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DEPENDENCE OVER MODULES

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1. INTRODUCTION

On the lines of the "classical" theory of linear dependence over vector spaces a satisfactory theory of dependence over abelian groups has been worked out in which the concept of rank plays a focal role (see R. BAER [1], T. SZELE [12], V. DLAB [2], [3]). The present paper is devoted to the problem of extending the theory of dependence over abelian groups to modules (in this way, both dependence over vector spaces and over abelian groups are studied within the frame of a general theory); the problem includes, in particular, the question on possibility of introducing an (invariant) rank of a module. This question has been dealt with in the paper [9] of L. FUCHS; the conditions assigned there to the rings of operators are rather of "commutative" character. In the present paper, we apply the theory of GA-dependence and LA-dependence structures of [4], [5] and [7] and get a theory of dependence over modules in which the results of [9] are included and generalized. In order to illustrate the character of our results, let us mention that we establish a necessary and sufficient condition for an (associative) ring R in order that modules over R admit a dependence theory similar to that of abelian groups, a necessary and sufficient condition for R in order that any two maximal independent subsets of an arbitrary module over R have the same cardinality, etc. (cf. [6]).

Here, a brief comment on the definition of the dependence relation over modules might be useful. Let M be a (left) module over an (associative) ring R. According to the commonly used definition, an element $x \in M$ is said to be dependent on a subset X of M if $\lambda, \lambda_i \in R$ and $x_i \in X$ for $1 \leqq i \leqq m$ exists such that

$$(1,1) \qquad 0 \neq \lambda x = \sum_{i=1}^{m} \lambda_i x_i .$$

This is a very natural definition in the case of unitary modules; otherwise, it relates rather to the behaviour of operators than ot the structure of a module. For, if R is unitary, (1,1) is equivalent to the fact that the intersection of the submodules generated by (x) and X, respectively, is non-zero; the latter condition seems to be an appropriate expression of what we agree to define as x depends on X in general. In this sense, the dependence relation over modules is used throughout the paper. Besides, within our theory, this concept covers also the linear dependence relation over modules (strong dependence in terminology of [9]) defined similarly to the above definition — where

$$\lambda x = \sum_{i=1}^{m} \lambda_i x_i \quad \text{with} \quad \lambda \neq 0$$

instead of (1,1) should only be read. Let us remark, however, that our method consists in translating the study of (general) modules to the study of unitary modules (see § 2). Finally, let us point out that — following the pattern of § 9 of [4] concerning abelian groups — dependence relations with respect to given submodules of a module M can be defined over M.

2. PRELIMINARIES

It is the purpose of this paragraph to summarize some concepts and results on algebraic dependence structures (cf. [4], [5] and [7]), on rings and modules, as well as to introduce terminology and notation explored in the present paper.

For a binary relation ϱ between elements x and subsets X of a non-void set $\bar{S}$ (in symbols, $[x, X] \in \varrho$ or $[x, X] \notin \varrho$), we define the subsets S_ϱ^N and S_ϱ^S of all *ϱ-neutral* and all *ϱ-singular* elements of $\bar{S}$ as the subsets of the elements $x \in \bar{S}$ such that $[x, X] \notin \varrho$ and $[x, X] \in \varrho$ for every $X \subseteq \bar{S}$, respectively. In what follows, we shall consider the ϱ-regular part S of $\bar{S}$: $S = S_\varrho = \bar{S} \setminus (S_\varrho^N \cup S_\varrho^S)$. Denote by $\mathscr{I}_\varrho$ the family of all *ϱ-independent* subsets of S, defined in the usual way. The relation ϱ is said to be an *algebraic dependence relation* (A-*dep. relation*) and (S, ϱ) to be an *algebraic dependence structure* (A-*dep. structure*) if the following conditions

(F) $$[x, X] \in \varrho \rightarrow \exists F(F \subseteq X \wedge F \textit{ finite} \wedge [x, F] \in \varrho),$$

(M) $$X_1 \subseteq X_2 \wedge [x, X_1] \in \varrho \rightarrow [x, X_2] \in \varrho$$

and

(E_r) $$x_1 \notin X \wedge X \in \mathscr{I}_\varrho \wedge [x_1, X] \notin \varrho \wedge [x_1, X \cup (x_2)] \in \varrho \rightarrow [x_2, X \cup (x_1)] \in \varrho$$

are satisfied. It is called *proper* if, moreover,

(I) $$x \in X \rightarrow [x, X] \in \varrho$$

holds. A subset $I \in \mathscr{I}_\varrho$ is called *ϱ-canonic* in S if

(T_r) $$x \notin X \wedge X \in \mathscr{I}_\varrho \wedge [x, I] \in \varrho \wedge \forall y(y \in I \rightarrow [y, X] \in \varrho) \rightarrow [x, X] \in \varrho .$$

By a *ϱ-canonic region* C of S we understand a substed $C \subseteq S$ such that every ϱ independent subset of C is ϱ-canonic in S. A ϱ-canonic region S^C of S is said to be a *ϱ-canonic zone* of S, or a *strict ϱ-canonic zone* of S, if for every $x \in S$ there is a ϱ-canonic subset $I \subseteq S^C$, or an element $y \in S^C$, such that

$$[x, I] \in \varrho, \quad \text{or} \quad [x, (y)] \in \varrho,$$

respectively. An A-dep. structure (S, ϱ) is called a GA-*dep. structure* (a *strict* GA-*dep. structure*) if there exists a ϱ-canonic zone of S (a strict ϱ-canonic zone of S); (S, ϱ) is called a LA-*dep. structure* if the entire S is a ϱ-canonic zone of S; then, ϱ is called a GA-*dep. relation* or a LA-*dep. relation*, respectively. Though, in general, a subset of a GA-dep. structure (S, ϱ) together with the corresponding relation induced on the subset by ϱ need not be a GA-dep. structure, a subset of a LA-dep. structure is, in the sense described above, always a LA-dep. structure – a LA-dep. substructure of the given LA-dep. structure.

In a GA-dep. structure, there are maximal independent subsets which are canonic. The cardinalities of any two such maximal canonic subsets are equal (the *rank* of the GA-dep. structure); they are no maximal independent subsets of a greater cardinality. In particular, all maximal independent subsets of an LA-dep. structure are of the same cardinality.

Let R be an (associative) ring. By the DORROH's extension R^* of R (see [8]) we understand the ring with identity $R^* = Z \times R$ of all pairs (m, μ), where m is an integer and μ an element of R, with the operations defined as follows:

$$(m, \mu) + (n, \nu) = (m + n, \mu + \nu),$$
$$(m, \mu) \, . \, (n, \nu) = (mn, m \times \nu + n \times \mu + \mu\nu).$$

We identify R with the subring of R^* of all elements of the form $(0, \mu)$, $\mu \in R$; in fact, R is a two-sided ideal of R^*.

Let M be a (left) module over R (briefly, an R-module). By $\{X\}_R$ denote the R-submodule of M generated by a subset $X \subseteq M$. The *order* (annihilator) of an element $x \in M$, i.e. the left ideal of R of all $\mu \in R$ such that $\mu x = 0$, will be denoted by $O(x)$ or, more precisely, by $O_R(x)$. Defining, for $(m, \mu) \in R^*$ and $x \in M$,

$$(m, \mu) \, x = m \times x + \mu x,$$

M can be considered also as a unitary (left) R^*-module. Then, we have $\{X\}_R = \{X\}_{R^*}$ for every $X \subseteq M$. Thus, the family of all R-submodules of M coincides with the family of all R^*-submodules of M.

In an R-module M, let us define the dependence relation δ in the following way:

(2,1) $$[x, X] \in \delta \leftrightarrow \{x\}_R \cap \{X\} \neq (0).$$

The subset of the δ-singular elements is void and there is a single δ-neutral element

$0 \in M$. In what follows, speaking about an R-module M as a dependence structure with respect to δ we shall always mean the structure $(M \setminus (0), \delta)$. The relation δ induces a relation δ_S on every subset S of M,

$$[x, X] \in \delta_S \leftrightarrow [x, X] \in \delta \quad \text{for} \quad x \in S \quad \text{and} \quad X \subseteq S\,,$$

and we can, thus, investigate also the structures (S, δ). [1]) The definition (2,1) of the relation δ on M does not depend on whether M is considered as an R-module or as an R^*-module. Hence, any such relation can be investigated as a relation over a unitary (left) module.

Unless otherwise stated, by an ideal of a ring R we understand always a left ideal. An ideal L of a ring R is said to be *irreducible* if, for any two ideals $L_1 \supsetneqq L$ and $L_2 \supsetneqq L$, the strict inclusion $L_1 \cap L_2 \supsetneqq L$ holds. An ideal $L \subseteq R$ is called *prime* if, for every $\mu \in R \setminus L$ and $\nu \in R \setminus L$, also $\mu\nu \in R \setminus L$. For an ideal $L \subseteq R$ and $\varkappa \in R$ we define the (*right*) *ideal-quotient* $L : \varkappa$ of L by $\varkappa$ as the set (left ideal) of all $\chi \in R$ such that $\chi\varkappa \in L$.

The following lemmas will be found useful in the next paragraphs.

Lemma 2.1 *An ideal L of a ring R is irreducible if and only if for every $\alpha \in R \setminus L$, $\beta \in R \setminus L$ there are $\mu \in R$, $\nu \in R$ such that*

$$(2,2) \qquad \begin{cases} (m \times \alpha + \mu\alpha) - (n \times \beta + \nu\beta) \in L \quad \text{and} \\ m \times \alpha + \mu\alpha \notin L \quad \text{for suitable integers } m, n. \end{cases}$$

If R is a ring with identity, then the condition (2,2) *is to be read*

$$\mu\alpha - \nu\beta \in L \quad \text{and} \quad \mu\alpha \notin L\,.$$

Proof. Let L be irreducible, $\alpha \notin L$, $\beta \notin L$. Then the intersection of the ideals $\{L, \alpha\}$ and $[L, \beta]$ contains an element $\varkappa \notin L$. Thus,

$$\varkappa = \lambda_1 + m \times \alpha + \mu\alpha = \lambda_2 + n \times \beta + \nu\beta \quad \text{with} \quad m \times \alpha + \mu\alpha \notin L\,.$$

Hence

$$(m \times \alpha + \mu\alpha) - (n \times \beta + \nu\beta) = \lambda_2 - \lambda_1 \in L\,,$$

as required.

On the other hand, if L is reducible, then

$$L = L_1 \cap L_2 \quad \text{with} \quad L_1 \neq L \neq L_2\,.$$

Take $\alpha \in L_1 \setminus L$ and $\beta \in L_2 \setminus L$ and consider an element

$$\lambda = (m \times \alpha + \mu\alpha) - (n \times \beta + \nu\beta) \in L\,.$$

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[1]) There is no danger of confusion in denoting the relation δ_S again simply by δ.

Then,

$$m \times \alpha + \mu\alpha = \lambda + (n \times \beta + \nu\beta) \in L_1 \cap L_2 = L.$$

The lemma follows.

Lemma 2.2. *If an ideal L of R is irreducible, then every ideal-quotient $L : \varkappa$ is irreducible.*

Proof. Let us give an indirect proof based on the previous Lemma 2,1. Assuming that, for a certain $\varkappa \in R$, $L : \varkappa$ is reducible, we deduce the existence of $\alpha \notin L : \varkappa$, $\beta \notin L : \varkappa$ such that

$$(m \times \alpha + \mu\alpha) - (n \times \beta + \nu\beta) \in L : \varkappa \quad \text{implies} \quad m \times \alpha + \mu\alpha \in L : \varkappa$$

for every $m, n \in Z$, $\mu, \nu \in R$. Hence, $\alpha\varkappa \notin L$, $\beta\varkappa \notin L$ and

$$(m \times (\alpha\varkappa) + \mu\alpha\varkappa) - (n \times (\beta\varkappa) + \nu\beta\varkappa) \in L \quad \text{implies} \quad m \times (\alpha\varkappa) + \mu\alpha\varkappa \in L$$

for every $m, n \in Z$, $\mu, \nu \in R$ — a contradiction, in view of Lemma 2,1, of irreducibility of L.

Lemma 2.3. *Let M be an R-module and $x \in M$. Then*

(i) *$O(\varkappa x) = O(x) : \varkappa$ for every $\varkappa \in R$;*
(ii) *$O(x)$ is a two-sided ideal if and only if $O(x) \subseteq O(\varkappa x)$ for every $\varkappa \in R$;*
(iii) *$O(x)$ is prime if and only if $O(x) \supseteq O(\varkappa x)$ for every $\varkappa \in R \setminus O(x)$.*

Proof. (i) is trivial. Further, $O(x)$ is two-sided if and only if, for every $\chi \in O(x)$ and $\varkappa \in R$, $\chi\varkappa \in O(x)$, i.e. $\chi \in O(x) : \varkappa = O(\varkappa x)$ holds. Also, $O(x)$ is prime if and only if, for every $\varkappa \in R \setminus O(x)$, $\chi \in O(\varkappa x)$, i.e. $\chi\varkappa \in O(x)$, implies $\chi \in O(x)$. The proof is completed.

Lemma 2.4. *Let L be an ideal of a ring R. Then there exists an R-module M containing an element $x \in M$ of order $O(x) = L$. If R is a ring with identity, then there is a unitary R-module M of that property.*

Proof. Let ε be the identity element of R. Then the coset represented by ε in the R-module of all cosets R mod L establishes the proof. In the general case, consider the Dorroh's extension R^* of R with the identity ε^*. Every ideal of R is an ideal of R^*. Again, it is easy to check that the order of the coset represented by ε^* in the R-module of all cosets R^* mod L is L.

Finally, let us introduce some definitions in order to simplify the formulations of the following paragraphs.

Definition 2.5. A ring R is said to have property $(\mathscr{I})$ if for every proper ideal L of R (i.e. $L \neq R$) there exists $\varkappa \in R \setminus L$ such that the ideal-quotient $L : \varkappa$ is irreducible. A ring is said to have property $(\mathscr{L})$ if its family of ideals is (linearly) ordered by inclusion.

Definition 2.6. A ring R with identity is said to possess property ($\check{G}$), or ($\check{L}$), if every unitary R-module is, with respect to the dependence relation δ, a GA-dep. structure, or a LA-dep. structure, respectively. A ring R is said to posses property (G) if every R-module is a GA-dep. structure.

Remark 2.7. Evidently, there is no ring R such that every R-module is a LA-dep. structure. For, any abelian group G can be considered as an R-module if $\chi g = 0$ is defined for any $\varkappa \in R$ and $g \in G$. Then, the family of all R-submodules coincides with the family of all subgroups and thus, the relation δ coincides with the dependence relation over abelian groups which is, in general, not a LA-dep. relation.

3. DEPENDENCE OVER UNITARY MODULES

Throughout this paragraph, R is always an (associative) ring with identity ε and M a unitary R-module. First, formulate the following basic

Theorem 3.1. *Let M be a unitary R-module, $x \in M$, $X \subseteq M$. Then, $[x, X] \in \delta$ if and only if there are $x_i \in X$ and $\lambda \in R$, $\lambda_i \in R$ $(1 \leqq i \leqq m)$ such that*

$$(3,1) \qquad 0 \neq \lambda x = \sum_{i=1}^{m} \lambda_i x_i \,.$$

With respect to this relation, M is an A-*dep. structure; moreover, δ is proper and satisfies* (E_r) *without the restriction $X \in \mathscr{I}_\delta$:*

$$(E) \qquad [x_1, X] \notin \delta \wedge [x_1, X \cup (x_2)] \in \delta \rightarrow [x_2, X \cup (x_1)] \in \delta \,.$$

A subset X of M is δ-independent if and only if $\{X\}_R$ is the direct sum

$$\{X\}_R = \sum_{x \in X} \{x\}_R = \sum_{x \in X} Rx \,.$$

Also, any subset $S \subseteq M$ is, with respect to the relation induced on S by δ, an A-*dep. structure* (S, δ).

The proof is of a routine nature and is therefore omitted. Now, our intention is to find some (necessary and sufficient) conditions for R guaranteeing that R-modules or some of their subsets are GA-dep. or LA-dep. structures; this will enable us to introduce the concept of rank in the theory of modules. We shall often refer to the following technical lemmas.

Lemma 3.2. *Let X be a δ-independent subset of M and $x \in M$ such that $[x, X] \in \delta$. Then there exist $x_j \in X$ and $\varkappa \in R$, $\varkappa_j \in R$ $(1 \leqq j \leqq n)$ such that*

$$(3,2) \qquad 0 \neq \varkappa x = \sum_{j=1}^{n} \varkappa_j x_j \quad \textit{with} \quad O(\varkappa x) = O(\varkappa_j x_j) \quad \textit{for} \quad 1 \leqq j \leqq n \,.$$

Proof. In view of our assumption, there are $x_i \in X$, $\lambda \in R$, $\lambda_i \in R$ $(1 \leqq i \leqq m)$ such that (3,1) holds; assume, moreover, that $\lambda_i x_i \neq 0$ for each $1 \leqq i \leqq m$. Consider the set of all ideals $O(\lambda_i x_i)$. This is either a single-point set – and, thus, (3,1) is of the form (3,2) –, or there are i_1, i_2 such that

$$O(\lambda_{i_1} x_{i_1}) \setminus O(\lambda_{i_2} x_{i_2}) \neq \emptyset .$$

In the latter case, multiply by an element μ of this set-theoretical difference the relation (3,1):

$$(3,3) \qquad \mu\lambda x = \sum_{i=1}^{m} \mu\lambda_i x_i ;$$

here $\mu\lambda_{i_1} x_{i_1} = 0$ and $\mu\lambda_{i_2} x_{i_2} \neq 0$. Hence, since X is δ-independent, $\mu\lambda x \neq 0$ and, furthermore, there are at most $m - 1$ non-zero members in the sum of the right-hand side of (3,3). Proceeding in a similar manner one can easily transform (3,1) into an expression of the form (3,2).

Lemma 3.3. *Let C be a subset of M such that the order $O(c)$ of each element $c \in C$ is an irreducible ideal. Then, C is a δ-canonic region of any* A-*dep. structure (S, δ) with $C \subseteq S \subseteq M$. On the other hand, if a subset $S \subseteq M$ satisfies the property that with an arbitrary $x \in S$ also $\lambda x \in S$ for every $\lambda \in R$,*[2]) *then the order of every element of a δ-canonic region C of S is irreducible.*

Proof. Evidently, it is sufficient to prove the first part of Lemma 3.3 for $S = M$ only. Thus, let $I \subseteq C$ be a δ-independent subset and let

$$[x, I] \in \delta \wedge \forall y(y \in I \rightarrow [y, X] \in \delta)$$

for a certain δ-independent subset $X \subseteq M$. Then, by Lemma 3,2, there are $y_j \in I$ and $\varkappa \in R$, $\varkappa_j \in R$, $\lambda_j \in R$ and $x_i \in X$, $\tau_{ji} \in R$ $(1 \leqq j \leqq n,\ 1 \leqq i \leqq m)$ such that

$$0 \neq \varkappa x = \sum_{j=1}^{n} \varkappa_j y_j \quad \text{with} \quad O(\varkappa x) = O(\varkappa_j y_j)$$

and

$$0 \neq \lambda_j y_j = \sum_{i=1}^{m} \tau_{ji} x_i \quad \text{for} \quad 1 \leqq j \leqq n .$$

Since $O(y_1)$ is irreducible, they exist, in view of Lemma 2,1, $\mu_1 \in R$, $\nu_1 \in R$ such that

$$\mu_1 \varkappa_1 - \nu_1 \lambda_1 \in O(y_1) \quad \text{and} \quad \mu_1 \varkappa_1 \notin O(y_1) .$$

Thus,

$$0 \neq \mu_1 \varkappa x = \sum_{j=1}^{n} \mu_1 \varkappa_j y_j$$

$$\text{with} \quad O(\mu_1 \varkappa x) = O(\mu_1 \varkappa_j y_j) \quad \text{for} \quad 1 \leqq j \leqq n$$

[2]) The proof suggests how this assumption could be weakened.

and

$$0 \neq \mu_1\varkappa_1 y_1 = \nu_1\lambda_1 y_1 = \sum_{i=1}^{m} \nu_1\tau_{1i}x_i .$$

Proceeding by induction, assume that (for $1 \leqq k < n$)

$$(3,4)\quad \begin{cases} 0 \neq \mu_k\mu_{k-1}\ldots\mu_1\varkappa x = \sum_{j=1}^{m} \mu_k\mu_{k-1}\ldots\mu_1\varkappa_j y_j \\ \text{with } O(\mu_k\mu_{k-1}\ldots\mu_1\varkappa x) = O(\mu_k\mu_{k-1}\ldots\mu_1\varkappa_j y_j) \text{ for } 1 \leqq j \leqq n \text{ and} \\ 0 \neq \mu_l\mu_{l-1}\ldots\mu_1\varkappa_l y_l = \nu_l\lambda_l y_l = \sum_{i=1}^{m} \nu_l\tau_{li}x_i \text{ for } 1 \leqq l \leqq k. \end{cases}$$

Applying again Lemma 2,1 to the irreducible ideal $O(y_{k+1})$, there exist $\mu_{k+1} \in R$, $\nu_{k+1} \in R$ such that

$$\mu_{k+1}\mu_k\ldots\mu_1\varkappa_{k+1} - \nu_{k+1}\lambda_{k+1} \in O(y_{k+1}) \quad\text{and}\quad \mu_{k+1}\mu_k\ldots\mu_1\varkappa_{k+1} \notin O(y_{k+1}) .$$

It is a routine to check that (3,4) holds for $k + 1$, and thus for $k = n$:

$$0 \neq \mu_n\mu_{n-1}\ldots\mu_1\varkappa x = \sum_{j=1}^{n} \mu_n\mu_{n-1}\ldots\mu_{j+1}\mu_j\mu_{j-1}\ldots\mu_1\varkappa_j y_j =$$
$$= \sum_{j=1}^{n} \mu_n\mu_{n-1}\ldots\mu_{j+1}\nu_j\lambda_j y_j = \sum_{j=1}^{n} \mu_n\mu_{n-1}\ldots\mu_{j+1}\nu_j\Big(\sum_{i=1}^{m}\tau_{ji}x_i\Big) =$$
$$= \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} \mu_n\mu_{n-1}\ldots\mu_{j+1}\nu_j\nu_j\tau_{ji}\Big)\, x_i ,$$

i.e. $[x, X] \in \delta$, as required. Consequently, C is a δ-canonic region and the proof is completed.

In order to prove the second part of Lemma 3,3 suppose that there is c in a δ-canonic region C of $S \subseteq M$ with a reducible order $O(c)$. Then, by Lemma 2,1, there are $\alpha \in R \setminus O(c)$, $\beta \in R \setminus O(c)$ such that

$$\mu\alpha - \nu\beta \in O(c) \quad\text{implies}\quad \mu\alpha \in O(c) \quad\text{for every}\quad \mu, \nu \in R .$$

Thus,

$$[\alpha c, (c)] \in \delta,\ [c, (\beta c)] \in \delta \quad\text{and}\quad [\alpha c, (\beta c)] \notin \delta$$

and since, in accordance with our assumption, $\alpha c \in S$, $\beta c \in S$, C is not a δ-canonic region of S.

As a simple consequence of Lemma 3,3 we get

Theorem 3.4. *A subset C of an R-module M is a δ-canonic region of M if and only if the order of each element of C is an irreducible ideal.*

Let us point out that the proof of Lemma 3,3 did not explore the δ-independence of $X \subseteq M$. Due to this fact, we deduce that the relation δ satisfies a stronger property than (T_r). In fact, δ has also another specific property expressed in Theorem 3,6 which is a consequence of Theorem 3,4 and the following

Lemma 3.5. *Let X be a δ-independent subset of M and $x \in M$ such that $[x, X] \in \delta$. If the order of every element of X is irreducible, then there exists an element $\varkappa \in R \setminus O(x)$ such that $O(\varkappa x) = O(x) : \varkappa$ is irreducible.*

Proof. The assertion follows readily from Lemmas 2,2 and 3,2.

Theorem 3.6. *If an R-module M is – with respect to δ – a* GA-*dep. structure, then it is a strict* GA-*dep. structure.*

Now, we are ready to state one of the main results of the present paper.

Theorem 3.7. *A ring has property* $(\check{\mathrm{G}})$ *if and only if it has property* $(\mathscr{I})$.

Proof. This is a consequence of Lemma 2,4. Theorem 3,4, Theorem 3,6 and Lemma 2,2 on the one hand and Theorem 3,4 on the other.

Corollary 3.8. *If a ring R has property $(\mathscr{I})$, then in any R-module*

(i) *there exists a maximal δ-independent subset of elements whose orders are irreducible;*

(ii) *any maximal δ-independent subset of non-zero elements whose orders are irreducible has the same cardinality – the rank* $\mathrm{r}(M)$ *of M;*

(iii) *the cardinality of any other maximal δ-independent subset of non-zero elements is less than or equal to* $\mathrm{r}(M)$.

The following statement is a simple consequence of Corollary 3,8 and Theorem 3,1:

Corollary 3.9. *Let $M = \sum_{\alpha \in A} Rx_\alpha = \sum_{\beta \in B} Ry_\beta$ be two direct decompositions of an R-module M into cyclic submodules of irreducible orders. Then,* $\mathrm{card}\,(A) = \mathrm{card}\,(B)$.

Theorem 3,4 together with Lemmas 2,4 and 2,1 yield also the following

Theorem 3.10. *Let $\mathscr{L}$ be a family of ideals of a ring R. If*

(i) *every ideal of $\mathscr{L}$ is irreducible, then*

(ii) *for any R-module M, the subset $S \subseteq M$ of all elements of M whose orders belong to $\mathscr{L}$ (and, thus, any subset $S \subseteq M$ such that the order $O(x)$ of each element $x \in S$ belong to $\mathscr{L}$, is – with respect to δ – a* LA-*dep. structure.*

If, moreover, the family $\mathscr{L}$ possesses the property that $L \in \mathscr{L}$ results in $L : \varkappa \in \mathscr{L}$ for every $\varkappa \in R \setminus L$, then (ii) *implies* (i).

Corollary3.11. *A ring has property* $(\check{\mathrm{L}})$ *if and only if it has property* $(\mathscr{L})$.

Proof. For, a ring has property $(\mathscr{L})$ if and only if every its ideal is irreducible.

Corollary 3.12. *Any two maximal δ-independent subsets of non-zero elements of an arbitrary R-module M have the same cardinality — the rank* $\mathrm{r}(M)$ *of M — if and only if R possesses property* $(\mathscr{L})$.

As far as the behaviour with respect to dependence is concerned, we have seen that R-modules resemble in many respects abelian groups. This parallelism can be extended also in the following way. We intend to define for an irreducible ideal L of a ring R, or more generally for a family $\mathscr{L}$ of irreducible ideals of R, the "L-rank" $\mathrm{r}_L(M)$, or the "$\mathscr{L}$-rank" $\mathrm{r}_{\mathscr{L}}(M)$, of an R-module M as the rank of the LA-dep. structure (S, δ), where S is the subset of all $x \in M$ such that $O(x) = L$, or $O(x) \in \mathscr{L}$, respectively. Naturally, this concept would be of little importance unless we prove that in any maximal δ-independent subset I of the whole R-module M (specified, possibly, in a certain way) the subset $S \cap I$ of I of those elements whose orders are L, or belong to $\mathscr{L}$, is always a maximal δ-independent subset of S and, thus, its cardinality equals to $\mathrm{r}_L(M)$, or $\mathrm{r}_{\mathscr{L}}(M)$. The following theorem is a general result in this direction reflecting the situation in abelian groups.

Theorem 3.13. *Let $\mathscr{L}$ be a family of ideals of a ring R. Consider the following three statements on a subfamily $\mathscr{L}_0 \subseteq \mathscr{L}$:*

(i′) *Every $L \in \mathscr{L}$ such that $L : \varkappa \in \mathscr{L}_0$ for a suitable $\varkappa \in R$, belongs to $\mathscr{L}_0$.*

(i″) *For every $L_0 \in \mathscr{L}_0$ and $\varkappa \in R \setminus L_0$, there exists $\lambda \in R$ such that $\lambda\varkappa \notin L_0$ and $L_0 : (\lambda\varkappa) \in \mathscr{L}_0$.*

(ii) *Every R-module M satisfies the following condition: Let S_0 be the subset of M of all elements whose orders belong to $\mathscr{L}_0$. If I is a maximal δ-independent subset of M consisting of elements of orders belonging to $\mathscr{L}$, then $S_0 \cap I$ is a maximal δ-independent subset of S_0.*

In particular, a consequence of (ii) *asserts*

(ii_p) *Every single element x of order $O(x) \in \mathscr{L}$ of an R-module M containing some elements of orders from $\mathscr{L}_0$, which forms a maximal δ-independent subset (x) of M, belongs to S_0 (i.e. $O(x) \in \mathscr{L}_0$).*

Always, (i′) *together with* (i″) *imply* (ii) *and, on the other hand,* (ii_p) *implies* (i′). *If, moreover, every ideal of $\mathscr{L}_0$ is irreducible and every R-module M possesses a maximal δ-independent subset consisting of elements whose orders belong to $\mathscr{L}$, then* (ii_p) *implies also* (i″). *In this case, S_0 is — with respect to δ — a* LA-*dep. structure and the conditions* (i′) *and* (i″) *are necessary and sufficient for the legitimacy of the following definition of the $\mathscr{L}_0/\mathscr{L}$-rank $\mathrm{r}_{\mathscr{L}_0/\mathscr{L}}(M)$ of M : $\mathrm{r}_{\mathscr{L}_0/\mathscr{L}}(M)$ is the cardinality of the set of all elements of orders belonging to $\mathscr{L}_0$ in a maximal δ-independent subset of M consisting of elements whose orders belong to $\mathscr{L}$.*

In particular, if $\mathscr{L}_0 \subseteq \mathscr{L}$ satisfy all the above requirements and R has, moreover, property $(\mathscr{I})$, then a family $\tilde{\mathscr{L}} \supseteq \mathscr{L}_0$ of irreducible ideals exists such that, in every R-module M, the subset of all elements of orders belonging to $\tilde{\mathscr{L}}$ is a δ-canonic zone M^C of M possessing the following property: The intersection $S_0 \cap I$

of the subset S_0 of all elements of M of orders belonging to $\mathscr{L}_0$ and a maximal δ-independent subset $I \subseteq M^C$ of M is a maximal δ-independent subset of S_0; we call the $\mathscr{L}_0/\tilde{\mathscr{L}}$-rank simply the $\mathscr{L}_0$-rank and denote it by $\mathrm{r}_{\mathscr{L}_0}(M)$.

Proof. First, (ii_p) follows from (ii) immediately.

Assume the validity of (i′) and (i″) and let S be the subset of an R-module M of all elements of orders from $\mathscr{L}_0$ and I be a maximal δ-independent subset of M consisting of elements of orders from $\mathscr{L}$; throughout this proof, let us call such subsets, for the sake of brevity, $\mathscr{L}$-subsets of M. Then, for every $x \in S$, there are, by Lemma 3,2, $x_j \in I$, $\varkappa \in R$ and $\varkappa_j \in R$ $(1 \leqq j \leqq n)$ such that (3,2) holds. Hence, by (i″), there is $\lambda \in R$ such that

$$0 \neq \lambda \varkappa x = \sum_{j=1}^{n} \lambda \varkappa_j x_j \quad \text{with} \quad O(\lambda \varkappa x) = O(\lambda \varkappa_j x_j) \quad \text{for} \quad 1 \leqq j \leqq n\,,$$

and, furthermore, such that $O(\lambda \varkappa x) = O(x) : (\lambda \varkappa) \in \mathscr{L}_0$. By virtue of (i′), we deduce from $O(\lambda \varkappa_j x_j) = O(x_j) : (\lambda \varkappa_j) \in \mathscr{L}_0$ and $O(x_j) \in \mathscr{L}$ that $O(x_j) \in \mathscr{L}_0$, i.e. $x_j \in S$ for $1 \leqq j \leqq n$. Since $S \cap I$ is evidently δ-independent, (ii) follows.

Now, let us prove that (i′) results from (ii_p). Let $L \in \mathscr{L}$ and $L : \varkappa \in \mathscr{L}_0$ for a suitable $\varkappa \in R$. Consider the R-module $M = R \bmod L$ of all cosets modulo L; the order of the coset containing $\varkappa$ is evidently $L : \varkappa \in \mathscr{L}_0$ and the coset containing ε (of order L) forms a single-point $\mathscr{L}$-subset of M. Thus, in view of (ii_p), necessarily $L \in \mathscr{L}_0$, as required.

In order to prove the implication $(\mathrm{ii}_\mathrm{p}) \to$ (i″), let $L_0 \in \mathscr{L}_0$ and $\varkappa \in R \setminus L_0$. If $L_0 : \varkappa \in \mathscr{L}_0$, (i″) holds for $\lambda = \varepsilon$. Otherwise, consider the R-module $M_1 = R \bmod (L_0 : \varkappa)$ of all cosets modulo $L_0 : \varkappa$; M_1 should possess, according to our additional assumption, an $\mathscr{L}$-subset and thus, there is an $\lambda \in R \setminus (L_0 : \varkappa)$, i.e. $\lambda \varkappa \in L_0$, such that the order $(L_0 : \varkappa) : \lambda = L_0 : (\lambda \varkappa)$ of the coset modulo $L_0 : \varkappa$ containing λ belongs to $\mathscr{L}$. Now, since L_0 is, according to the other additional assumption, irreducible, the coset modulo L_0 containing $\lambda \varkappa$ forms an $\mathscr{L}$-subset of the R-module $M_2 = R \bmod L_0$ of all cosets modulo L_0 (its order is $L_0 : (\lambda \varkappa) \in \mathscr{L}$). Meanwhile, the order of the coset containing ε is $L_0 \in \mathscr{L}_0$ and hence, in view of (ii_p), $L_0 : (\lambda \varkappa) \in \mathscr{L}_0$. Hence, (i″) holds.

The next assertion on S_0 and the $\mathscr{L}_0/\mathscr{L}$-rank of M are simple consequences of Theorem 3,10 and the preceding part of the present proof.

In order to complete the proof, assume that R possesses, moreover, property $(\mathscr{I})$. Then, for every $L \in \mathscr{L}$, there exists $\varkappa \in R \setminus L$ such that $L : \varkappa$ is irreducible. Denote by $\tilde{\mathscr{L}}$ the family consisting of all irreducible ideals of $\mathscr{L}$ and of all irreducible ideals $\tilde{L}$ such that $\tilde{L} = L : \varkappa$ for suitable reducible $L \in \mathscr{L}$ and $\varkappa \in R \setminus L$. Let M be an R-module. First, we are going to prove that the subset M^C of all elements of M of orders belonging to $\tilde{\mathscr{L}}$ is a δ-canonic zone of M. In fact, in view of Lemma 3,3, only the proof of maximality in M is needed. Thus, let $0 \neq x \in M$. Since R has property $(\mathscr{I})$, there is $\varkappa_0 \in R \setminus O(x)$ such that $O(\varkappa_0 x) = O(x) : \varkappa$ is irreducible. Write $x_0 = \varkappa_0 x \neq 0$.

Because of existence of an $\mathscr{L}$-subest of M, there are, by Lemma 3,3, $x_j \in M$ and $\varkappa \in R$, $\varkappa_j \in R$ $(1 \leqq j \leqq n)$ such that

$$0 \neq \varkappa x_0 = \sum_{j=1}^{n} \varkappa_j x_j \quad \text{with} \quad O(x_j) \in \mathscr{L} \quad \text{and}$$

$$O(\varkappa x_0) = O(\varkappa_j x_j) = O(x_j) : \varkappa_j \quad \text{for} \quad 1 \leqq j \leqq n\,.$$

By Lemma 2,2, $O(\varkappa x_0) = O(x_0) : \varkappa$ is irreducible and thus, $O(x_j) : \varkappa_j$ are irreducible. Let $O(x_j)$ be for $1 \leqq j \leqq t$ reducible and for $t + 1 \leqq j \leqq n$ irreducible ideals. Then, $O(\varkappa_j x_j)$ for $1 \leqq j \leqq t$ and $O(x_j)$ for $t + 1 \leqq j \leqq n$ belong to $\tilde{\mathscr{L}}$ and, thus,

$$C = (\varkappa_1 x_1, \varkappa_2 x_2, \ldots, \varkappa_t x_t, x_{t+1}, \ldots, x_n)$$

is a δ-independent subset of M^C. Furthermore, $[x, C] \in \delta$, as required.

The property of M^C described at the end of our theorem follows from the following observation: The families $\mathscr{L}_0 \subseteq \tilde{\mathscr{L}}$ satisfy the conditions (i′) and (i″). Only (i′) should be checked. Let $\tilde{L} \in \tilde{\mathscr{L}}$ and $\tilde{L} : \varkappa \in \mathscr{L}_0$ for a suitable $\varkappa \in R$. If $\tilde{L} \in \mathscr{L}$, then evidently $L \in \mathscr{L}_0$ (by (i′) applied to $\mathscr{L}_0 \subseteq \mathscr{L}$). Otherwise, $\tilde{L} = L : \lambda$ for suitable reducible $L \in \mathscr{L}$ and $\lambda \in R \setminus L$. From here,

$$L : (\varkappa\lambda) = (L : \lambda) : \varkappa = \tilde{L} : \varkappa \in \mathscr{L}_0\,,$$

and thus, again by (i′) applied to $\mathscr{L}_0 \subseteq \mathscr{L}$, $L \in \mathscr{L}_0$ contradicting the hypothesis that all ideals of $\mathscr{L}_0$ are irreducible.

This completes the proof of Theorem 3,13.

Now, formulate some corollaries of Theorem 3,13.

Corollary 3.14. (Fuchs [9]). *The subset of elements of order* (0) *in any two maximal δ-independent subsets of an R-module M have the same cardinality if and only if the ideal* $(0) \subseteq R$ *is irreducible and R is without zero-divisors* (*i.e.* (0) *is prime*).

Proof. We apply Theorem 3,13 for the family $\mathscr{L}$ of all ideals of R and $\mathscr{L}_0$ consisting of the single zero-ideal (0). The assumption on equal cardinalities of our corollary implies readily (ii_p). Therefore, (i′) holds, i.e. every relation $L : \varkappa = (0)$ results in $L = (0)$. This is equivalent to the fact that (0) is irreducible; for, $L : \varkappa = (0)$ means precisely that $\varkappa \neq 0$ and $L \cap R\varkappa = (0)$. Furthermore since $\mathscr{L}$ consists of all ideals and (0) is irreducible, also (i″) holds, i.e. for every $\varkappa \neq 0$, $\lambda \in R$ exists such that $\lambda\varkappa \neq 0$ and $(0) : (\lambda\varkappa) = (0)$. This statement is, in the presence of (i′), evidently equivalent to the fact that R has no (non-trivial) zero-divisors; for, $(0) = (0) : (\lambda\varkappa) = ((0) : \varkappa) : \lambda$ implies $(0) : \varkappa = (0)$. The corollary follows.

Consequently, if (0) is an irreducible and prime ideal of R, we can define on the basis of Corollary 3,14 the r_0-rank for R-modules; this question will be treated shortly in the final remark of this paragraph. We cannot expect a similar result in such a strong version for an ideal $L_0 \neq (0)$ of R. Nevertheless, we can either confine only to some types of rings or to specify in a certain way the family $\mathscr{L}$; after all, such is the situation even in abelian groups. Always, Theorem 3,13 represents a general pattern. Though a number of corollaries could be formulated, we introduce here only a simple

Corollary 3.15. (cf. [9]) *Let $\mathscr{L}$ be a family of two-sided prime ideals of R such that maximal δ-independent subsets of elements of orders belonging to $\mathscr{L}$ exist in any R-module M. Then the subsets of elements of order $L_0 \in \mathscr{L}$ in any two such maximal δ-independent subsets of an R-module M have the same cardinality if and only if L_0 is irreducible.*

Proof. Theorem 3,13 and Theorem 3,10 yield the statement immediately.

Remark 3.16. Both Theorem 3,10 and Corollary 3,14 relate to the concept of the *linear dependence relation* δ_0 in$R$-modules M (strong dependence of Fuchs [9]) defined in analogy to (3,1) by: *For $x \in M$ and $X \subseteq M$,*

$$[x, X] \in \delta_0 \leftrightarrow \varkappa x = \sum_{i=1}^{m} \varkappa_i x_i$$

for certain $0 \neq \varkappa \in R$, $\varkappa_i \in R$ *and* $x_i \in X$ $(1 \leqq i \leqq m)$. It is easy to check that the set of δ_0-neutral elements is void and that an element $x \in M$ is δ_0-singular if and only if its order differs from (0). Thus, we consider the (regular) subset $M_0 \subseteq M$ of all elements of order (0), the conditions (F), (M), (E) and (I) can be easily verified for δ_0. Hence, (M_0, δ_0) is always an A-dep. structure. Since, for an element x of order (0), $\varkappa x \neq 0$ is equivalent to $\varkappa \neq 0$, δ_0 is identical with δ on M_0. Thus, from this point of view the relation δ_0 is a derived relation from δ:

(i) for $x \in M_0$, $[x, X] \in \delta_0$ if and only if $[x, X \cap M_0] \in \delta$ and

(ii) for $x \in M \setminus M_0$, $[x, X] \in \delta_0$ for every $X \subseteq M$.

Therefore, the study of δ_0 is a part of the theory of δ. In particular, Theorem 3,10 for $\mathscr{L} = (0)$ reads as follows (cf. Kertész [11], Fuchs [9]): *If the zero-ideal (0) is irreducible in R, then, for any R-module M, (M_0, δ_0) is a* LA-*dep. structure and thus, all maximal δ_0-independent subsets of M have the same cardinality. If, moreover, (0) is prime in R (i.e. R is without zero-divisors) the converse holds, as well.* Then, however, the strengthened statement of Corollary 3,14 holds. Let us recall, at this point, that the condition for (0) in R to be irreducible and prime is necessary and sufficient for the subset of all elements of non-zero orders T in every R-module M to be a submodule and the quotient module M/T to be torsion-free. This suggests an alternative way of treating the relation δ_0 within the study of the relation δ.

In the preceding paragraph, necessary and sufficient conditions were established for a ring R with identity in order that all unitary R-modules become GA-dep. or LA-dep. structures with respect to the relation δ (properties ($\check{G}$) and ($\check{L}$)). As we have already pointed out (§ 2), there are no rings such that every R-module be a LA-dep. structure with respect to δ. However, the related question for GA-dep. structures appears to be non-trivial and is investigated and solved in the present paragraph: We give here a necessary and sufficient condition for a ring R in order to possess property (G).

First, we need a generalization of the concept of an ideal-quotient.

Definition 4.1. Let R be a ring, L a (left) ideal of R. By a generalized ideal-quotient $L:\{k, \varkappa\}$, for an integer $k \in Z$ and an element $\varkappa \in R$, we understand the (left) ideal defined by

$$\lambda \in L:\{k, \varkappa\} \quad \text{if and only if} \quad k \times \lambda + \lambda\varkappa \in L.$$

Evidently, every ideal-quotient is a generalized ideal-quotient and, on the other hand, every generalized ideal-quotient in a ring with identity is an ideal-quotient (for, $L:\{k, \varkappa\} = L:(k \times \varepsilon + \varkappa)$).

Furthermore, for the proof of the main theorem of this paragraph we shall need a series of lemmas on the relations between the ideals L of a ring R and the ideals L^* of the Dorroh's extension R^* of R. It turns out that every ideal of R is an ideal of R^* and, of course, if L^* is an ideal of R^*, then $L^* \cap R$ is an ideal of R.

Lemma 4.2. *R is an irreducible (two-sided) ideal of R^*.*

Proof. Let $(m, \mu) \in L_1^* \setminus R$ and $(n, \nu) \in L_2^* \setminus R$ be two elements of ideals $L_1^* \supseteq R$ and $L_2^* \supseteq R$ of R^*. Then, $(m, 0) \in L_1^*$, $(n, 0) \in L_2^*$ and, thus, $(m, n, 0) \in (L_1^* \cap L_2^*) \setminus R$.

Lemma 4.3. *Let L^* be an irreducible ideal of R^*. Then $L = L^* \cap R$ is an irreducible ideal of R.*

Proof. Suppose, on the contrary, that $L = L_1 \cap L_2$ for suitable ideals L_1 and L_2 of R such that $L \subsetneqq L_1$ and $L \subsetneqq L_2$. Then, the ideals L_1^* and L_2^* of R^* generated by L^*, L_1 and L^*, L_2, respectively, satisfy the relation $L_1^* \cap L_2^* = L^*$. For, $(m, \mu) \in \in L_1^* \cap L_2^*$ means that

$$(m, \mu) = (l_1^*, \lambda_1^* + \lambda_1) = (l_2^*, \lambda_2^* + \lambda_2),$$

$$\text{i.e.} \quad l_1^* = l_2^* \quad \text{and} \quad \lambda_1 = \lambda_2^* + \lambda_2 - \lambda_1^*$$

with $l_i^* \in Z$, $\lambda_i^* \in L^*$ and $\lambda_i \in L_i$ for $i = 1, 2$; from here, $\lambda_1 \in L_1 \cap L_2 = L$ and thus, $(m, \mu) \in L^*$.

Lemma 4.4. *Let L be an ideal of R. If the ideal-quotient $L:\tau \neq R$ for every $\tau \in R \setminus L$, then also $(L:\{k, \varkappa\}):\sigma \neq R$ for every $k \in Z$, $\varkappa \in R$ and $\sigma \in R \setminus (L: :\{k, \varkappa\})$.*

Proof. Since $k \times \sigma + \sigma\varkappa \notin L$,

$$(L:\{k, \varkappa\}):\sigma = L:(k \times \sigma + \sigma\varkappa) \neq 0\,,$$

as required.

Lemma 4.5 *Let L be an ideal of R such that $L:\tau \neq R$ for every $\tau \in R \setminus L$. Then, there exists a uniquely determined ideal $\tilde{L}$ of R^* satisfying $\tilde{L} \cap R = L$ and containing any other ideal L^* of R for which $L^* \cap R = L$.*

Proof. Let us define the subset $\tilde{L}$ of R^* as follows: $(m, \mu) \in \tilde{L}$ if and only if $m \times \lambda + \lambda\mu \in L$ for every $\lambda \in R$. It is a routine to check that $\tilde{L}$ is a (left) ideal of R^*. Moreover, $\tilde{L} \cap R = L$; for, $\tilde{L} \cap R \supseteq L$ is trivial and the other inclusion holds due to our hypothesis on L ($\lambda\mu \in L$ for every $\lambda \in R$ implies $\mu \in L$). Finally, let (n, ν) be an element of an ideal L^* of R^* for which $L^* \cap R = L$. Then, $(0, \lambda)(n, \nu) = (0, n \times \lambda + \lambda\nu) \in L^*$, i.e. $n \times \lambda + \lambda\nu \in L$ for every $\lambda \in R$, and thus, $(n, \nu) \in \tilde{L}$.

Lemma 4.6. *Let L be an irreducible ideal of R such that $L:\tau \neq R$ for every $\tau \in R \setminus L$. Then the ideal $\tilde{L} \subseteq R^*$ of Lemma 4,5 is irreducible, too.*

Proof. This follows immediately from the fact that $L^* \cap R \supsetneqq L$ for every ideal L^* of R^* such that $L^* \supsetneqq \tilde{L}$.

Lemma 4.7. *Let L be an ideal of R such that $L:\tau \neq R$ for every $\tau \in R \setminus L$ and $\tilde{L}$ be the ideal of R^* defined in Lemma 4,5. Then, by Lemma 4,4, the ideal $\widetilde{L:\{k, \varkappa\}}$ of R^* is well-defined and equals to the ideal-quotient $\tilde{L}:\{k, \varkappa\}$ in R^* for every $k \in Z$ and $\varkappa \in R$.*

Proof. Let $(m, \mu) \in \widetilde{L:\{k, \varkappa\}}$; then, $m \times \lambda + \lambda\mu \in L:\{k, \varkappa\}$, i.e.

$$k(m \times \lambda + \lambda\mu) + (m \times \lambda + \lambda\mu)\varkappa \in L \quad \text{for every} \quad \lambda \in R\,.$$

Hence,

$$mk \times \lambda + \lambda(k \times \mu + m \times \varkappa + \mu\varkappa) \in L \quad \text{for every} \quad \lambda \in R\,, \quad \text{i.e.}$$

$$(mk, m \times \varkappa + k \times \mu + \mu\varkappa) = (m, \mu)(k, \varkappa) \in \tilde{L}$$

and, thus,

$$(m, \mu) \in \tilde{L}:(k, \varkappa)\,.$$

All these implications can be reversed and the lemma follows.

Now, we are ready to formulate and prove the following

Theorem 4.8. *A ring R has property* (G) *if and only if, for every ideal $L \subsetneqq R$, there exist*

(i) *either $\varkappa \in R \setminus L$ such that $L : \varkappa = R$,*

(ii) *or $k \in Z$ and $\varkappa \in R$ such that $L : \{k, \varkappa\}$ is irreducible and different from R.*

In this case, the set of all non-zero elements x of an R-module M such that either

(a) *$O(x) = R$ and $n \times x = 0$ only for $n = 0$, or*

(b) *$O(x) = R$ and $p^t \times x = 0$ for a certain power of a prime p, or*

(c) *$O(x) \neq R$ is irreducible and there are no $k \in Z$ and $\varkappa \in R$ such that $k \times x + \varkappa x \neq 0$ and $O(k \times x + \varkappa x) = R$, forms a (strict) δ-canonic zone of M.*

Proof. Our proof will be based on the related Theorem 3,7. The necessity of the conditions follows quite easily: In view of the one-to-one correspondence between R-modules and unitary R^*-modules, R^* has property ($\check{G}$) and Theorem 3,7 can be applied: Let $L \subsetneqq R$ be an ideal. If there exists an element $\tau \in R \setminus L$ such that $L : \tau = R$, we take simply $\varkappa = \tau$ and (i) follows. Otherwise, there is, in view of Lemma 4,5, the uniquely determined ideal $\tilde{L} \supseteq L$ of R^* of all (m, μ) such that $m \times \lambda + \lambda\mu \in L$ for every $\lambda \in R$. By Theorem 3,7, there exists $(k, \varkappa) \in R^* \setminus \tilde{L}$ such that $\tilde{L} : (k, \varkappa)$ is irreducible. Hence, by Lemmas 4,4, 4,5, 4,7 and 4,3,

$$L : \{k, \varkappa\} = \widetilde{L : \{k, \varkappa\}} \cap R = [\tilde{L} : (k, \varkappa)] \cap R \neq R$$

is irreducible.

Now, we are going to prove that the conditions (i) and (ii) are sufficient. Taking again into account the one-to-one correspondence between R-modules and unitary R^*-modules, we need only to show that R^* has property ($\mathscr{I}$). Thus, let $L^* \subsetneqq R^*$ be an ideal of R^*. Write $L = L^* \cap R$. If $L = R$, then, with every $(m, \mu) \in L^*$, also $(m, 0) \in L^*$. Suppose that $L^* \neq R$ and let t be the least natural number such that $(t, 0) \in L^*$; then $(m, 0) \in L^*$ if and only if m is a multiple of t. If $t = pt_0$ with a prime p, then $L^* : (t_0, 0) \supseteq L^*$ is a maximal, and thus irreducible, ideal of R^*.

Also, if there is an element $\tau \in R \setminus L$ satisfying $L : \tau = R$, i.e.

$$R \subseteq L : (0, \tau) \subseteq L^* : (0, \tau) \neq R^* ,$$

then we conclude, as above, that $(t_0, 0)$ exists such that

$$[L^* : (0, \tau)] : (t_0, 0) = L^* : (0, t_0 \times \tau) \neq R$$

is irreducible.

Otherwise (i.e. if $L : \tau \neq R$ for every $\tau \in R \setminus L$), the uniquely determined ideal $\tilde{L}$ of Lemma 4,5 exists with

$$L = L^* \cap R = \tilde{L} \cap R \quad \text{and} \quad L^* \subseteq \tilde{L}.$$

In the case that $L^* \neq \tilde{L}$, there is $(l, \lambda) \in \tilde{L} \setminus L^*$ and, by the definition of $\tilde{L}$,

$$R \subseteq L^* : (l, \lambda) \neq R^* ;$$

again, $t_0 \in Z$ exists such that

$$[L^* : (l, \lambda)] : (t_0, 0) = L^* : (t_0 l, t_0 \times \lambda) \neq R^*$$

is irreducible. The case $L^* = \tilde{L}$ remains to be considered. Here, we use (ii) and apply Lemmas 4,6 and 4,7: There are $k \in Z$ and $\varkappa \in R$ such that $L : \{k, \varkappa\} \neq R$ is irreducible; hence,

$$\widetilde{L : \{k, \varkappa\}} = \tilde{L} : (k, \varkappa) = L^* : (k, \varkappa) \neq R^*$$

is irreducible.

The validity of the second part of Theorem 4,8 can readily be seen if we consider M as an R^*-module and realize that in every non-zero R^*-submodule of M generated by a single element, a non-zero element x satisfying one of the conditions (a), (b), (c) exists and its order in R^* is irreducible.

The proof of Theorem 4,8 is completed.

Corollary 4.9. *For a ring R with identity all three properties* ($\breve{G}$), (G) *and* $\mathcal{I}$ *are equivalent.*

Proof. This follows immediately from Theorem 4,8, Theorem 3,7 and the fact that $L : \varkappa \neq R$ for $\varkappa \in R \setminus L$ and $L : \{k, \varkappa\} = L : (k \times \varepsilon + \varkappa)$, where ε is the identity element of R.

5. SOME FINAL REMARKS

In this short final paragraph we intend to point out some classes of rings with properties ($\breve{G}$) and (G) and to give also an example of a commutative ring with identity which does not possess these properties.

Lemma 5.1. *Let the ascending chain condition for left ideals hold in a ring R. Then R possesses property* ($\mathcal{I}$).

Lemma 5.2. *Let the descending chain condition for left ideals hold in a ring R. Then R possesses property* ($\mathcal{I}$).[3]

Proof of Lemmas 5,1 and 5,2. We give here an indirect proof of both statements: *Assuming that property* ($\mathcal{I}$) *is not satisfied in R, we shall construct an (infinite) strictly increasing and decreasing sequences of left ideals in R.* Our assumption yields the existence of a reducible left ideal L of R with the additional property that $L : \varkappa$ is also reducible for every $\varkappa \in R \setminus L$.

[3]) For a ring R with (a one-sided) identity, Lemma 5,2 follows from Lemma 5,1 (see Hopkins [10]).

We are going to show the existence of two sequences of elements of R:

$$\alpha_1, \alpha_2, \dots, \alpha_k, \dots \quad \text{and} \quad \beta_1, \beta_2, \dots, \beta_k, \dots$$

such that, for every k,

$$(5,1) \qquad \alpha_k\alpha_{k-1} \dots \alpha_1 \in R \setminus L \quad \text{and} \quad \beta_k\alpha_{k-1} \dots \alpha_1 \in R \setminus L$$

and, moreover, any relation

$$(5,2) \qquad \begin{aligned}&(m \times \alpha_k\alpha_{k-1} \dots \alpha_1 + \mu\alpha_k\alpha_{k-1} \dots \alpha_1) - \\ &- (n \times \beta_k\alpha_{k-1} \dots \alpha_1 + \nu\beta_k\alpha_{k-1} \dots \alpha_1) \in L\end{aligned}$$

with $m, n \in Z$ and $\mu, \nu \in R$ implies

$$(5,3) \qquad m \times \alpha_k\alpha_{k-1} \dots \alpha_1 + \mu\alpha_k\alpha_{k-1} \dots \alpha_1 \in L.$$

Since L is reducible, the existence of $\alpha_1 \in R \setminus L$ and $\beta_1 \in R \setminus L$ with the required properties follows immediately from Lemma 2,1. Further, proceed by induction; suppose that up to a certain natural k the sequences of

$$\alpha_1, \alpha_2, \dots, \alpha_k \quad \text{and} \quad \beta_1, \beta_2, \dots, \beta_k$$

with the required properties have been defined. Hence $\alpha_k\alpha_{k-1} \dots \alpha_1 \in R \setminus L$, and thus $L_k = L : (\alpha_k\alpha_{k-1} \dots \alpha_1)$ is reducible. Again, in view of Lemma 2,1, we deduce the existence of $\alpha_{k+1} \in R \setminus L_k$ and $\beta_{k+1} \in R \setminus L_k$ such that any relation

$$(m \times \alpha_{k+1} + \mu\alpha_{k+1}) - (n \times \beta_{k+1} + \nu\beta_{k+1}) \in L_k$$

with $m, n \in Z$ and $\mu, \nu \in R$ implies

$$m \times \alpha_{k+1} + \mu\alpha_{k+1} \in L_k\,,$$

i.e. α_{k+1} and β_{k+1} exist such that $\alpha_{k+1}\alpha_k \dots \alpha_1 \in R \setminus L$, $\beta_{k+1}\alpha_k \dots \alpha_1 \in R \setminus L$ and any relation

$$\begin{aligned}&(m \times \alpha_{k+1}\alpha_k \dots \alpha_1 + \mu\alpha_{k+1}\alpha_k \dots \alpha_1) - \\ &- (n \times \beta_{k+1}\alpha_k \dots \alpha_1 + \nu\beta_{k+1}\alpha_k \dots \alpha_1) \in L\end{aligned}$$

with $m, n \in Z$ and $\mu, \nu \in R$ implies

$$m \times \alpha_{k+1}\alpha_k \dots \alpha_1 + \mu\alpha_{k+1}\alpha_k \dots \alpha_1 \in L,$$

as required.

Now, for every natural k, consider the following two left ideals of R, generated by L and the elements indicated:

$$L_k^* = \{L, \beta_1, \beta_2\alpha_1, \dots, \beta_k\alpha_{k-1} \dots \alpha_1\} \quad \text{and} \quad L_{*k} = \{L, \alpha_k\alpha_{k-1} \dots \alpha_1\}\,.$$

Clearly,

$$L_1^* \subseteq L_2^* \subseteq \dots \subseteq L_k^* \subseteq \dots \quad \text{and} \quad L_{*1} \supseteq L_{*2} \supseteq \dots \supseteq L_{*k} \supseteq \dots$$

All the inclusions in both sequences are strict. For, suppose, first, that

$$L^*_{k-1} = L^*_k \quad \text{for a certain natural } k\,.$$

Then,

$$\beta_k\alpha_{k-1}\dots\alpha_1 = \lambda + (n_1 \times \beta_1 + \nu_1\beta_1) + \dots + \\ + (n_{k-1} \times \beta_{k-1}\alpha_{k-2}\dots\alpha_1 + \nu_{k-1}\beta_{k-1}\alpha_{k-2}\dots\alpha_1)$$

with a suitable $\lambda \in L$, i.e.

$$[\beta_k\alpha_{k-1}\dots\alpha_2 - (n_2 \times \beta_2 + \nu_2\beta_2 + \dots + n_{k-1} \times \\ \times \beta_{k-1}\alpha_{k-2}\dots\alpha_2 + \nu_{k-1}\beta_{k-1}\alpha_{k-2}\dots\alpha_2)]\,\alpha_1 - (n_1 \times \beta_1 + \nu_1\beta_1) \in L,$$

and thus, by (5,2) and (5,3) for $k = 1$, we get $n_1 \times \beta_1 + \nu_1\beta_1 \in L$. Using the same argument (for $k = 2$), we can see that also $n_2 \times \beta_2\alpha_1 + \nu_2\beta_2\alpha_1 \in L$ and, by induction, we get finally

(5,4) $$\beta_k\alpha_{k-1}\dots\alpha_1 \in L,$$

a contradiction of (5,1).

Similarly, suppose that

$$L_{*k-1} = L_{*k} \quad \text{for a certain } k.$$

Then,

$$\beta_k\alpha_{k-1}\dots\alpha_1 = \lambda + (m \times \alpha_k\alpha_{k-1}\dots\alpha_1 + \mu\alpha_k\alpha_{k-1}\dots\alpha_1)$$

with a suitable $\lambda \in L$, i.e.

$$(m \times \alpha_k\alpha_{k-1}\dots\alpha_1 + \mu\alpha_k\alpha_{k-1}\dots\alpha_1) - \beta_k\alpha_{k-1}\dots\alpha_1 \in L.$$

Hence, by (5,2) and (5,3), we get again the contradiction (5,4) of (5,1). The proof is completed.

From here and Theorem 4,8 we conclude

Theorem 5.3. *Every ring in which either the maximum or the minimum condition for left ideals holds possesses property* (G).

As a particular result we get

Corollary 5.4. *Every (left) Noetherian ring possesses property* (G).

The following is an example of a commutative ring with identity which does not possess property (Ğ).

Example 5.5. Denote by X the set of all real numbers α such that $0 < \alpha < 1$ and α is not a rational number of the form $t/2^n$. By an X-interval $(t_1/2^m, t_2/2^n)$, where $t_1/2^m \leqq t_2/2^n$, we understand the set of all those $\alpha \in X$ which, moreover, satisfy $t_1/2^m < \alpha < t_2/2^n$; X itself is also an X-interval.

Consider the family R of all finite (disjoint) set-theoretical unions of X-intervals. Then, R with respect to addition and multiplication defined by the set-theoretical symmetric difference and intersection, respectively, is a commutative (Boolean) ring with identity. Take an element $a \in R$, $a \neq 1$ (i.e. a finite set-theoretical union of X-intervals different from X) and denote by L_a the family of all those elements of R which are subsets of a. We can verify readily that $L_a \neq R$ is a principal ideal of $R : L_a = Ra$. Moreover, L_a is evidently a reducible ideal of R; in fact, it is always an intersection of two principal ideals different from L_a. If b is an arbitrary element of $R \setminus L_a$, then $L_a : b = L_{1+b+ab} \neq R$ is again reducible. Thus, R does not possess the property $(\mathscr{I})$ and hence, by Theorem 3,7, does not possess property $(\breve{G})$.

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Резюме

ЗАВИСИМОСТЬ В МОДУЛЯХ

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Статья посвящена проблеме распространения теории зависимости в абелевых группах на модули (таким способом можно зависимость в векторных пространствах и в абелевых группах изучать в рамках общей теории); в проблеме, в частности, содержится вопрос о возможности определения ранга модуля. В работе применяются результаты теории GA-зависимых и LA-зависимых структур из [4], [5] и [7].

Пусть M – (левый) модуль над (ассоциативным) кольцом R. Мы скажем,

что элемент $x \in M$ зависит от подмножества $X \subseteq M$, если существуют $\lambda, \lambda_i \in R$ и $x_i \in X$ $(1 \leqq i \leqq m)$ так, что

$$0 \neq \lambda x = \sum_{i=1}^{m} \lambda_i x_i$$

(это отношение мы обозначим символом δ). В работе приведено необходимое и достаточное условие, которому должно удовлетворять кольцо R, чтобы каждый R-модуль был по отношению к зависимости δ GA-зависимой структурой (Теорема 3.7 для унитарных и Теорема 4.8 для общих модулей); в случае унитарного модуля это следующее условие: для любого собственного левого идеала $L \subsetneqq R$ существует $\kappa \in R \setminus L$ так, что идеаловый множитель $L:\kappa$ является неприводимым (т.е. из $L_1 \supsetneqq L:\kappa$ и $L_2 \supsetneqq L:\kappa$ вытекает $L_1 \cap L_2 \supsetneqq L:\kappa$). Итак, если это условие выполнено, то

(i) существует максимальное δ-независимое подмножество элементов, порядки которых неприводимы;

(ii) каждое такое максимальное δ-независимое подмножество ненулевых элементов имеет одну и ту же мощность — ранг $\mathrm{r}(M)$ модуля M;

(iii) мощность любого максимального δ-независимого подмножества ненулевых элементов меньше или равна $\mathrm{r}(M)$ (Следствие 3.8). В качестве следствия получаем утверждение, что мощность множества слагаемых в прямом разложении произвольного унитарного модуля M в циклические подмодули неприводимых порядков является инвариантом модуля M (Следствие 3.9).

Каждый R-модуль является по отношению к δ LA-зависимой структурой тогда и только тогда, когда все левые идеалы кольца R образуют цепь (Следствие 3.11). Это, следовательно, представляет необходимое и достаточное условие для того, чтобы два любых максимальных δ-независимых подмножества ненулевых элементов (произвольного) R-модуля M имели ту же мощность — ранг $\mathrm{r}(M)$ модуля M (Следствие 3.12).

Теорема 3.13 обобщает некоторые свойства зависимости в абелевых группах на модули. Частными следствиями этой теоремы, равно как теоремы 3.10, служат результаты, касающиеся линейной зависимости в R-модулях: $x \in M$ зависит линейно от $X \subseteq M$, если существуют $0 \neq \kappa$, $\kappa_i \in R$ и $x_i \in X$ $(1 \leqq i \leqq m)$ так, что

$$\kappa x = \sum_{i=1}^{m} \kappa_i x_i$$

(Следствие 3.14, Замечание 3.16.)

В последнем параграфе показано, что кольца, удовлетворяющие максимальному или минимальному условию для левых идеалов (в частности нетеровские кольца) обладают тем свойством, что ко всякому собственному левому идеалу $L \subsetneqq R$ существует $\kappa \in R \setminus L$ так, что $L:\kappa$ является неприводимым (Теорема 5.3, Следствие 5.4). Произвольное коммутативное кольцо этим свойством не обладает (Пример 5.5).