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ALGEBRAIC DEFINITION OF CENTRAL DISPERSIONS OF THE 1st KIND OF THE DIFFERENTIAL EQUATION $y'' = Q(t)\,y$

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1. Introduction. Let $Q(t)$ be a real function of a real variable defined and continuous in interval $J = (A, B)$, where $A < B$, and let the symbol (A, B) signifiy an interval with the left-hand end-point A ($A = -\infty$ is admitted) and with the right-hand end-point B ($B = \infty$ is admitted). The interval can be either closed $[A, B]$, or open $]A, B[$ or half-closed $[A, B[$, $]A, B]$. In what follows the word "interval" will signify an interval containing at least two points.

If a differential equation

$$y'' = Q(t)\,y \tag{Q}$$

is given we shall denote the set of all its solutions in the interval J by the symbol (Q); nontrivial solutions of differential equation (Q) in the interval J will be called integrals.

In his paper [1] O. BORŮVKA has defined dispersions of the 1st kind of differential equation (Q) as the largest solution $X(t)$ of the differential equation

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + Q(X)\,X'^2 = Q(t)\,.$$

They are functions either increasing or decreasing everywhere. Note that the introduced differential equation in the real domain is equivalent to two differential equations,

$$\sqrt{-X'}\left(\frac{1}{\sqrt{-X'}}\right)'' + Q(X)\,X'^2 \,\mathrm{x}= Q(t)$$

and

$$\sqrt{X'}\left(\frac{1}{\sqrt{X'}}\right)'' + Q(X)\,X'^2 = Q(t)\,. \tag{Q,Q}$$

The first equation is the differential equation of all decreasing dispersions whereas equation (Q,Q) is the differential equation of all increasing dispersions. As central dispersions are increasing we confine ourselves, in what follows, on increasing dispersions $\zeta(t)$ – further dispersions only – that are solutions of differential equation (Q, Q) not admitting any prolongation in the interval $J \times J$ and thus passing boundary to boundary; the set of all dispersions (i.e. increasing dispersions) of differential equation (Q) will be denoted by the symbol (Q, Q), too.

In what follows, we shall denote the definition-domain of an arbitrary function f by the symbol $\operatorname{Dom} f$ and the set of its values by $\operatorname{Im} f$. Thus, for every $\zeta \in (Q, Q)$ $\operatorname{Dom} \zeta$ and $\operatorname{Im} \zeta$ are certain subintervals of interval J.

Similarly as we have confined ourselves on increasing dispersions we can do it, without loss of generality, for increasing phases of differential equation (Q), see [1]. For that reason, under a phase of differential equation (Q) we understand any solution $\alpha(t)$ of the differential equation

$$(-1, Q) \qquad \sqrt{\alpha'}\left(\frac{1}{\sqrt{\alpha'}}\right)'' - \alpha'^2 = Q(t)\,.$$

in the interval J. The set of all phases of differential equation (Q) will be denoted by the symbol $(-1, Q)$, too.

By the amplitude of differential equation (Q) we mean any solution $\varrho(t)$ of the dif. equation

$$({}^{\Delta}Q) \qquad \varrho'' = Q(t)\,\varrho + \frac{\Delta^2}{\varrho^3}$$

in the interval J, where $\Delta \neq 0$ is an arbitrary constant; the set of all solutions of differential equation $({}^{\Delta}Q)$ in interval J will be denoted by the symbol $({}^{\Delta}Q)$, too. The set of all amplitudes of differential equation (Q) is then a union $\bigcup_{\Delta \neq 0} ({}^{\Delta}Q)$. In the paper [2] there has been proved, resp. will be analogously proved, that every dispersion $\zeta \in (Q, Q)$ transforms every $u \in (Q)$, $\varrho \in ({}^{\Delta}Q)$, respectively, on a solution U of differential equation (Q), or on a solution P of differential equation $({}^{\Delta}Q)$ in $\operatorname{Dom} \zeta$, according to the formula

$$U(t) = \frac{u[\zeta(t)]}{\sqrt{\zeta'(t)}}\,, \quad \text{resp.} \quad P(t) = \frac{\varrho[\zeta(t)]}{\sqrt{\zeta'(t)}}\,.$$

If we assign to any $u \in (Q)$ the solution $U \in (Q)$ fulfilling the first of the above formulas in $\operatorname{Dom} \zeta$, then it is possible to interpret any dispersion as a linear operator on the set (Q).

The first definition of central dispersions (of the 1^{st} kind) of differential equation (Q) is an application of the original Borůvka's definition on the general case. Let n be an arbitrary integer. Let $t \in J$ be an arbitrary number. Let y be an integral of differential equation (Q) such that $y(t) = 0$. If we take t as a zero root of the integral y and if we assign positive indices to the roots lying right from t, we define the value of a central dispersion φ_n at the point t as the n-th root of the integral y as far as this root exists in J.

The number $\varphi_n(t)$ is, sometimes, denoted by t_n and we call it the n-th conjugated number with the number $t_0 = \varphi_0(t) = t$.

Irrespective of the trivial case that some central dispersion exists at most at one point of interval J, every central dispersion φ exists in a certain subsinterval of interval J, is increasing and fulfills the differential equation (Q, Q) there, while it does not admit an extension in $J \times J$, so that $\varphi \in (Q, Q)$.

A central dispersion $\varphi_0(t) = t$ exists always in the entire interval J. The inverse function to an arbitrary central dispersion is again a central dispersion while $\varphi_n^{-1} = \varphi_{-n}$. The set of all central dispersions of differential equation (Q) is at most enumerable; we denote it by C.

In the paper [4], there is proved that the following definition is equivalent to the above mentioned definition. Let C_1 be a set of all dispersions $\varphi \in (Q, Q)$ that transform every $u \in (Q)$ on $\pm u$, i.e. for which the formula

$$\frac{u[\varphi(t)]}{\sqrt{\varphi'(t)}} = \pm\, u(r)$$

holds for every $u \in (Q)$. There holds $C_1 = C$, so that it is possible to define central dispersions of differential equation (Q) as elements of the set C_1.

Define in the third way central dispersions of differential equation (Q) as elements of the set C_2, where C_2 is the set of all $\varphi \in (Q, Q)$ such that for every $\zeta \in (Q, Q)$, for which the composed functions $\zeta\varphi$ and $\varphi\zeta$ have some common interval of existence, there holds $\zeta\varphi = \varphi\zeta$ in this interval.

In the paper [4] it is proved that $C_2 \subset C$. The equivalence of this definition to the preceding definitions of central dispersions depends on the inclusion $C_2 \subset C$, the proof of which we present in this paper.

2. Representation of dispersions by means of matrices. For an arbitrary $\zeta \in (Q, Q)$ let M_ζ be a set of all $\tilde{\zeta} \in (Q, Q)$ for which $\tilde{\zeta}\zeta$ and $\zeta\tilde{\zeta}$ have some common interval of existence, i.e. Dom $\zeta \cap$ Dom $\tilde{\zeta}$ is an interval. The set M_ζ contains an identical mapping e of the interval J on itself for every ζ.

Let N_ζ be a set of all $\tilde{\zeta} \in M_\zeta$ for which $\tilde{\zeta}\zeta = \zeta\tilde{\zeta}$ holds in the corresponding common interval of existence of both composed functions. The set N_ζ is never empty because $e \in N_\zeta$ for every $\zeta \in (Q, Q)$. In general, $N_\zeta \neq M_\zeta$. But there exist $\zeta \in (Q, Q)$ with the property $N_\zeta = M_\zeta$. e.g. $\zeta = e$.

The set C_2 is evidently the set of all $\varphi \in (Q, Q)$ for which $N_\varphi = M_\varphi$.

Let us choose fixed ordered pair of linearly independent integrals u, v of differential equation (Q). Let ζ be an arbitrary dispersion. Then we can univocally assign to the dispersion ζ or to its arbitrary part, in an interval, a matrix

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

by means of the formulas

$$\text{(1)} \qquad \frac{u(\zeta)}{\sqrt{\zeta'}} = \alpha u + \beta v\,, \quad \frac{v(\zeta)}{\sqrt{\zeta'}} = \gamma u + \delta v$$

and we shall write $\zeta \to A$.

The transformation property of the dispersion ζ, namely, that an integral $\tilde{y} = = \tilde{a}u + \tilde{b}v$ is assigned to any integral $y = au + bv$ according to the formula $y(\zeta)/\sqrt{\zeta'} = \tilde{y}$, can be expressed by means of the matrix, corresponding to the dispersion ζ, by the formula in coefficients

$$\text{(2)} \qquad \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Let us remark that one matrix can be assigned to more different dispersions. Transformation equations (1) can be written in the form

$$\text{(3)} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{u(\zeta)}{\sqrt{\zeta'}} \\ \dfrac{v(\zeta)}{\sqrt{\zeta'}} \end{pmatrix}.$$

The differentiation of the equation (3) gives

$$\text{(3')} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \left(\dfrac{1}{\sqrt{\zeta'}}\right)' u(\zeta) + \sqrt{\zeta'}\, u'(\zeta) \\ \left(\dfrac{1}{\sqrt{\zeta'}}\right)' v(\zeta) + \sqrt{\zeta'}\, v'(\zeta) \end{pmatrix}.$$

Recall that the Wronskian $uv' - u'v$ of the ordered pair of integrals u, v is a constant $\Delta \neq 0$. From this there follows the relation $\alpha\delta - \beta\gamma = 1$; thus, a matrix assigned to an arbitrary dispersion is unimodular.

If $\zeta \to A$, then $\zeta^{-1} \to A^{-1}$. If $\zeta_i \to A_i$, $i = 1, 2$, then to $\zeta_2\zeta_1$, as far as it exists, the product of matrices in inverse order is assigned, i.e. $\zeta_2\zeta_1 \to A_1A_2$.

Lemma 1. *Central dispersions $\varphi \in C$ are characterized by the fact that their matrix is*

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. Let $\varphi \in C$; let $\varphi \to \begin{pmatrix} \varkappa & \mu \\ \lambda & \nu \end{pmatrix}$. As for any $y \in (Q)$ we have $y(\varphi)/\sqrt{\varphi'} = \pm y$, the elements of the matrix $\begin{pmatrix} \varkappa & \mu \\ \lambda & \nu \end{pmatrix}$ with arbitrary numbers a, b satisfy the equations

$$a(\varkappa \mp 1) + b\mu = 0\,, \quad a\lambda + b(\nu \mp 1) = 0\,.$$

For $a = 0$, $b = 1$ we get $\mu = 0$, $\nu = \pm 1$; for $a = 1$, $b = 0$ it follows $\varkappa = \pm 1$, $\lambda = 0$; thus

$$\varphi \to \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Conversely, let

$$\varphi \to \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $\varphi \in (Q, Q)$. Then for any a, b we have

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix},$$

or, according to (2), for any $y \in (Q)$ there holds $y(\varphi)/\sqrt{\varphi'} = \pm y$ which, in addition, follows from (1), too – and thus $\varphi \in C$.

The problem of validity of the inclusion $C_2 \subset C$ mentioned in the introduction can be interpreted in the following way. Let be $\varphi \in C_2$, while $\varphi \to \begin{pmatrix} \varkappa & \mu \\ \lambda & \nu \end{pmatrix}$. Then for any $\zeta \in M_\varphi$, where $\zeta \to \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ there holds $\zeta \in N_\varphi$ and thus $\zeta\varphi = \varphi\zeta$ in a certain interval dependent on φ and ζ; thus, the elements of a matrix $\begin{pmatrix} \varkappa & \mu \\ \lambda & \nu \end{pmatrix}$ satisfy the system of equations

$$(\sigma) \qquad \begin{aligned} -\gamma\lambda \qquad + \beta\mu \qquad\qquad &= 0 \\ \gamma\varkappa \qquad + (\delta - \alpha)\,\mu - \gamma\nu &= 0 \\ -\beta\varkappa + (\alpha - \delta)\,\lambda \qquad\qquad + \beta\nu &= 0 \end{aligned}$$

with coefficients α, β, γ, δ that can vary in a certain way in dependence on $\zeta \in M_\varphi$.

In order to deduce some conclusions on numbers $\varkappa$, λ, μ, ν from the system of equations (σ), we need to know a little more about the set M_ζ for arbitrary $\zeta \in (Q, Q)$.

The circumstance that the identity belongs to M_φ gives the system of equations (σ) with coefficients $\alpha = \delta = 1$, $\beta = \gamma = 0$, which for the numbers $\varkappa$, λ, μ, ν gives only the relation that they are arbitrary.

3. Continuous increasing mappings. Let ζ be an arbitrary increasing and continuous mapping of some interval $(a, b) \subset J$ on some interval $(c, d) \subset J$ such that the graph of a mapping ζ passes from one boundary to another in the square $J \times J$; let $\mathscr{P}$ be a set of all such mappings ζ. Let K_ζ be a set of all $\tilde{\zeta} \in \mathscr{P}$ such that the functions $\tilde{\zeta}\zeta$ and

$\zeta\tilde{\zeta}$ have some common interval of existence. Evidently $M_\zeta = (Q, Q) \cap K_\zeta$ at $\zeta \in \in (Q, Q)$.

In this paragraph, we shall first solve the following problem to find a convenient, necessary and sufficient condition for $\tilde{\zeta} \in \mathcal{P}$ to belong to K_ζ at a given $\zeta \in \mathcal{P}$.

A mappings $\tilde{\zeta} \in \mathcal{P}$ belongs to K_ζ if and only if Dom $\tilde{\zeta}\zeta \cap$ Dom $\zeta\tilde{\zeta}$ is an interval, i.e. according to the remark in the introduction, an interval with non-empty interior, For thet reason, it is necessary for Dom $\tilde{\zeta}\zeta$ and Dom $\zeta\tilde{\zeta}$ not to be empty or one-point sets. As Dom $\tilde{\zeta}\zeta = \zeta^{-1}(\text{Dom}\,\tilde{\zeta} \cap \text{Im}\,\zeta)$ and Dom $\zeta\tilde{\zeta} = \tilde{\zeta}^{-1}$ (Dom $\zeta \cap$ Im $\tilde{\zeta}$), this necessary condition will be fulfilled exactly if Dom $\tilde{\zeta} \cap$ Im ζ and Dom $\zeta \cap$ Im $\tilde{\zeta}$ are intervals (with non-empty interior); then with notation Dom $\zeta = (a, b)$, Im $\zeta = = (c, d)$, Dom $\tilde{\zeta} = (\tilde{a}, \tilde{b})$, Im $\tilde{\zeta} = (\tilde{c}, \tilde{d})$ we can write Dom $\tilde{\zeta} \cap$ Im $\zeta = (c_1, d_1)$ where $c_1 = \max(\tilde{a}, c)$ and $d_1 = \min(\tilde{b}, d)$ and similarly Dom $\zeta \cap$Im $\tilde{\zeta} = (\tilde{c}_1, \tilde{d}_1)$ where $\tilde{c}_1 = \max(a, \tilde{c})$ and $\tilde{d}_1 = \min(b, \tilde{d})$; consider that $c_1 < d_1$, $\tilde{c}_1 < \tilde{d}_1$ and denote in this case Dom $\tilde{\zeta}\zeta = (a_1, b_1)$ and Dom $\zeta\tilde{\zeta} = (\tilde{a}_1, \tilde{b}_1)$ so that $a_1 < b_1$, and $\tilde{a}_1 < \tilde{b}_1$. Then Dom $\tilde{\zeta}\zeta \cap$ Dom $\zeta\tilde{\zeta}$ is an interval if and only if for $a_0 = \max(a_1, \tilde{a}_1)$ and $b_0 = \min(b_1, \tilde{b}_1)$ the inequality $a_0 < b_0$ holds; then Dom $\tilde{\zeta}\zeta \cap$ Dom $\zeta\tilde{\zeta} = = (a_0, b_0)$.

Hence we get the following simple criterion:

Lemma 2. *If and only if $c_1 < d_1$ and $\tilde{c}_1 < \tilde{d}_1$, each of the functions $\tilde{\zeta}\zeta$ and $\zeta\tilde{\zeta}$ exists in an interval; for $\tilde{\zeta} \in K_\zeta$ to be valid it is necessary and sufficient that*

$$a_0 = \max(a_1, \tilde{a}_1) < \min(b_1, \tilde{b}_1) = b_0 .$$

Note that $\zeta(a_0, b_0) \subset (\tilde{a}, \tilde{b})$, $\tilde{\zeta}(a_0, b_0) \subset (a, b)$ and that $(a_0, b_0) \subset (a, b) \cap (\tilde{a}, \tilde{b})$ so that $a_0 \geqq \max(a, \tilde{a})$, $b_0 \leqq \min(b, \tilde{b})$. If we denote $(c_2, d_2) = \tilde{\zeta}(c_1, d_1)$ and similarly $(\tilde{c}_2, \tilde{d}_2) = \zeta(\tilde{c}_1, \tilde{d}_1)$, then Im $\tilde{\zeta}\zeta = (c_2, d_2)$ and Im $\zeta\tilde{\zeta} = (\tilde{c}_2, \tilde{d}_2)$. Let us remark that if $\tilde{\zeta} \in K_\zeta$, then $\zeta \in K_{\tilde{\zeta}}$. If $\tilde{\zeta} \in K_\zeta$, then $t \in (a_0, b_0)$ exist such that $\tilde{\zeta}$ is defined at t and simultaneously at $\zeta(t)$, and at the same time ζ is defined at points t and $\tilde{\zeta}(t)$, too, If we are to construct some $\tilde{\zeta} \in K_\zeta$ to a given $\zeta \in \mathcal{P}$, it is possible e.g. to choose $t \in]a, b[$ and $\tilde{\zeta}(t) \doteq t$, and to insure that $\tilde{\zeta}$ is defined for the number $\zeta(t)$. For example $\tilde{\zeta} = e$ has these properties for every $\zeta \in \mathcal{P}$.

Let $J = (A, B)$ be an interval; let us remind that A can be $-\infty$, B can be ∞ and J an open, closed or half-closed interval. Let $\zeta \in \mathcal{P}$ so that $\zeta : (a, b) \to (c, d)$ while it holds

$$a = A \text{ or } c = A \quad \text{and} \quad b = B \text{ or } d = B .$$

There exist four types of a mapping $\zeta \in \mathcal{P}$:

(I)	$a = A, \quad d = B$
(II)	$c = A, \quad b = B$
(III)	$a = A, \quad b = B$
(IV)	$c = A, \quad d = B$

These types are not quite disjunct in the sense that there exist mappings $\zeta \in \mathscr{P}$ that are simultaneously either of the type (I) and (III), or (I) and (IV) or (II) and (III), or (II) and (IV) or of all types at the same time.

This classification of functions $\zeta \in \mathscr{P}$ by types (I)–(IV) permits us to investigate the relation $\tilde{\zeta} \in K_\zeta$ and by means of Lemma 2 to get necessary and sufficient conditions for this relation.

Let $\zeta \in \mathscr{P}$ and $\zeta : (a, b) \to (c, d)$ with boundary condition ($a = A$ or $c = A$) and ($b = B$ or $d = B$).

Let $\tilde{\zeta} \in \mathscr{P}$, and $\tilde{\zeta} : (\tilde{a}, \tilde{b}) \to (\tilde{c}, \tilde{d})$ with boundary conditions ($\tilde{a} = A$ or $\tilde{c} = A$) and ($\tilde{b} = B$ or $\tilde{d} = B$).

Theorem 1'. *If $\zeta \in \mathscr{P}$ is of the type* (I) *then $\tilde{\zeta} \in \mathscr{P}$ fulfils the relation $\tilde{\zeta} \in K_\zeta$ if and only if $\tilde{a} < b$, $\tilde{c} < b$; $\tilde{b} > c$, $\tilde{d}$ arbitrary, and the following conditions are satisfied (as far as they are meaningful): $\zeta(\tilde{a}) < \tilde{b}$, $\zeta^{-1}(\tilde{a}) < \tilde{\zeta}^{-1}(b)$, $\zeta^{-1}(\tilde{a}) < \tilde{b}$.*

Proof. With the notation introduced in this paragraph, $\mathrm{Dom}\, \tilde{\zeta} \cap \mathrm{Im}\, \zeta = (c_1, d_1)$ etc., we have $d_1 = \tilde{b}$, $\tilde{c}_1 = \tilde{c}$, so that $\mathrm{Dom}\, \tilde{\zeta}\zeta = (a_1, \zeta^{-1}(\tilde{b}))$ and $\mathrm{Dom}\, \zeta\tilde{\zeta} = (\tilde{a}, \tilde{b}_1)$. By Lemma 2 $\tilde{\zeta} \in K_\zeta$ if and only if $a_1 < \tilde{b}_1$, $\tilde{a} < \zeta^{-1}(\tilde{b})$; then, from the inclusion $(a_0, b_0) \subset (A, b) \cap (\tilde{a}, \tilde{b})$ it follows that $\max(A, \tilde{a}) < \min(b, \tilde{b})$ and therefore $\tilde{a} < b$ is the necessary condition for $\tilde{\zeta} \in K_\zeta$. According to Lemma 2 there is a further necessary condition for $\tilde{\zeta} \in K_\zeta$ $c_1 < d_1$, $\tilde{c}_1 < \tilde{d}_1$ or $\max(\tilde{a}, c) < \min(\tilde{b}, B)$, $\max(A, \tilde{c}) < \min(b, \tilde{b})$ so that inequalities $c < \tilde{b}$, $\tilde{c} < b$ are some part of the above mentioned necessary and sufficient condition.

Thus, we have proved that $\tilde{\zeta} \in K_\zeta \Leftrightarrow a_1 < \tilde{b}_1$, $\tilde{a} < \zeta^{-1}(\tilde{b})$; $\tilde{a} < b$, $\tilde{c} < b$, $c < \tilde{b}$. In order to give the inequality $a_1 < \tilde{b}_1$ a little more illustrative content in the system of all other inequalities, let us recall that $c_1 = \max(\tilde{a}, c)$, $\tilde{d}_1 = \min(b, \tilde{d})$; there occur 4 possibilities (1a,) (1b), (2a), (2b), where

(1) $\tilde{a} \leqq c$ and therefore $c_1 = c$, $a_1 = A$

(2) $\tilde{a} > c$ and therefore $c_1 = \tilde{a}$, $a_1 = \zeta^{-1}(\tilde{a})$

(a) $b < \tilde{d}$ and therefore $\tilde{d}_1 = b$, $\tilde{b}_1 = \tilde{\zeta}^{-1}(b)$

(b) $b \geqq \tilde{d}$ and therefore $\tilde{d}_1 = \tilde{d}$, $\tilde{b}_1 = \tilde{b}$.

In case (1a) we have $a_1 = A$, $\tilde{b}_1 = \tilde{\zeta}^{-1}(b)$ and the inequality $a_1 < \tilde{b}_1$ is fulfilled automatically because $b > \tilde{c}$. In case (1b) we have $a_1 = A$, $\tilde{b}_1 = \tilde{b}$ and the inequality $a_1 < \tilde{b}_1$ is again fulfilled automatically.

In case (2a) there follows from $\tilde{a} < b$ that $c < b$, and from boundary conditions for $\tilde{\zeta}$ that $\tilde{c} = A$; we have $a_1 = \zeta^{-1}(\tilde{a})$, $\tilde{b}_1 = \tilde{\zeta}^{-1}(b)$ and the inequality $a_1 < \tilde{b}_1$ gives the condition $\zeta^{-1}(\tilde{a}) = \tilde{\zeta}^{-1}(b)$.

In case (2b) there is again necessarily $c < b$, $\tilde{c} = A$; we have $a_1 = \zeta^{-1}(\tilde{a})$, $\tilde{b}_1 = \tilde{b}$ so that the inequality $a_1 < \tilde{b}_1$ gives the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$.

Theorem 1′ is proved because $a_1 < \tilde{b}_1$, $\tilde{a} < \zeta^{-1}(\tilde{b})$ is according to the accomplished considerations equivalent to the complex of inequalities from theorem 1′, as far as they are meaningful.

Theorem 1″. *Let $\zeta \in \mathscr{P}$ be of type* (II). *Then for $\tilde{\zeta} \in \mathscr{P}$ there holds $\tilde{\zeta} \in K_\zeta$ if and only if $\tilde{a} < d$, $\tilde{c}$ arbitrary; $\tilde{b} > a$, $\tilde{d} > a$ and the conditions $\tilde{a} < \zeta(\tilde{b})$, $\tilde{\zeta}^{-1}(a) < \zeta^{-1}(\tilde{b})$ must be fulfilled.*

Proof. In this case we have $b = B$, $c = A$ and therefore $c_1 = \tilde{a}$, $\tilde{d}_1 = \tilde{d}$ so that $\mathrm{Dom}\, \tilde{\zeta}\zeta = (\zeta^{-1}(\tilde{a}), b_1)$, $\mathrm{Dom}\, \zeta\tilde{\zeta} = (\tilde{a}_1, \tilde{b})$. According to Lemma 2 the equivalence $\tilde{\zeta} \in K_\zeta \Leftrightarrow \tilde{a}_1 < b_1$, $\zeta^{-1}(\tilde{a}) < \tilde{b}$ holds; this necessary and sufficient condition leaks a telling meaning and for that reason we shall specify it in details by explicit stressing of its illustrative parts contained in it implicitly.

Similarly as in the proof of Theorem 1′ an illustrative limiting of area $\tilde{a} < d$, $\tilde{b} > a$, $\tilde{d} < a$ follows from the inequalities $\tilde{a}_1 < b_1$, $\zeta^{-1}(\tilde{a}) < \tilde{b}$. We get further illustrative necessary conditions contained in the inequality $\tilde{a}_1 < b_1$ from relations $\tilde{c}_1 = \max(a, \tilde{c})$, $d_1 = \min(\tilde{b}, d)$; there are four possibilities (1a), (1b), (2a), (2b) where

(1)	$a \leqq \tilde{c}$ and therefore $\tilde{c}_1 = \tilde{c}$, $\tilde{a}_1 = \tilde{a}$
(2)	$a > c$ and therefore $\tilde{c}_1 = a$, $\tilde{a}_1 = \tilde{\zeta}^{-1}(a)$
(a)	$d \leqq \tilde{b}$ and therefore $d_1 = d$, $b_1 = B$
(b)	$d > b$ and therefore $d_1 = \tilde{b}$, $b_1 = \zeta^{-1}(\tilde{b})$.

In case (1a) the inequality $\tilde{a}_1 < b_1$ is fulfilled automatically as well as in case (1b) and (2a).

In case (2b) we have for a, d a restriction $a < d$, $\tilde{d} = B$. The inequality $\tilde{a}_1 < b_1$ gives a condition $\tilde{\zeta}^{-1}(a) < \zeta^{-1}(\tilde{b})$. Thereby the proof of Theorem 1″ is finished.

Theorem 1‴. *Let $\zeta \in \mathscr{P}$ be of type* (III). *Then for $\tilde{\zeta} \in \mathscr{P}$ there holds $\tilde{\zeta} \in K_\zeta$ if and only if $\tilde{a} < d$, $\tilde{b} > c$; $\tilde{c}, \tilde{d}$ arbitrary and $\zeta(\tilde{a}) < \tilde{b}$, $\tilde{a} < \zeta(\tilde{b})$.*

Proof. In this case we have $a = A$, $b = B$ so that $\tilde{c}_1 = \tilde{c}$, $\tilde{d}_1 = \tilde{d}$ and therefore $\mathrm{Dom}\, \tilde{\zeta}\zeta = (a_1, b_1)$, $\mathrm{Dom}\, \zeta\tilde{\zeta} = (\tilde{a}, \tilde{b})$. According to Lemma 2, the equivalence $\tilde{\zeta} \in K_\zeta \Leftrightarrow a_1 < \tilde{b}$, $\tilde{a} < b_1$.

First of all, holds from the found condition the illustrative restrictions $\tilde{a} < d$, $\tilde{b} > c$ follow; its further consequence can be obtained by relations $c_1 = \max(\tilde{a}, c)$, $d_1 = \min(\tilde{b}, d)$, which give four cases (1a), (1b); (2a), (2b):

(1)	$\tilde{a} \leqq c$ and therefore $c_1 = c$, $a_1 = A$
(2)	$\tilde{a} > c$ and therefore $c_1 = \tilde{a}$, $a_1 = \zeta^{-1}(\tilde{a})$
(a)	$\tilde{b} < d$ and therefore $d_1 = \tilde{b}$, $b_1 = \zeta^{-1}(\tilde{b})$
(b)	$\tilde{b} \geqq d$ and therefore $d_1 = d$, $b_1 = B$.

In case $(1a)$ the inequality $a_1 < \tilde{b}$ is fulfilled automatically and the inequality $\tilde{a} < b_1$ gives the condition $\tilde{a} < \zeta^{-1}(\tilde{b})$.

In case $(1b)$ we do not get any new condition.

In case $(2a)$ we get two conditions $\zeta^{-1}(\tilde{a}) < \tilde{b}$, $\tilde{a} < \zeta^{-1}(\tilde{b})$.

In case $(2b)$ the inequality $\tilde{a} < b$ is fulfilled automatically and the inequality $a_1 < \tilde{b}$ gives again the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$. Thereby Theorem 1‴ is proved.

Theorem 1^{IV}. *Let $\zeta \in \mathcal{P}$ be of type* (IV). *For $\xi \in \mathcal{P}$ there holds $\xi \in K_\zeta$ if and only if $\tilde{a} < b$, $\tilde{c} < b$; $\tilde{b} > a$, $\tilde{d} > a$ and if conditions $\zeta^{-1}(\tilde{a}) < \xi^{-1}(b)$, $\xi^{-1}(a) < \zeta^{-1}(\tilde{b})$ are fulfilled.*

Proof. Now we have $c = A$, $d = B$ and consequently $c_1 = \tilde{a}$, $d_1 = \tilde{b}$; Dom $\xi\zeta = (\zeta^{-1}(\tilde{a}), \zeta^{-1}(\tilde{b}))$, Dom $\zeta\xi = (\tilde{a}_1, \tilde{b}_1)$. According to Lemma 2 we have

$$\xi \in K_\zeta \Leftrightarrow \zeta^{-1}(\tilde{a}) < \tilde{b}_1\,, \quad \tilde{a}_1 < \zeta^{-1}(\tilde{b})\,.$$

From the found condition it is possible to deduce its immediate illustrative part: $\tilde{a} < b$, $a < \tilde{b}$, $\tilde{c} < d$, $a < \tilde{d}$ and by means of relations $\tilde{c}_1 = \max(a, \tilde{c})$, $d_1 = \min(b, \tilde{d})$ the remainder of the illustrative part by discerning the four cases $(1a)$, $(1b)$, $(2a)$, $(2b)$, where

(1) $\qquad a \leqq \tilde{c}$ and therefore $\tilde{c}_1 = \tilde{c}$, $\tilde{a}_1 = \tilde{a}$

(2) $\qquad a > c$ and therefore $\tilde{c}_1 = a$, $\tilde{a}_1 = \xi^{-1}(a)$

(a) $\qquad b < \tilde{d}$ and therefore $\tilde{d}_1 = b$, $\tilde{b}_1 = \xi^{-1}(b)$

(b) $\qquad b \geqq \tilde{d}$ and therefore $\tilde{d}_1 = \tilde{d}$, $\tilde{b}_1 = \tilde{b}$.

In case $(1a)$ we get a condition $\zeta^{-1}(\tilde{a}) < \xi^{-1}(b)$, in case $(1b)$ we do not obtain any new condition, in case $(2a)$ we get both conditions $\zeta^{-1}(\tilde{a}) < \xi^{-1}(b)$ and $\xi^{-1}(a) < \zeta^{-1}(\tilde{b})$ and in case $(2b)$ we have a condition $\xi^{-1}(a) < \zeta^{-1}(\tilde{b})$. Thereby the proof of Theorem 1^{IV} is finished.

Results given in Theorems 1′ to 1^{IV} may be expressed in one theorem.

Theorem 1. *Let $\zeta \in \mathcal{P}$ be arbitrary. Then for $\xi \in \mathcal{P}$ there holds that $\xi \in K_\zeta$, if and only if there hold the inequalities (as far as they are meaningful)*

$$\tilde{a} < \min(b, d)\,, \; \tilde{c} < b\,, \quad \tilde{b} > \max(a, c)\,, \; \tilde{d} > a \tag{4}$$

$$\zeta^{-1}(\tilde{a}) < \xi^{-1}(b)\,, \quad \xi^{-1}(a) < \zeta^{-1}(\tilde{b})\,. \tag{5}$$

Proof. The inequalities for $\tilde{a}$ up to $\tilde{d}$ are evident. Under the assumption of Theorem 1′ we have $a = A$; for $\tilde{a} = A$ the condition $\zeta(\tilde{a}) < \tilde{b}$ reduces to $c < \tilde{b}$ which is fulfilled; for $\tilde{a} > A$ we have $\xi^{-1}(a) = \tilde{a}$ and the condition $\zeta(\tilde{a}) < \tilde{b}$ or $\tilde{a} < \zeta^{-1}(\tilde{b})$ is identical with the condition $\xi^{-1}(a) < \zeta^{-1}(\tilde{b})$. The condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$ occurred n case $(2b)$ provided $\tilde{d} \leqq b$; if $b = B$ and $\tilde{b} = B$ then the condition $\zeta^{-1}(\tilde{a}) < b$ i reduces to $\zeta^{-1}(\tilde{a}) < B$ or to $\tilde{a} < \zeta(B) = B$ which is, naturally, fulfilled; for $b = B$, $\tilde{b} < B$ we have $\tilde{\zeta}^{-1}(b) = \tilde{b}$ and the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$ is identical with the condition $\zeta^{-1}(\tilde{a}) < \tilde{\zeta}^{-1}(b)$; for $b < B$ we have $\tilde{b} = B$ with respect to boundary conditions for $\tilde{\zeta}$ and with respect to the inequality $\tilde{d} \leqq b$; the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$ is, thus, trivial.

In Theorem 1″ we have the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$; of course, even here $b = B$ is true so that $\tilde{\zeta}^{-1}(b) = \tilde{\zeta}^{-1}(B) = \tilde{b}$ as soon as $\tilde{d} = B$ and in this case the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$ is identical with $\zeta^{-1}(\tilde{a}) < \tilde{\zeta}^{-1}(b)$; if however, $\tilde{d} < B$, then necessarily $\tilde{b} = B$ and the condition is trivial.

In Theorem 1‴ we have the condition $\tilde{a} < \zeta^{-1}(\tilde{b})$; but here, of course, $a = A$; for $\tilde{a} = A$ the condition is trivial because it means that $c < \tilde{b}$, and $\tilde{\zeta}^{-1}(a) = \tilde{a}$ for $\tilde{a} > A$ and the condition identifies with $\tilde{\zeta}^{-1}(a) < \zeta^{-1}(\tilde{b})$. Analogously the condition $\zeta^{-1}(\tilde{a}) < \tilde{b}$ is a special case of the inequality $\zeta^{-1}(\tilde{a}) < \tilde{\zeta}^{-1}(b)$.

Thereby the theorem is proved.

The results concerning the behaviour of curves $\tilde{\zeta} \in K_\zeta$ have the disadvantage that conditions (4) do not suffice and that it is necessary to respect (5). For this reason, we need some sufficient condition free from conditions (5) which would possess more or less the character of (4). In fact, the matter is whether for a given $\zeta \in \mathscr{P}$ some fixed limits, dependent on ζ, exists such that whenever $\tilde{\zeta}$ is in these limits, then necessarily $\tilde{\zeta} \in K_\zeta$; now, we are going to deal with this problem.

Theorem 2. *Let $\zeta \in \mathscr{P}$ be arbitrary. Let $x \leqq y$ be arbitrary numbers such that $A < x < \min(b, d)$, $\max(a, c) < y < B$. Let E be an arbitrary number fulfilling the relations*

$$0 < E < \min\left(b - \zeta^{-1}(x),\ \zeta^{-1}(y) - a,\ B - y,\ x - A\right).$$

Let $K = x - \frac{1}{2}E$, $L = y + \frac{1}{2}E$. Let $0 < \varepsilon < \frac{1}{2}E$. Let $D_\zeta = \{(t, z)/K \leqq t \leqq L, |z - t| \leqq \varepsilon\}$. Then $D_\zeta \subset J^\circ \times J^\circ$ where J^0 denotes the interior of interval J, and any $\tilde{\zeta} \in \mathscr{P}$, for which $\operatorname{Dom} \tilde{\zeta} \supset [K, L]$ and for which $|\tilde{\zeta}(t) - t| \leqq \varepsilon$ holds in interval $[K, L]$, belongs necessarily to K_ζ.

Proof. The existence of the number E depends on the fact whether or not $\min(b - \zeta^{-1}(x), \zeta^{-1}(y) - a, B - y, x - A) > 0$. For a chosen x in the open interval $]A, \min(b, d)[$ $\zeta^{-1}(x)$ need not exist; then simply the element $b - \zeta^{-1}(x)$ is not taken into consideration. If $\zeta^{-1}(x)$ exists then necessarily $\zeta^{-1}(x) < b$ because $x < \zeta(b) = d$. Likely, for a chosen $y \in]\max(a, c), B[$ $\zeta^{-1}(y)$ need not exist; then the element $\zeta^{-1}(y) - a$ in the definition of E is simply failing. If $\zeta^{-1}(y)$ exists then necessarily $\zeta^{-1}(y) > a$, because $y > \zeta(a) = c$. Thus, $\min(b - \zeta^{-1}(x), \zeta^{-1}(y) - a, B - y, x - A) > 0$ and it is possible to choose $E > 0$ less than this minimum.

As $E < x - A$, a straight line $t = K$, $|\zeta - K| \leqq \varepsilon$ lies inside the square $(A, x) \times (A, x)$; similarly, a straight line $t = L$, $|\zeta - L| \leqq \varepsilon$ lies inside the square $(y, B) \times (y, B)$; consequently for $x \leqq y$ the rhomboid $K \leqq t \leqq L$, $|\zeta - t| \leqq \varepsilon$ lies then inside the square $J^0 \times J^0$.

As $x < b$ and $\zeta^{-1}(x)$ – if it exists – is also less than b, every $\tilde{\zeta} \in \mathscr{P}$ for which $\tilde{a} < x$ and for which $\tilde{\zeta}^{-1}(b) > \zeta^{-1}(x)$ fulfills the sooner condition $\zeta^{-1}(\tilde{a}) < \tilde{\zeta}^{-1}(b)$ – as far as $\zeta^{-1}(a)$ has a sense at all.

As $y > a$ and $\zeta^{-1}(y) > a$, too, as far as $\zeta^{-1}(y)$ exists, and $\tilde{\zeta} \in \mathscr{P}$ for which $\tilde{b} > y$ and for which $\tilde{\zeta}^{-1}(a) < \zeta^{-1}(y)$ fulfils the sooner condition $\tilde{\zeta}^{-1}(a) < \zeta^{-1}(\tilde{b})$ – as far as $\zeta^{-1}(\tilde{b})$ is defined at all.

Hence, from this it follows that for an arbitrary $\tilde{\zeta} \in \mathscr{P}$, the graph of which cuts simultaneously all rectangles $]A, x[\times]A, x[$; $]A, \zeta^{-1}(y)[\times]a, \zeta^{-1}(y)[$; $]\zeta^{-1}(x), B[\times]\zeta^{-1}(x), b[$; $]y, B[\times]y, B[$, provided they exist, there holds $\tilde{\zeta} \in K_\zeta$ according to Theorem 1.

Particularly, any $\tilde{\zeta} \in \mathscr{P}$ from Theorem 2 has the property that $\tilde{\zeta} \in K_\zeta$, because passing through a rhomboid it necessarily cuts all rectangles, if they exist. Thereby Theorem 2 is proved.

Theorem 2 serves as a preparation for a closer investigation of properties of the set M_ζ at arbitrary $\zeta \in (Q, Q)$ if we recall that $M_\zeta = (Q, Q) \cap K_\zeta$. Many functions $\tilde{\zeta} \in K_\zeta$ pass through the rhomboid D_ζ; the matter is whether also solutions of diff. equation (Q, Q) pass through D_ζ.

4. Dependence of dispersions on initial conditions. Every dispersion $\zeta \in (Q, Q)$ is well defined by the initial conditions $\zeta(t_0) = \zeta_0$, $\zeta'(t_0) = \zeta'_0$, $\zeta''(t_0) = \zeta''_0$, where $(t_0, \zeta_0, \zeta'_0, \zeta''_0)$ is an arbitrary point of the set

$$\omega = \{(t_0, \zeta_0, \zeta'_0, \zeta''_0)/t_0 \in J, \zeta_0 \in J, \zeta'_0 > 0 \quad \text{arbitrary}, \ \zeta''_0 \ \text{arbitrary}\}.$$

To any $p \in \omega$ there exists just one $\zeta \in (Q, Q)$ and an interval $I_\zeta = \mathrm{Dom}\,\zeta \subset J$ (while $t_0 \in I_\zeta$) so that $\zeta(t)$ satisfies differential equation (Q, Q) in interval I_ζ and fulfils the corresponding initial conditions, see [2].

The dispersion $\zeta = \zeta(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)$ as a function of the variable t and of the initial conditions is thus defined in a certain set $\Omega \subset J \times \omega$, where $\Omega = \{(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)/(t_0, \zeta_0, \zeta'_0, \zeta''_0) \in \omega, \ t \in I_\zeta\}$.

Let $\zeta \in (Q, Q)$ be given, $\zeta = \zeta(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)$. To an arbitrary $\Delta \neq 0$ let us take an arbitrary solution $\varrho(t)$ of the differential equation $({}^\Delta Q)$ in interval J. Then we define the solution $P(t)$ of the differential equation $({}^\Delta Q)$ in interval J by means of the initial conditions

$$(6) \qquad P(t_0) = \frac{\varrho(\zeta_0)}{\sqrt{\zeta'_0}} \, (= P_0), \quad P'(t_0) = \varrho'(\zeta_0) \sqrt{(\zeta'_0)} - \tfrac{1}{2} P(t_0) \frac{\zeta''_0}{\zeta'_0} \, (= P'_0).$$

Then the dispersion ζ with the upper initial conditions is the solution in interval I_ζ of the separated differential equation

$$(7) \qquad \zeta' = \frac{\varrho^2(\zeta)}{P^2(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)}$$

with the initial condition $\zeta(t_0) = \zeta_0$; instead of $P(t)$ we write more explicitly $P(t;\ t_0, \zeta_0, \zeta'_0, \zeta''_0)$ in order to stress the dependence of the right side in (7) on parameters $(t_0, \zeta_0, \zeta'_0, \zeta''_0) \in \omega$.

Properties of the function $\zeta(t;\ t_0, \zeta_0, \zeta'_0, \zeta''_0)$ depend on the property of the function

$$G = \frac{\varrho^2(\zeta)}{P^2(t;\ t_0, \zeta_0, \zeta'_0, \zeta''_0)}.$$

The function $P(t)$ is an amplitude of some pair of linearly independent integrals U, V of the differential equation (Q) in interval J, i.e. $P^2 = U^2 + V^2$; the integrals U, V possess the following initial conditions

$$U(t_0) = U_0\,,\ U'(t_0) = U'_0\,;\quad V(t_0) = V_0\,,\ V'(t_0) = V'_0\,,$$

where

(8) $$U_0 = P_0 \cos\alpha\,,\quad U'_0 = P'_0 \cos\alpha - \frac{\Delta}{P_0} \sin\alpha\,,$$

$$V_0 = P_0 \sin\alpha\,,\quad V'_0 = P'_0 \sin\alpha + \frac{\Delta}{P_0} \cos\alpha\,.$$

Here α is an arbitrary, fixedly chosen number independent of $p \in \omega$.

The integrals $U = U(t;\ t_0, U_0, U'_0)$, $V = V(t;\ t_0, V_0, V'_0)$ are defined by relations (6), (8) as functions of variables $t, t_0, \zeta_0, \zeta'_0, \zeta''_0$ on the set $J \times \omega$; properties of the function P and thereby properties of the function G, too depend thus on properties of functions $U = U(t;\ t_0, \zeta_0, \zeta'_0, \zeta''_0)$, $V = V(t;\ t_0, \zeta_0, \zeta'_0, \zeta''_0)$.

An arbitrary solution y of differential equation (Q) in interval J with initial conditions $y(t_0) = y_0$, $y'(t_0) = y'_0$ is a function of variables t, t_0, y_0, y'_0 on the set $\Theta = J \times \vartheta$ where

$$\vartheta = \{(t_0, y_0, y'_0)/t_0 \in J, \text{arbitrary}, y'_0 \text{ arbitrary}\}\,.$$

With the notation $\boldsymbol{Y} = \begin{pmatrix} y \\ y' \end{pmatrix}$, $\boldsymbol{Q}^*(t) = \begin{pmatrix} 0 & 1 \\ Q(t) & 0 \end{pmatrix}$ differential equation (Q) is equivalent to the differential equation

$(\boldsymbol{Q})^*$ $$\boldsymbol{Y}' = \boldsymbol{Q}^*(t)\,\boldsymbol{Y}\,.$$

The solution $\boldsymbol{Y}$ of differential equation $(\boldsymbol{Q})^*$ is, in fact, a function $\boldsymbol{Y}(t;\ t_0, y_0, y'_0)$ defined on the set Θ; we shall show that it is continuous in the domain Θ^0 (= interior of Θ). Denote $R =]-\infty, \infty[$. The function $\boldsymbol{Q}^*(t)\,\boldsymbol{Y}$ is continuous on the set $J \times R \times R$ but it is not bounded there.

Choose an arbitrary but fixed point $(t;\ t_0, y_0, y'_0) \in \Theta^0$; then particularly, $t \in J^0$, $t_0 \in J^0$. Let $\overline{t_0, t}$ be a minimal closed interval containing point t_0, t; we have $\overline{t_0, t} \subset J^0$. Consequently, numbers $A_1 < A_2$ and $B_1 > B_2$ exist such that $A < A_1 < A_2 < \min(t_0, t)$, $\max(t_0, t) < B_2 < B_1 < B$.

Let us denote $J_1 = [A_1, B_1]$, $J_2 = [A_2, B_2]$. Let $R_1 = [y_0 - 1, y_0 + 1]$, $R_2 = [y'_0 - 1, y'_0 + 1]$. Then, in the domain $J_1^0 \times R_1^0 \times R_2^0$, $\boldsymbol{Q}^*(t)\,\boldsymbol{Y}$ is continuous and bounded.

Let us denote $\boldsymbol{Y}(t) = \boldsymbol{Y}(t; t_0, y_0, y'_0)$, $\tilde{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t; \tilde{t}_0, \tilde{y}_0, \tilde{y}'_0)$. Solutions $\boldsymbol{Y}(t)$ and and $\tilde{\boldsymbol{Y}}(t)$ of differential equation $(\boldsymbol{Q})^*$ exist in interval J_2 and according to theorems on dependence of solutions of differential equations on initial conditions, for $(\tilde{t}_0, \tilde{y}_0, \tilde{y}'_0) \to (t_0, y_0, y'_0)$ we have $\tilde{\boldsymbol{Y}}(t) \rightrightarrows \boldsymbol{Y}(t)$ in interval J_2, see [3].

From this it follows that for $(t, \tilde{t}_0, \tilde{y}_0, \tilde{y}'_0) \to (t, t_0, y_0, y'_0)$ we have $\tilde{\boldsymbol{Y}}(\tilde{t}) \to \boldsymbol{Y}(t)$ which proves the continuity of function $\boldsymbol{Y}(t; t_0, y_0, y'_0)$ in the domain Θ^o.

Especially, the general solution of differential equation (Q) $y(t;, t_0, y_0, y'_0)$ is a continuous function in the domain Θ^0.

As P_0 and P'_0 are continuous functions of variables ζ_0, ζ'_0, ζ''_0 on the set $J \times]0, \infty[\times] -\infty, \infty[$, functions U_0, U'_0; V_0, V'_0 of variables ζ_0, ζ'_0, ζ''_0 are also continuous here. Then functions $U(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)$, $V(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)$ are continuous in the domain $J^0 \times \omega^0$ and $P^2(t; t_0, \zeta_0, \zeta'_0, \zeta''_0)$ is continuous, too. As $P^2 > 0$, we get finally the result that the function $G = G(t; \zeta; t_0, \zeta_0, \zeta'_0, \zeta''_0)$ on the right-hand side of differential equation (7) is continuous in the domain $J^0 \times J^0 \times \omega^0$.

Theorem 3. *Let $J_2 = [A_2, B_2]$ be an arbitrary closed subinterval of interval J^0. Choose $\varepsilon > 0$ such that $\varepsilon = \min(A_2 - A, B - B_2)$. Denote $D = \{(t, z)/t \in J_2, |z - t| \leqq \varepsilon\}$. Then, for an arbitrary $t_0 \in J_2^0$ an $\eta > 0$ exists such that for all $|\tilde{t}_0 - t_0| < \eta$, $|\tilde{\zeta}_0 - t_0| < \eta$, $|\tilde{\zeta}'_0 - 1| < \eta$, $|\tilde{\zeta}''_0| < \eta$ we have:*

1° *$\tilde{\zeta}(t)$ exists in J_2 and a graph of $\tilde{\zeta}$ runs in D*

2° *$(\tilde{t}_0, \tilde{\zeta}_0, \tilde{\zeta}'_0, \tilde{\zeta}''_0) \to (t_0, t_0, 1, 0)$ implies $\tilde{\zeta}(t) \rightrightarrows t$ in interval J_2.*

Proof. Choose $J_1 = [A_1, B_1]$ such that $A < A_1 < A_2 - \varepsilon$, $B_2 + \varepsilon < B_1 < B$. Then $D \subset J_1^0 \times J_1^0$. In the domain $t \in J_1^0$, $\zeta \in J_1^0$, $\tilde{t}_0 \in J_1^0$, $\tilde{\zeta}_0 \in J_1^0$, $|\tilde{\zeta}'_0 - 1| < \frac{1}{2}$, $|\tilde{\zeta}''_0| < \frac{1}{2}$ the function G is continuous and bounded. Using the theorems on dependence of solutions of differential equations on initial conditions and parametres, we conclude the proof.

At the end of paragraph 2, the demanded detailed knowledge of set M_ζ is given by

Theorem 4. *Let $\zeta \in (Q, Q)$ be an arbitrary dispersion. Let D_ζ be the closed set from Theorem 2. Then, for an arbitrary $t_0 \in]K, L[$ an $\eta > 0$ exists such that for all $|\tilde{t}_0 - t_0| < \eta$, $|\tilde{\zeta}_0 - t_0| < \eta$, $|\tilde{\zeta}'_0 - 1| < \eta$, $|\tilde{\zeta}''_0| < \eta$ the dispersion $\tilde{\zeta}$ with initial conditions $\tilde{\zeta}(\tilde{t}_0) = \tilde{\zeta}_0$, $\tilde{\zeta}'(\tilde{t}_0) = \tilde{\zeta}'_0$, $\tilde{\zeta}''_0(\tilde{t}_0) = \tilde{\zeta}''_0$ belongs to M_ζ.*

Proof. Let us put $J_2 = [K, L]$ in Theorem 3. We shall show that ε from Theorem 2 fulfils, the inequalities $\varepsilon < K - A$, $\varepsilon < B - L$: thus, we have $K = x - \frac{1}{2}E$, $E < x - A$, $\varepsilon < \frac{1}{2}E$; hence $\varepsilon < x - K < K - A$; similarly, $L = y + \frac{1}{2}E$, $E < B - y$, $\varepsilon < \frac{1}{2}E$ implies $\varepsilon < L - y < B - L$. For this reason our D_ζ possesses properties of D from Theorem 3; by Theorem 3 it follows that $\tilde{\zeta} \in (Q, Q)$ runs in D_ζ, so that according to Theorem 2 we have $\tilde{\zeta} \in K_\zeta$ and consequently $\tilde{\zeta} \in M_\zeta$.

The results obtained so far, form a basis for the investigation of equation system (σ) from paragraph 2.

5. System of equations (σ). The dependence of numbers α, β, γ, δ on an arbitrary $\zeta \in (Q, Q)$ can be obtained explicitly from equations (3), (3'). If Δ is a Wronskian of the ordered pair of integrals u, v i.e. $\Delta = uv' - u'v$, then

$$
\begin{aligned}
(R) \qquad \Delta\alpha &= \frac{u(\zeta)\, v'(t)}{\sqrt{\zeta'}} + \frac{u(\zeta)\, v(t)\, \zeta''}{2\zeta'^{3/2}} - u'(\zeta)\, v(t) \sqrt{\zeta'}\,, \\
\Delta\beta &= -\frac{u(\zeta)\, u'(t)}{\sqrt{\zeta'}} - \frac{u(\zeta)\, u(t)\, \zeta''}{2\zeta'^{3/2}} + u'(\zeta)\, u(t) \sqrt{\zeta'}\,, \\
\Delta\gamma &= \frac{v(\zeta)\, v'(t)}{\sqrt{\zeta'}} + \frac{v(\zeta)\, v(t)\, \zeta''}{2\zeta'^{3/2}} - v'(\zeta)\, v(t) \sqrt{\zeta'}\,, \\
\Delta\delta &= -\frac{v(\zeta)\, u'(t)}{\sqrt{\zeta'}} - \frac{v(\zeta)\, u(t)\, \zeta''}{2\zeta'^{3/2}} + v'(\zeta)\, u(t) \sqrt{\zeta'}\,.
\end{aligned}
$$

These equations are dependent because $\alpha\delta - \beta\gamma = 1$. The expressions on the left hand sides and then also these at the right-hand sides of (R) do not depend on t: If we choose $t = t_0 \in J_0$ arbitrarily but fixed, the matter is to find which values assume coefficients α, β, γ, δ as functions of initial conditions t_0, ζ_0, ζ'_0, ζ''_0 if the dispersion $\zeta \in M_\varphi$ where $\varphi \in C_2$.

Adding the second and the third equation of system (σ) we get the equation

$$(\sigma') \qquad (\gamma - \beta)(\varkappa - \nu) + (\delta - \alpha)(\mu - \lambda) = 0\,.$$

Let

$$\varphi \in C_2\,, \quad \varphi \to \begin{pmatrix} \varkappa & \mu \\ \lambda & \nu \end{pmatrix};$$

then, for the elements of this matrix at every

$$\zeta \in M_\varphi\,, \quad \zeta \to \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

there hold the equations (σ), (σ') with coefficients α, β, γ, δ which are given by formulas (R) in which $t = t_0$ is an arbitrary fixed number in a certain interval, $\zeta = \zeta_0$ can be chosen equal to t_0, $\zeta' = \zeta'_0$ is an arbitrary number in a small neighbourhood of number one and $\zeta'' = \zeta''_0$ is an arbitrary number in a small neighbourhood of zero. At this special choice $\zeta_0 = t_0$ in (R) we substract the second equation from the third one and the fourth from the first one.

We get

$$(S) \qquad \Delta(\gamma - \beta) = \frac{u_0^2 + v_0^2}{2\zeta_0'^{3/2}} \zeta_0'' + (u_0 u_0' + v_0 v_0')\left(\frac{1}{\sqrt{\zeta_0'}} - \sqrt{\zeta_0'}\right),$$

$$\Delta(\alpha - \delta) = \frac{u_0 v_0}{\zeta_0'^{3/2}} \zeta_0'' + (u_0 v_0' + u_0' v_0)\left(\frac{1}{\sqrt{\zeta_0'}} - \sqrt{\zeta_0'}\right).$$

For ζ_0' in the neighbourhood of one $\xi = (1/\sqrt{\zeta_0'}) - \sqrt{\zeta_0'}$ lies in the neighbourhood of zero, and in these neighbourhoods the correspondence between ζ_0' and ξ is one-to-one.

System (S) taken as the system of linear equations for ζ_0'' and ξ has the determinant

$$\begin{vmatrix} \dfrac{u_0^2 + v_0^2}{2\zeta_0'^{3/2}} & u_0 u_0' + v_0 v_0' \\ \\ \dfrac{u_0 v_0}{\zeta_0'^{3/2}} & u_0 v_0' + u_0' v_0 \end{vmatrix} = \frac{\Delta}{2\zeta_0'^{3/2}} (u_0^2 - v_0^2).$$

As $u_0 = u(t_0)$, $v_0 = v(t_0)$ and t_0 can be chosen arbitrarily in a certain interval, we can arrange it in such a way that $u_0 \neq 0$, $v_0 \neq 0$ and $u_0^2 - v_0^2 \neq 0$. This means that due to simultaneous validity of $\gamma - \beta = 0$ and $\delta - \alpha = 0$ we have necessarily $\zeta_0'' = 0$, $\zeta_0' = 1$, whereas in the neighbourhood of $\zeta_0'' = 0$, $\zeta_0' = 1$ it is possible to choose values ζ_0' and ζ_0'' such that $\gamma - \beta = 0$ and at the same time $\delta - \alpha \neq 0$ and there is also possible to choose values ζ_0' near to one and ζ_0'' near to zero in such a way that $\gamma - \beta \neq 0$ and at the same time $\delta - \alpha = 0$; from equation (σ') it follows $\mu = \lambda$ in the first case, $\nu = \varkappa$ in the second one. Thus, the dispersion $\varphi \in C_2$ has a matrix

$$\begin{pmatrix} \varkappa & \lambda \\ \lambda & \varkappa \end{pmatrix}.$$

Then for $\beta - \gamma \neq 0$ the first equation in system (σ) gives $\lambda = 0$, so that $\varphi \rightarrow \begin{pmatrix} \varkappa & 0 \\ 0 & \varkappa \end{pmatrix}$. The determinant must be equal to 1 and hence we get $\varkappa = \pm 1$. Thus,

$$\varphi \rightarrow \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $\varphi \in C$.

The inclusion $C_2 \subset C$ is proved.

Thus, it is possible to define in an equivalent way central dispersions of the differential equation (Q) as dispersions $\varphi \in (Q, Q)$ with the property that for any $\zeta \in \in (Q, Q)$ for which both composed functions $\zeta\varphi$, $\varphi\zeta$ exist in some common interval, there holds $\zeta\varphi = \varphi\zeta$.

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Резюме

АЛГЕБРАИЧЕСКОЕ ОПРЕДЕЛЕНИЕ ЦЕНТРАЛЬНЫХ ДИСПЕРСИЙ ПЕРВОГО ПОРЯДКА ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = Q(t)\, y$

ЭРИХ БАРВИНЕК (Erich Barvínek), Брно

Мы рассматриваем дифференциальное уравнение

$$(Q) \qquad y'' = Q(t)\, y\,,$$

где $Q(t)$ действительная функция от действительной переменной определена и непрерывна в интервале $J = (A, B)$, где $A < B$ и символ (A, B) обозначает интервал с левым концом A (или $A = -\infty$) и с правым концом B (или $B = \infty$), причем интервал может быть замкнутым $[A, B]$ или открытым $]A, B[$ или полуоткрытым $[A, B[$, $]A, B]$. Мы ничего не предполагаем о колебании решения дифференциального уравнения (Q).

Первое определение центральных дисперсий (первого порядка) дифференциального уравнения (Q) представляет собой сведение к общему случаю первоначального определения Борувки, см. [1]: Пусть n любое целое число. Пусть $t \in J$ любое число. Пусть y представляет собой ненулевое решение дифференциального уравнения (Q), именно такое, что $y(t) = 0$. Если точку t взять в качестве нулевого корня интеграла y и если корням интеграла справа от t поставить в соответствие положительные индексы, то значение центральной дисперсии φ_n в точке t можно определить как n-ый корень решения y, поскольку этот корень существует в J.

Множество всех центральных дисперсий дифференциального уравнения обозначаем через C. Множество C счетно или конечно.

Назовем дисперсией (ее более точное название „возрастающая дисперсия первого порядка") дифференциального уравнения (Q) любое решение дифференциального уравнения

$$(Q, Q) \qquad \sqrt{\zeta'}\left(\frac{1}{\sqrt{\zeta'}}\right)'' + Q(\zeta)\,\zeta'^2 = Q(t)$$

определенное в подинтервале интервала J, график которого проходит в интервале $J \times J$ от края до края. Множество всех дисперсий дифференциального уравнения (Q) мы будем также обозначать символом (Q, Q). Напомним, что $C \subset (Q, Q)$.

Пусть C_1 есть множество всех дисперсий $\varphi \in (Q, Q)$, которые преобразовывают каждое решение u дифференциального уравнения (Q) в $\pm u$ по уравнению

$$\frac{u[\varphi(t)]}{\sqrt{\varphi'(t)}} = \pm u(t)\,.$$

В работе (4) доказано, что $C_1 = C$, так что центральные дисперсии дифференциального уравнения (Q) можно определить эквивалентным образом как элементы множества C_1.

Пусть C_2 представляет собой множество всех $\varphi \in (Q, Q)$ так что для каждого $\zeta \in (Q, Q)$, для которого обе сложные фунцкии $\zeta\varphi$ и $\varphi\zeta$ имеют некоторый общий интервал существования, имеет место в этом интервале $\zeta\varphi = \varphi\zeta$.

В работе (4) доказано, что $C \subset C_2$. В настоящей статье доказывается обратное включение $C_2 \subset C$, так что центральные дисперсии дифференциального уравнения (Q) можно определить третьим эквивалентным способом как элементы множества C_2.