for fixed* ξ.

9) *There holds* $F(\xi, \theta) - F(\xi, -\varrho\sqrt{\xi}) = \int_{-\varrho\sqrt{\xi}}^{\theta} G(\xi, \gamma)\,\mathrm{d}\gamma$, *where* $G(\xi, \gamma)$ *is a solution of* (13,1) *with initial condition* $G(1, \theta) = (\mathrm{d}F/\mathrm{d}\theta)(1, \theta)$ *and boundary conditions* $G(\xi, -\varrho\sqrt{\xi}) = G(\xi, \sqrt{\xi}) = 0$ *for* $\xi \geqq 1$.

10) $F(\xi, \sqrt{\xi}) - F(\xi, -\varrho\sqrt{\xi})$ *is a non-increasing function.*

Proof. The proofs of 1) 2) 3) 8) 9) are the same as in Lemma 15. Item 10) is proved with the difference that as the auxiliery function we now take $\hat{F}(\xi, \theta)$, the solution of (13,1) with initial condition $\hat{F}(\xi_1, \theta) = F(\xi_1, \theta)$ for $-\varrho\sqrt{\xi_1} \leqq \theta \leqq \sqrt{\xi_1}$, $\hat{F}(\xi_1, \theta) = F(\xi_1, -\varrho\sqrt{\xi_1})$ for $\theta \leqq -\varrho\sqrt{\xi_1}$ and $\hat{F}(\xi_1, \theta) = F(\xi_1, \sqrt{\xi_1})$ for $\theta \geqq \sqrt{\xi_1}$, and with boundary conditions $(\partial\hat{F}/\partial\theta)(\xi, -\varrho\sqrt{\xi_2}) = (\partial\hat{F}/\partial\theta)(\xi, \sqrt{\xi_2}) = 0$ for $\xi \geqq \xi_1$. The proof of 4 is now easier than in the case of Lemma 15. The proofs of 7 and 2 are also similar to those in Lemma 15. There is a change in the case of item 5 (and hence also in the case of item 6). We apply formula (21,1); passing to the limit for $n \to \infty$ we obtain that the expression

$$F^+_a(1 + a, \theta) - F^+_a(1 + a, -\varrho\sqrt{(1 + a)}) - F^+_{-a}(1 + a, \theta) +$$
$$+ F^+_{-a}(1 + a, -\varrho\sqrt{(1 + a)})$$

may be estimated by means of $\tilde{F}(1 + a, \beta\sqrt{(1 + a)} + \theta) - \tilde{F}(1 + a, \beta + \theta)$ and by one expression $U_k = \tilde{F}(1 + a, 2(k + 1)(1 + \varrho)\sqrt{(1 + a)} + \theta) - \tilde{F}(1 + a, 2(k + 1) + 2k\varrho - \theta)$, where $\tilde{F}$ corresponds to $F(1, \theta)$ by the Poisson formula, β are suitable combinations of $j_1 + j_2\varrho$ (j_1, j_2 are integers) and $\lim_{k\to\infty} U_k = 0$. We obtain a finite number of these new expressions and we can estimate them as in Lemma 15.

The following lemma must be proved because its proof differs from that of Lemma 16.

Lemma 23. *Let* v_n *be given by* (12,1) (*where* d, $\xi^{(i)}$, v, δ, K, λ *depend on* n) *and* v_n *is bounded by* (20,1) *at* $\xi_n^{(i)}$. *Assume that the* $F_n(1, \theta)$ *converge uniformly to* $F(1, \theta)$ *and that* $F(1, \theta)$ *has continuous derivatives at* $\theta = -\varrho$, $\theta = 1$ *and* $(\mathrm{d}F/\mathrm{d}\theta)(1, -\varrho) = (\mathrm{d}F/\mathrm{d}\theta)(1, 1) = 0$. *Set* $F_n(\xi, \theta) = P(v_n(\xi) \leqq \theta)$. *Then the functions*

$$F_n(\xi, \theta) - F_n\left(\xi, -\left(\frac{v_n^{(1)}}{v_n^{(2)}} - 3\,\frac{K_n d_n}{v_n^{(2)}}\right)\sqrt{\xi}\right)$$

converge to $F(\xi, \theta) - F(\xi, -\varrho\sqrt{\xi})$, *where* $F(\xi, \theta)$ *is the bounded solution of* (13,1) *defined in the region decribed by* $\xi \geqq 1$, $-\varrho\sqrt{\xi} \leqq \theta \leqq \sqrt{\xi}$ *with initial condition* $F(1, \theta)$ *and with boundery conditions* $(\partial F/\partial\theta)(\xi, -\varrho\sqrt{\xi}) = (\partial F/\partial\theta)(\xi, \sqrt{\xi}) = 0$. *The convergence of*

$$F_n(\xi, \theta) - F_n\left(\xi, -\left(\frac{v_n^{(1)}}{v_n^{(2)}} - 3\frac{K_n d_n}{v_2^{(n)}}\right)\sqrt{\xi}\right)$$

is uniform with respect to ξ, θ *in the region* $1 + h \leqq \xi \leqq H$, $-\varrho\sqrt{\xi} \leqq \theta \leqq \sqrt{\xi}$. *If* $F(1, \theta)$ *is continuous then the convergence is uniform in the region* $1 \leqq \xi \leqq H$, $-\varrho\sqrt{\xi} \leqq \theta \leqq \sqrt{\xi}$.

Proof. Let us choose the numbers $\eta > 0$, $h > 0$. By Lemma 22 (5) a number $a > 0$ exists such that

$$(23,1) \qquad |F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho\sqrt{\xi}) - F_{-a}^+(\xi, \theta) + F_a^+(\xi, -\varrho\sqrt{\xi})| < \eta$$

for $\xi \in \langle 1 + h, H\rangle$. According to Lemma 22 (4) we can choose $\varepsilon > 0$ such that

$$(23,2) \qquad F_\varepsilon(\xi, \theta) - F_\varepsilon(\xi, -\varrho\sqrt{(\xi + \varepsilon)}) \leqq F_a^+(\xi, \theta) - F_a^+(\xi, -\varrho\sqrt{\xi}) + \eta$$

$$F_{-a}^+(\xi, \theta) - F_{-a}^+(\xi, -\varrho\sqrt{\xi}) \leqq F_{-\varepsilon}(\xi, \theta) - F_{-\varepsilon}(\xi, -\varrho\sqrt{(\xi - \varepsilon)}) + \eta$$

Next, let $\zeta_k : \zeta_0 = 1$, $\zeta_k = 1 + a + (k - 1)\varepsilon/2$ for $k \geqq 1$

$$g_k^{(1)} : g_0^{(1)} = \sqrt{(1 + a)},\ g_k^{(1)} = \sqrt{(\zeta_k + \varepsilon)} \quad \text{for} \quad k \geqq 1\,;$$
$$g^{(1)}(\xi) : g^{(1)}(\xi) = g_k^{(1)} \quad \text{for} \quad \xi \in \langle\zeta_k, \zeta_{k+1})$$

$$g_k^{(2)} : g_0^{(2)} = \varrho\sqrt{(1 + a)},\ g_k^{(2)} = \varrho\sqrt{(\zeta_k + \varepsilon)} \quad \text{for} \quad k \geqq 1\,;$$
$$g^{(2)}(\xi) : g^{(2)}(\xi) = g_k^{(2)} \quad \text{for} \quad \xi \in \langle\zeta_k, \zeta_{k+1})$$

$$g_k^{(3)} : g_0^{(3)} = 1,\ g_k^{(3)} = \sqrt{(\zeta_k - \varepsilon/2)} \quad \text{for} \quad k \geqq 1\,;$$
$$g^{(3)}(\xi) : g^{(3)}(\xi) = g_k^{(3)} \quad \text{for} \quad \xi \in \langle\zeta_k, \zeta_{k+1})$$

$$g_k^{(4)} : g_0^{(4)} = \varrho,\ g_k^{(4)} = \varrho\sqrt{(\zeta_k - \varepsilon/2)} \quad \text{for} \quad k \geqq 1\,;$$
$$g^{(4)}(\xi) : g^{(4)}(\xi) = g_k^{(4)} \quad \text{for} \quad \xi \in \langle\xi_k, \zeta_{k+1})\,.$$

The region Q is the set of points $[\xi, \theta]$ such that $-g^{(2)}(\xi) \leqq \theta \leqq g^{(1)}(\xi)$. The region I is the set of point $[\xi, \theta]$ such that $-g^{(4)}(\xi) \leqq \theta \leqq g^{(3)}(\xi)$. The process v_n is defined in accordance with (12,1) and it is bounded by (20,1) at $\xi_n^{(i)}$. The processes v_n^Q and v_n^I are also defined by means of (12,1) but they are bounded by $-g^{(2)}(\xi)$, $g^{(1)}(\xi)$ or by $-g^{(4)}(\xi)$, $g^{(3)}(\xi)$ respectively at $\zeta^{(i)}$. The distributions F_n, F_n^I, F_n^Q, F_I, F_Q are defined in the same way as in Lemma 16. For sufficiently large n (such that $3K_n d_n/v_n^{(2)} < \varepsilon/4H$, $|(v_n^{(1)} - \varrho v_n^{(2)} - 3K_n d_n)/v_n^{(2)}| < \varrho\varepsilon/4H$), the inequalities

$$(23,3) \qquad F_n^I(\xi, \theta) - F_n^I(\xi, -g^{(4)}(\xi)) \leqq F_n(\xi, \theta) - F_n(\xi, -f_1(\xi)) \leqq$$
$$\leqq F_n^Q(\xi, \theta) - F_n^Q(\xi, -g^{(2)}(\xi))\,.$$

evidently hold. By Lemma 21 there exists a number n such that

$$(23,4)\qquad \left|F_n^I(\xi,\theta) - F_n^I(\xi,-g^{(4)}(\xi)) - F_I(\xi,\theta) + F_I(\xi,-g^{(4)}(\xi))\right| < \eta$$
$$\left|F_n^Q(\xi,\theta) - F_n^Q(\xi,-g^{(2)}(\xi)) - F_Q(\xi,\theta) + F_Q(\xi,-g^{(2)}(\xi))\right| < \eta\,.$$

According to the definition of the regions I, Q we obtain

$$(23,5)\qquad F_{-\varepsilon}(\xi,\theta) - F_{-\varepsilon}(\xi,-\varrho\sqrt{(\xi-\varepsilon)}) \leqq F_I(\xi,\theta) - F_I(\xi,-g^{(4)}(\xi))$$
$$F_Q(\xi,\theta) - F_Q(\xi,-g^{(2)}(\xi)) \leqq F_\varepsilon(\xi,\theta) - F_\varepsilon(\xi,-\varrho\sqrt{(\xi+\varepsilon)}) \quad \text{for} \quad \xi \geqq 1 + a$$

Lemma 23 follows immediately from the inequalities (23,1)–(23,5) and Lemma 22 (7). If $F(1,\theta)$ is continuous we apply 6) of Lemma 22 to the inequality (23,1), and to the inequality (23,2) we may apply that the equations $F_\varepsilon = F_a^+$, $F_{-\varepsilon} = F_{-a}^+$ hold for $\xi \in \langle 1, 1 + a\rangle$.

From the preceding Lemmas we may conclude the following theorems.

Theorem 3. *Let there be given sequences of numbers δ_n, K_n, $v_n^{(1)}$, $v_n^{(2)}$, d_n, T_n, λ_n and a sequence of distributions $F_n(1,\theta)$. Assume that the number sequences satisfy* (11)–(15) *and assume that the distributions $F_n(1, \theta\, v_n^{(2)})$ converge uniformly to $F(\theta)$. Let the function $F(\theta)$ have a continuous derivative at $\theta = -\varrho$ and at $\theta = 1$ and assume that $(\mathrm{d}F/\mathrm{d}\theta)(1,-\varrho) = (\mathrm{d}F/\mathrm{d}\theta)(1,1) = 0$ hold (we consider now processes belonging to $X(F_n(1,\theta), \delta_n, K_n, d_n, \lambda_n)$. Set $P_3^{(n)} = P_3(\delta_n, K_n, v_n^{(1)}, v_n^{(2)}, d_n, T_n, \lambda_n)$. Then the limit $\lim_{n\to\infty} P_{(3)}^n$ exists and the formula*

$$\lim_{n\to\infty} P_{(3)}^n = 1 - F(e^{2T}, e^T) + F(e^{2T}, -\varrho e^T)$$

holds, where $F(\xi,\theta)$ is the bounded solution of the equation

$$\frac{\partial F}{\partial \xi} = \frac{A^2}{4}\frac{\partial^2 F}{\partial \theta^2}$$

in the region $\xi > 1$, $-\varrho\sqrt{\xi} < \theta < \sqrt{\xi}$ with initial condition $F(\theta)$ and with boundary conditions $(\partial F/\partial\theta)(\xi, -\varrho\sqrt{\xi}) = (\partial F/\partial\theta)(\xi,\sqrt{\xi}) = 0$.

By means of the transformation $\tau = \lg \xi/2T$, $\mu = \theta/\sqrt{\xi}$, $G(\tau,\mu) = F(\xi,\theta)$ it is easy to prove that Theorem 3 is equivalent to the following theorem.

Theorem 4. *Let the conditions of Theorem 3 be satisfied; then $\lim_{n\to\infty} P_3^{(n)} = 1 - G(1,1) + G(1,-\varrho)$, where $G(\tau,\mu)$ is the bounded solution of the equation*

$$\frac{\partial G}{\partial \tau} = \mu T \frac{\partial G}{\partial \mu} + \frac{TA^2}{2}\frac{\partial^2 G}{\partial \mu^2}$$

in the region $\tau > 0$, $-\varrho < \mu < 1$ *with initial condition* $G(0, \mu) = F(\mu)$ *and with boundary conditions* $(\partial G/\partial \mu)(\tau, -\varrho) = (\partial G/\partial \mu)(\tau, 1) = 0$.

Remark 5. If the limit $\lim_{n\to\infty} (\delta_n K_n d_n T_n/(v_n^{(2)})^2) = a^2$ exists, then as in the Remark 4, we obtain the equation $\partial G/\partial \tau = (a^2/2)(\partial^2 G/\partial \mu^2)$.

II

In this part we shall deal with the application of the results obtained above. These were proved in such a manner that we can proceed in two different ways. First we may consider the discrete case when the perturbations are effective only at certain instants $t^{(k)}$ so that $x(t^{(k)} + 0) - x(t^{(k)})$ may be different from zero. Secondly we can consider the continuous case when the perturbations act in the entire intervals $\langle t^{(k)}, t^{(k+1)})$ and they are bounded.

In the discrete case we would need the theory of distributions if we wanted to use differential equations of the type (1). Nevertheless we can utilize the fact that in this case we may describe the perturbations S by means of the differences $x(t^{(k)} + 0, \omega) - x(t^{(k)}, \omega)$. Since at the interior points of $\langle t^{(k)}, t^{(k+1)}\rangle$ the perturbations are not effective we may define the solution in the following way. The process $x(t, \omega)$ is a solution of the equation

$$(1') \qquad \dot{x} = -\lambda x + S(t, x, \omega)$$

if $x(t, \omega)$ is a solution of the equation $\dot{x} = -\lambda x$ for almost all fixed ω and for $t \in (t^{(k)}, t^{(k+1)})$ and if the distributions of $x(t^{(k)} + 0, \omega) - x(t^{(k)}, \omega)$ are prescribed.

The following assumptions are similar to those in [1].

1) The perturbations are in a certain sense small:

$$(16) \qquad E(|x(t^{(k)} + 0, \omega) - x(t^{(k)}, \omega)| \mid x(t^{(k)}, \omega), \ldots, x(0, \omega)) \leqq \delta d\,.$$

The conditional expectation means that the estimate holds independently of the behaviour in the past. (i.e. for $t < t^{(k)}$) The coefficient d in (16) guarantees that the influence of an individual perturbation (that is in one moment $t^{(k)}$), decreases if the density of the points $t^{(k)}$ in $\langle 0, T\rangle$ increases.

The assumption that "systematic error" cannot arise we can again write as

$$(17) \qquad E(x(t^{(k)} + 0, \omega) - x(t^{(k)}, \omega) \mid x(t^{(k)}, \omega), \ldots, x(0, \omega)) = 0\,.$$

The last condition has the same meaning as that in [1]

$$(18) \qquad P(|x(t^{(k)} + 0, \omega) - x(t^{(k)}, \omega)| > Kd \mid x(t^{(k)}, \omega), \ldots, x(0, \omega)) = 0\,.$$

By a slight modification we can easily prove that the conditions (2)–(4) follow from (16)–(18).

Assume that we have a sequence of differential equations (1). If the assumptions (16)–(18) are satisfied then Theorems 1–4 hold for solutions of (1).

The continuous case. First we must assume that:

$S(t, x, \omega)$ is a random variable on Ω for any t, x.

$S(t, x, \omega)$ satisfies Carathéodory's conditions for almost all ω.

$S(t, x, \omega)$ is mesurable in $E \times \Omega$ for every x.

These conditions are suffisant for the existence of solutions of equation (1').

Consider now the simplest case, i.e. that the perturbations $S(t, x, \omega)$ depend only on ω and on the intervals $\langle t^{(k)}, t^{(k+1)})$. Set $S_k(\omega) = S(t, x, \omega)$ for $t \in \langle t^{(k)}, t^{(k+1)})$. $S_k(\omega)$ are now random variables. Then it is natural to require the following conditions

$$(19) \qquad E(|S_k(\omega)|) \leqq \delta\,, \quad E(S_k(\omega)) = 0\,, \quad |S_k(\omega)| \leqq K\,.$$

If we add the assumption that the random variables $S_k(\omega)$, $k = -1, 0, 1, \ldots$ are independent ($S_{-1}(\omega)$ is the random variable $x(0, \omega)$), we may derive the conditions (2)–(4) from (19) ane Theorems 1–4 hold for solutions of (1).

The conditions are easily formulated, too, if S depends only on t, ω. In this case we shall assume that

$$(20) \qquad E(|S(t, \omega)|) \leqq \delta\,, \quad E(S(t, \omega)) = 0\,, \quad |S(t, \omega)| \leqq K\,.$$

Denote by $\mathscr{F}^{(k)}$ the least σ-field which corresponds to random variables $S(\tau, \omega)$, where τ is a number from $\langle t^{(k)}, t^{(k+1)})$; $\mathscr{F}^{(-1)}$ is the least σ-field which corresponds to $x(0, \omega)$. Now we shall assume that the σ-fields are independent (i.e. $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$ if $A_i \in \mathscr{F}^{(i)}$).

It is possible to investigate the general case similarly to that in [1].

References

[1] *I. Vrkoč:* Über eine Klasse der zufälligen Prozesse mit absorbierenden Barrieren. Čas. pěst. mat. *89* (1964), 402–425.

[2] *М. Иржина* (M. Jiřina): Условия вероятности на алгебрах со счетным базисом. Чех. мат. ж. 1954, *79*, 372–380.

[3] *Ф. Трикоми* (F. Tricomi): Лекции по уравнениях в частных производных. Изд. ин. лит. Москва 1957.

[4] Р. *Выборны* (R. Výborný): О некоторых основных свойствах решения краевых задач для дифференциальных уравнений в частных проиводных параболического типа. Чех. мат. ж. т. *8* (*83*) 1958, 537–551.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).

Резюме

ОБ ОДНОРОДНОМ ЛИНЕЙНОМ ДИФФЕРЕНЦИАЛЬНОМ УРАВНЕНИИ СО СЛУЧАЙНЫМИ ВОЗМУЩЕНИЯМИ

ИВО ВРКОЧ, (Ivo Vrkoč), Прага

В работе находятся предельное выражения для

$$\sup_{x_n} \bar{P}\big(\sup_{\langle 0, T_n \rangle} x_n(\tau, \omega) \geqq v_n \big), \quad \sup_{x_n} \bar{P}\big(\sup_{\langle 0, T_n \rangle} |x_n(\tau, \omega)| \geqq v_n \big),$$

где $\bar{P}(\sup_{\langle 0, T_n \rangle} x_n(\tau, \omega) \geqq v_n)$, $\bar{P}(\sup_{\langle 0, T_n \rangle} |x_n(\tau, \omega)| \geqq v_n)$ значат вероятности с которыми процесс x_n или $|x_n|$ переходит по крайной мере один раз границу v_n в итервале $\langle 0, T_n \rangle$, и x_n пробегает группу случайных процессов $X(F_n, \delta_n, K_n, d_n, \lambda_n)$. В конце работы используются эти результаты для дифференциальных уравнений $\dot{x} = -\lambda_n x + S_n(t, x, \omega)$, причем $S_n(t, x, \omega)$ случайный процесс, который выражает возмущения. Приведены условия при которых можно здесь использовать теоремы 1.—4.