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AXIOMATIC TREATMENT OF BASES IN ARBITRARY SETS

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1. INTRODUCTION

In his paper [5], H. WHITNEY has defined a matroid in several ways. In particular, the primitive notion of "an independent set" being chosen, a matroid is a finite set S together with a family of independent sets (subsets of S) satisfying the following two postulates:

(I_1) *Any subset of an independent set is an independent set.*

(I_2) *If I_1 and I_2 are two independent sets with n and $n + 1$ elements, respectively, then there exists an element $x \in I_2 \setminus I_1$ such that $I_1 \cup (x)$ is an independent set.*

The definition of a matroid in terms of "bases" consists in specifying a family of bases (subsets) of a (finite) set S with the following two properties:

(B_1) *No proper subset of a base is a base.*

(B_2) *If B_1 and B_2 are two bases, then for any element $b_1 \in B_1$ there exists an element $b_2 \in B_2$ such that $[B_1 \setminus (b_1)] \cup (b_2)$ is a base.*

The correspondence expressed by

"*Every maximal independent set is a base*"

and

"*Every subset of a base is an independent set*"

then easily establishes the equivalence of both concepts.

The definition of a matroid in terms of independent sets was later extended to infinite sets introducing an additional condition of "finite character property" for the family of independent sets:

(I_3) *If every finite subset of a set I is an independent set, then I is an independent set.*

This generalized concept coincides with the author's concept of a LA-dependence structure in [2], where also some other conditions equivalent to (I_2) are introduced.

The present paper offers a generalization of the definition of a matroid in terms of "bases" to sets of an arbitrary cardinality. An attempt in this direction was also the paper [4] of E. SZODORAY. As a particular result we shall prove that the properties (B_1), (B_2) together with the following condition expressing „finite character property"

(B_3) *If every finite subset of a set I is a subset of a suitable base, then I is a subset of a base.*[1])

give a complete characterization of the maximal independent sets (bases) of LA-dependence structures (generalized matroids).

2. PRELIMINARIES

Throughout the paper, the terminology introduced in [1] and [2] will be used. Let us recall that, in terms of independent sets, an A-dependence structure is a pair $(S, \mathscr{I})$ of a set S and an A-*independence net* of S, i.e. a family $\mathscr{I} \neq \emptyset$ of subsets of S satisfying (I_1) and (I_3). Denoting by $\mathscr{M}_{\mathscr{I}}$ the family of all maximal independent sets of $(S, \mathscr{I})$, i.e. the family defined by

$$(\mathscr{I} \to \mathscr{M}) \qquad M \in \mathscr{M}_{\mathscr{I}} \leftrightarrow M \in \mathscr{I} \wedge \forall X(X \supsetneq M \to X \notin \mathscr{I})$$

one can easily prove that $\mathscr{M}_{\mathscr{I}}$ satisfies the conditions (B_1) and (B_3), where "X is a base" should be read "$X \in \mathscr{M}_{\mathscr{I}}$". On the other hand, let $\mathscr{M} \neq \emptyset$ be a family of subsets of a given set S such that $\mathscr{M}$ satisfies both (B_1) and (B_3) (again, "X is a base" should be read as "$X \in \mathscr{M}$"). A family of this kind will be called an A-*independence covering* of S. Then, defining $\mathscr{I}_{\mathscr{M}}$ by

$$(\mathscr{M} \to \mathscr{I}) \qquad I \in \mathscr{I}_{\mathscr{M}} \leftrightarrow \exists M(M \supseteq I \wedge M \in \mathscr{M})$$

it turns out immediately that $\mathscr{I}_{\mathscr{M}}$ possesses the properties (I_1) and (I_3) (reading $\mathscr{I}_{\mathscr{M}}$ for $\mathscr{I}$). Also, combining the correspondences $(\mathscr{I} \to \mathscr{M})$ and $(\mathscr{M} \to \mathscr{I})$, the following equalities hold

$$\mathscr{M}_{\mathscr{I}_{\mathscr{M}}} = \mathscr{M} \quad \text{and} \quad \mathscr{I}_{\mathscr{M}_{\mathscr{I}}} = \mathscr{I}\,.$$

This yields the following basic result:

Theorem. *The concepts of an* A-*dependence structure* $(S, \mathscr{I})$, *where* $\mathscr{I}$ *is an* A-*independence net of* S *and the concept of an* A-*dependence structure* $(S, \mathscr{M})$, *where* $\mathscr{M}$ *is an* A-*independence covering of* S *are equivalent, the equivalence being established by the mappings* $(\mathscr{I} \to \mathscr{M})$ *and* $(\mathscr{M} \to \mathscr{I})$.

Furthermore, $\mathscr{I}$ being an A-independence net of a set S recall at least the following two concepts of [2]:

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[1]) Another, equivalent, formulation of (B_3) is the following one: *A set which is not contained in any base possesses a finite subset with the same property.*

The *closure operation* $C_{\mathscr{I}}$ on $\mathscr{I}$ is defined by

$$(C) \qquad C_{\mathscr{I}}(I) = I \cup \bigcup_{I \cup (x) \notin \mathscr{I}} I \cup (x)$$

and the family $\mathscr{C}_{\mathscr{I}}$ of *canonic sets* by

$$(\mathscr{C}) \qquad I \in \mathscr{C}_{\mathscr{I}} \leftrightarrow I \in \mathscr{I} \wedge \forall X[X \in \mathscr{I} \wedge I \subseteq C_{\mathscr{I}}(X) \rightarrow C_{\mathscr{I}}(I) \subseteq C_{\mathscr{I}}(X)] .$$

In what follows, by a *base* of an A-dependence structure always a maximal independent set which is canonic will be understood. Let us introduce also the relation $\varepsilon_{\mathscr{I}} \subseteq \mathscr{I} \times \mathscr{I}$ defined by

$$(\varepsilon) \qquad [I_1, I_2] \in \varepsilon_{\mathscr{I}} \leftrightarrow I_1 \subseteq C_{\mathscr{I}}(I_2) \wedge I_2 \subseteq C_{\mathscr{I}}(I_1) .$$

Our investigations will be based on the following two lemmas on A-independent nets of a set S (see [2]):

Lemma A. *Let $\mathscr{I}$ be an A-independent net of S. Then, for any $I_1 \in \mathscr{I}$ and $I_2 \in \mathscr{I}$ with $I_1 \subseteq C_{\mathscr{I}}(I_2)$ there exists $I_0 \subseteq I_2 \setminus I_1$ such that*

$$I_1 \cup I_0 \in \mathscr{I} \quad \textit{and} \quad I_2 \subseteq C_{\mathscr{I}}(I_1 \cup I_0) .$$

Lemma B. *Let $\mathscr{I}$ be an A-independent net of S. Let $I_1 \in \mathscr{I}$ and $I_2 \in \mathscr{C}_{\mathscr{I}}$ such that $[I_1, I_2] \in \varepsilon_{\mathscr{I}}$. Then,*

$$\operatorname{card}(I_1 \setminus I_2) \leqq \operatorname{card}(I_2 \setminus I_1) .$$

3. SOME PROPERTIES OF BASES

In this short paragraph, let $(S, \mathscr{I})$ be a (fixed) LA-dependence structure (i.e. $\mathscr{C}_{\mathscr{I}} = \mathscr{I}$).[2] We are going to show that, besides (B_1) and (B_3), also (B_2) and some further properties hold for bases (i.e. maximal independent sets) of $(S, \mathscr{I})$ (comp. [2]).

First, state the consequences of Lemmas A and B for bases of $(S, \mathscr{I})$.

Statement A. *Let B be a base and I an independent subset of S. Then there exists a base B_0 such that*

$$I \subseteq B_0 \quad \textit{and} \quad B_0 \setminus I \subseteq B .$$

Statement B. *Let B_1 and B_2 be two bases of S. Then*

$$\operatorname{card}(B_1 \setminus B_2) = \operatorname{card}(B_2 \setminus B_1) .\text{[3]}$$

Now, formulate the following

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[2]) Though some statements can be formulated more generally for GA-dependence structures.

[3]) And, hence, card (B_1) = card (B_2). Of course, both equalities are equivalent for a finite set S.

Theorem. *Let B_1 and B_2 be two bases of S. Then, besides* (B_2), *also the following statements hold*:

(B'_2) *For any element $b_1 \in B_1 \setminus B_2$ there exists an element $b_2 \in B_2 \setminus B_1$ such that $[B_1 \setminus (b_1)] \cup (b_2)$ is a base of S.*

(B_{2f}) *For any finite subset $B'_1 \subseteq B_1$ there exists a subset $B'_2 \subseteq B_2$ of the same number of elements such that $(B_1 \setminus B'_1) \cup B'_2$ is a base of S.*

(B'_{2f}) *For any finite subset $B'_1 \subseteq B_1 \setminus B_2$ there exists a subset $B'_2 \subseteq B_2 \setminus B_1$ of the same number of elements such that $(B_1 \setminus B'_1) \cup B'_2$ is a base of S.*

(B_{2g}) *For any subset $B'_1 \subseteq B_1$ there exists a subset $B'_2 \subseteq B_2$ such that* $\operatorname{card}(B'_1) = \operatorname{card}(B'_2)$ *and $(B_1 \setminus B'_1) \cup B'_2$ is a base of S.*

(B'_{2g}) *For any subset $B'_1 \subseteq B_1 \setminus B_2$ there exists a subset $B'_2 \subseteq B_2 \setminus B_1$ such that* $\operatorname{card}(B'_1) = \operatorname{card}(B'_2)$ *and $(B_1 \setminus B'_1) \cup B'_2$ is a base of S.*

$(\tilde{\mathrm{B}}_2)$ *For any element $b_1 \in B_1$ there exists an element $b_2 \in B_2$ such that $(b_1) \cup [B_2 \setminus (b_2)]$ is a base of S.*

$(\tilde{\mathrm{B}}'_2)$ *For any element $b_1 \in B_1 \setminus B_2$ there exists an element $b_2 \in B_2 \setminus B_1$ such that $(b_1) \cup [B_2 \setminus (b_2)]$ is a base of S.*

$(\tilde{\mathrm{B}}_{2f})$ *For any finite subset $B'_1 \subseteq B_1$ there exists a subset $B'_2 \subseteq B_2$ of the same number of elements such that $B'_1 \cup (B_2 \setminus B'_2)$ is a base of S.*

$(\tilde{\mathrm{B}}'_{2f})$ *For any finite subset $B'_1 \subseteq B_1 \setminus B_2$ there exists a subset $B'_2 \subseteq B_2 \setminus B_1$ of the same number of elements such that $B'_1 \cup (B_2 \setminus B'_2)$ is a base of S.*

$(\tilde{\mathrm{B}}_{2g})$ *For any subset $B'_1 \subseteq B_1$ there exists a subset $B'_2 \subseteq B_2$ such that* $\operatorname{card}(B'_1) = \operatorname{card}(B'_2)$ *and $B'_1 \cup (B_2 \setminus B'_2)$ is a base of S.*

$(\tilde{\mathrm{B}}'_{2g})$ *For any subset $B'_1 \subseteq B_1 \setminus B_2$ there exists a subset $B'_2 \subseteq B_2 \setminus B_1$ such that* $\operatorname{card}(B'_1) = \operatorname{card}(B'_2)$ *and $B'_1 \cup (B_2 \setminus B'_2)$ is a base of S.*

Proof. Since fourteen implications in the following two diagrams

$$\begin{array}{ccc} (\mathrm{B}'_{2g}) & \rightarrow & (\mathrm{B}_{2g}) \\ \downarrow & & \downarrow \\ (\mathrm{B}'_{2f}) & \rightarrow & (B_{2f}) \\ \downarrow & & \downarrow \\ (\mathrm{B}'_2) & \rightarrow & (\mathrm{B}_2) \end{array} \qquad \begin{array}{ccc} (\tilde{\mathrm{B}}'_{2g}) & \rightarrow & (\tilde{\mathrm{B}}_{2g}) \\ \downarrow & & \downarrow \\ (\tilde{\mathrm{B}}'_{2f}) & \rightarrow & (\tilde{\mathrm{B}}_{2f}) \\ \downarrow & & \downarrow \\ (\tilde{\mathrm{B}}'_2) & \rightarrow & (\tilde{\mathrm{B}}_2) \end{array}$$

are quite evident, we are going to prove (B'_{2g}) and $(\tilde{\mathrm{B}}'_{2g})$.

Thus, let $B'_1 \subseteq B_1 \setminus B_2$. Then, in view of Statement A applied to $I = B_1 \setminus B'_1$, there exists $B'_2 \subseteq B_2 \setminus B_1$ such that $(B_1 \setminus B'_1) \cup B'_2$ is a base of S. Moreover, making use of Statement B,

$$\operatorname{card}(B'_1) = \operatorname{card}(B_1 \setminus [(B_1 \setminus B'_1) \cup B'_2]) = \operatorname{card}([(B_1 \setminus B'_1) \cup B'_2] \setminus B_1) = \\ = \operatorname{card}(B'_2).$$

Hence, the property (B'_{2g}) for the bases is established.

In order to prove $(\tilde{B}'_{2g})$ consider again $B'_1 \subseteq B_1 \setminus B_2$. Now, applying Statement A to $I = B'_1 \cup (B_1 \cap B_2)$ we get the existence of $B''_2 \subseteq B_2 \setminus B_1$ such that

$$B'_1 \cup (B_1 \cap B_2) \cup B''_2$$

is a base of S. Denote by B'_2 the complement of $(B_1 \cap B_2) \cup B''_2$ in B_2; hence

$$B'_2 \subseteq B_2 \setminus B_1 \quad \text{and} \quad B'_1 \cup (B_2 \setminus B'_2) = B'_1 \cup (B_1 \cap B_2) \cup B''_2 .$$

Also, by Statement B,

$$\operatorname{card}(B'_1) = \operatorname{card}([B'_1 \cup (B_2 \setminus B'_2)] \setminus B_2) = \operatorname{card}(B_2 \setminus [B'_1 \cup (B_2 \setminus B'_2)]) = \\ = \operatorname{card}(B'_2) ,$$

as required.

The proof of Theorem is completed.

4. EQUIVALENCE OF SOME PROPERTIES OF § 3

The aim of this paragraph is to established some simple relations among the properties (B_1) and those of Theorem in § 3.

Thus, let S be a given set and $\mathcal{M}$ a family of its subsets. In what follows, the phrase "X is a base" in the formulation of the properties under consideration should be read, as before, "$X \in \mathcal{M}$".

First, we have the implication

"*If* $\mathcal{M}$ *possesses the property* (B'_{2g}), *then it possesses* (B_{2g})"

and, similarly, the other thirteen trivial ones of the diagrams in the proof of Theorem in § 3. Further, one can see immediately that if $\mathcal{M}$ possesses any one of the properties (B'_{2g}), (B'_{2f}), (B'_2), $(\tilde{B}'_{2g})$, $(\tilde{B}'_{2f})$ or $(\tilde{B}'_2)$, then is possesses also the property (B_1). On the other hand, we have

Lemma 1. *If* $\mathcal{M}$ *possesses* (B_1) *and* (B_2), *or* (B_{2f}), *or* (B_{2g}), *then it possesses* (B'_2), *or* (B'_{2f}), *or* (B'_{2g}), *respectively.*

Proof. The first two statements are consequences of the last one. In order to prove it, let $B_1 \in \mathcal{M}$, $B_2 \in \mathcal{M}$ and $B'_1 \subseteq B_1 \setminus B_2$. By (B_{2g}), there is $B'_2 \subseteq B_2$ such that

$$\operatorname{card}(B'_1) = \operatorname{card}(B'_2) \quad \text{and} \quad (B_1 \setminus B'_1) \cup B'_2 \in \mathcal{M} .$$

Denote the difference $B'_2 \setminus B_1 \subseteq B_2 \setminus B_1$ by B''_2. Since

$$(B_1 \setminus B'_1) \cup B''_2 = (B_1 \setminus B'_1) \cup B'_2$$

belongs to $\mathcal{M}$, it suffices to prove that $\operatorname{card}(B''_2) = \operatorname{card}(B'_1)$; this will establish the property (B'_{2g}) for $\mathcal{M}$.

Thus, let $\text{card}(B_2'') < \text{card}(B_1')$ (because of the inclusion $B_2'' \subseteq B_2'$ always $\text{card}(B_2'') \leqq \text{card}(B_2') = \text{card}(B_1')$). Applying (B_{2g}) to B_1, $(B_1 \setminus B_1') \cup B_2'$ and $B_2'' \subseteq$ $\subseteq [(B_1 \setminus B_1') \cup B_2''] \setminus B_1$ we deduce the existence of a subset $B_1'' \subseteq B_1$ such that

$$\text{card}(B_1'') = \text{card}(B_2'') \quad \text{and} \quad \{[(B_1 \setminus B_1') \cup B_2''] \setminus B_2''\} \cup B_1'' \in \mathcal{M} .$$

But

$$\{[(B_1 \setminus B_1') \cup B_2''] \setminus B_2''\} \cup B_1'' = (B_1 \setminus B_1') \cup B_1'' .$$

Since

$$\text{card}(B_1'') = \text{card}(B_2'') < \text{card}(B_1') ,$$

we get a proper inclusion $(B_1 \setminus B_1') \cup B_1'' \subsetneqq B_1$

which is a contradiction of the assumption (B_1) to be satisfied for $\mathcal{M}$. The proof of Lemma 1 is completed.

Remark. Although there is a similarity between the first six properties and the other six ones (denoted by $(\tilde{\ })$) of Theorem in § 3, we are going to show that the related statements to those of Lemma 1 do not hold for the latter properties.

Let $S = (x_1, x_2, x_3, x_4, x_5)$ and

$$\mathcal{M} = \{(x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_2, x_5), (x_1, x_3, x_4), (x_2, x_3, x_4), (x_3, x_4, x_5)\} .$$

It is a matter of routine to check that $\mathcal{M}$ satisfies $(\tilde{\mathrm{B}}_2)$; evidently, (B_1) is satisfied. On the other hand, taking $B_1 = (x_1, x_2, x_5)$, $B_2 = (x_3, x_4, x_5)$ and $x_1 \in B_1$, neither (x_1, x_3, x_5) nor (x_1, x_4, x_5) belong to $\mathcal{M}$. Thus $(\tilde{B}_2')$ is not satisfied.

Lemma 2. *If $\mathcal{M}$ possesses (B_2'), or $(\tilde{\mathrm{B}}_2')$, then it possesses (B_{2f}'), or $(\tilde{\mathrm{B}}_{2f}')$, respectively.*

Proof. Both assertions can be proved easily by induction.

Lemma 3. *If $\mathcal{M}$ possesses (B_{2g}'), then it possesses $(\tilde{\mathrm{B}}_2')$. Also, if $\mathcal{M}$ possesses $(\tilde{\mathrm{B}}_{2g}')$, then it possesses (B_2').*

Proof. Let us prove the first statement; the proof of the other one follows the same line. Let $B_1 \in \mathcal{M}$, $B_2 \in \mathcal{M}$ and $b_1 \in B_1 \setminus B_2$. Consider the set

$$B_1' = B_1 \setminus [(B_1 \cap B_2) \cup (b_1)] .$$

Since $B_1' \subseteq B_1 \setminus B_2$, there exists by (B_{2g}') a subset $B_2' \subseteq B_2 \setminus B_1$ such that

$$(B_1 \setminus B_1') \cup B_2' \in \mathcal{M} .$$

As a consequence of (B_1) (implied by (B_{2g}')) the subset

$$B_2'' = (B_2 \setminus B_1) \setminus B_2' \subseteq B_2 \setminus B_1$$

is non-empty. Now, apply (B'_{2g}) once again (in fact, (B'_2) would be sufficient at this point) to

$(b_1) \cup (B_2 \setminus B''_2) = (B_1 \setminus B'_1) \cup B'_2 \in \mathcal{M}$, $B_2 \in \mathcal{M}$ and $b_1 \in [(b_1) \cup (B_2 \setminus B''_2)] \setminus B_2$ we deduce the existence of an element $b_2 \in B''_2$ such that $(b_1) \cup (B_2 \setminus (b_2)) \in \mathcal{M}$. Since

$$(b_1) \cup (B_2 \setminus (b_2)) \supseteq (b_1) \cup (B_2 \setminus B''_2),$$

we get, in view of (B_1), the equality and thus $B''_2 = (b_2)$, as required.

Now, Lemmas of this paragraph yield the following

Theorem. *Let S be a finite set and $\mathcal{M}$ a family of its subsets satisfying (B_1). Then the nine properties (B_2), (B'_2), (B_{2f}), (B'_{2f}), (B_{2g}), (B'_{2g}), $(\tilde{B}'_2)$, $(\tilde{B}'_{2f})$ and $(\tilde{B}'_{2g})$ are equivalent one to the other.*

5. EXTENSION OF THE RESULTS OF § 4 TO GENERAL CASE

In this paragraph, the assertion of Theorem in § 4 will be proved for an arbitrary set S and a family $\mathcal{M}$ satisfying, besides (B_1), the additional property (B_3); in [3], two examples have been given showing the necessity of assuming (B_3) for our investigations. The mentioned proof will explore the results of § 2.

Lemma 1. *Let $(S, \mathcal{I})$ be an A-dependence structure. If the subfamily $\mathcal{M}_{\mathcal{I}} \subseteq \mathcal{I}$ of all maximal independent sets possesses the property (B'_2), then every maximal independent set is canonic.*

Proof. Let $B \in \mathcal{M}_{\mathcal{I}}$ and $I \in \mathcal{I}$ such that $B \subseteq C_{\mathcal{I}}(I)$. We are going to prove that $C_{\mathcal{I}}(I) = S$; then, $S = C_{\mathcal{I}}(B) \subseteq C_{\mathcal{I}}(I) = S$ and Lemma 1 will follow.

Let us give an indirect proof of the equality $C_{\mathcal{I}}(I) = S$. Thus, let $I \subsetneqq B'$, where $B' \in \mathcal{M}_{\mathcal{I}}$ and take $b' \in B' \setminus I$; evidently, because of $B \subseteq C_{\mathcal{I}}(I)$, $b' \notin B$. Applying (B'_2) to B', B and $b' \in B' \setminus B$, we deduce the existence of $b \in B \setminus B'$ such that

$$(B \setminus (b')) \cup (b) \in \mathcal{M}_{\mathcal{I}}.$$

Hence, $b \in B \setminus I$ and $b \notin C_{\mathcal{I}}(B \setminus (b')) \supseteq C_{\mathcal{I}}(I)$, in contradiction to the hypothesis $B \subseteq C_{\mathcal{I}}(I)$.

Lemma 2. *Let $(S, \mathcal{I})$ be an A-dependence structure. If $\mathcal{M}_{\mathcal{I}}$ possesses the property (B'_2), then it possesses also (B'_{2g}).*

Proof. Let $B_1 \in \mathcal{M}_{\mathcal{I}}$, $B_2 \in \mathcal{M}_{\mathcal{I}}$ and $B'_1 \subseteq B_1 \setminus B_2$. Since $B_1 \setminus B'_1 \subseteq C_{\mathcal{I}}(B_2)$, there exists, in accordance with Lemma A, a subset $B'_2 \subseteq B_2 \setminus (B_1 \setminus B'_1) = B_2 \setminus B_1$ such that

$$(B_1 \setminus B'_1) \cup B'_2 \in \mathcal{I} \quad \text{and} \quad B_2 \subseteq C_{\mathcal{I}}[(B_1 \setminus B'_1) \cup B'_2].$$

In view of Lemma 1, B_2 is canonic, and thus $S = C_{\mathscr{I}}(B_2) \subseteq C_{\mathscr{I}}[(B_1 \setminus B_1') \cup B_2']$; we conclude that the set $(B_1 \setminus B_1') \cup B_2'$ belongs to $\mathscr{M}_{\mathscr{I}}$ and, moreover, again by Lemma 1, that it is canonic. This fact unables us to apply Lemma B to B_1 and $(B_1 \setminus B_1') \cup B_2'$ resulting in the equality

$$\operatorname{card}(B_1') = \operatorname{card}(B_1 \setminus [(B_1 \setminus B_1') \cup B_2']) = \operatorname{card}([(B_1 \setminus B_1') \cup B_2'] \setminus B_1) = \\ = \operatorname{card}(B_2').$$

The property $(\mathrm{B}_{2\mathrm{g}}')$ for $\mathscr{M}_{\mathscr{I}}$ is thus established.

Lemma 3. *Let $(S, \mathscr{I})$ be an* A*-dependence structure. If $\mathscr{M}_{\mathscr{I}}$ possesses the property $(\tilde{\mathrm{B}}_{2\mathrm{f}}')$, then the* A*-independence net $\mathscr{I}$ possesses the property (I_2) and $(S, \mathscr{I})$ is thus a* LA*-dependence structure.*

Proof. Consider $I_1 \in \mathscr{I}$ and $I_2 \in \mathscr{I}$ with n and $n + 1$ elements, respectively. Let $B_1 \in \mathscr{M}_{\mathscr{I}}$ and $B_2 \in \mathscr{M}_{\mathscr{I}}$ be maximal independent sets such that $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$.

If $(I_2 \cap B_1 \cap B_2) \setminus I_1 \neq \emptyset$, then taking an element of this set we have $x \in I_2 \setminus I_1$ and, since $I_1 \cup (x) \subseteq B_1$, also $I_1 \cup (x) \in \mathscr{I}$.

Thus, we can assume that

$$(I_2 \cap B_1 \cap B_2) \setminus I_1 = \emptyset;$$

hence, every element of I_2 which lies in $B_1 \cap B_2$ belongs to I_1. Therefore, if $I_1 \setminus B_2$ has $k \leqq n$ elements, i.e. if $I_1 \cap B_1 \cap B_2$ has $n - k$ elements, then $I_2 \cap B_1 \cap B_2$ has at most $n - k$ elements, i.e. $I_2 \setminus B_1$ has at least $k + 1$ elements.

Now, make use of the property $(\tilde{\mathrm{B}}_{2\mathrm{f}}')$ applied to B_1, B_2 and $I_1 \setminus B_2 \subseteq B_1 \setminus B_2$. It guarantees the existence of $B_2' \subseteq B_2 \setminus B_1$ such that B_2' has k elements and

$$(I_1 \setminus B_2) \cup (B_2 \setminus B_2') \in \mathscr{M}_{\mathscr{I}}.$$

But, evidently, $B_2 \setminus B_2' \subseteq B_1 \cap B_2$ and, further, $(I_2 \setminus B_1) \setminus B_2' \neq \emptyset$; this follows from the fact that $I_2 \setminus B_1$ has at least $k + 1$ while B_2' has only k elements. Hence,

$$I_1 \subseteq (I_1 \setminus B_2) \cup (B_2 \setminus B_2'),$$

and taking an element x of $(I_2 \setminus B_1) \setminus B_2'$ we have $x \in I_2 \setminus I_1$ and, because of the inclusion $I_1 \cup (x) \subseteq (I_1 \setminus B_2) \cup (B_2 \setminus B_2')$, also $I_1 \cup (x) \in \mathscr{I}$.

The proof of Lemma 3 is completed.

Combining the results of § 2 together with Lemma 2, Lemma 3 and Theorem of § 3 we get the following two lemmas:

Lemma 4. *Let a family $\mathscr{M}$ of subsets of a set S possess the properties (B_2') and (B_3). Then it possesses also $(\mathrm{B}_{2\mathrm{g}}')$.*

Lemma 5. *If $\mathscr{M}$ possesses the properties $(\tilde{B}'_{2f})$ and (B_3), then it possesses any one of the twelve properties introduced in Theorem of* § 3.

The arrows (with the appropriate quotation) in the following diagram indicate the implications which have been proved under the assumption of validity (B_1) and (B_3) for a family $\mathscr{M}$ of subsets of a set S:

$$\begin{array}{ccccc}
(B_{2g}) & \xleftarrow{\ \S 5,\ \textit{Lemma 5}\ } & (\tilde{B}'_{2f}) & \xrightarrow{\ \S 5,\ \textit{Lemma 5}\ } & (\tilde{B}'_{2g}) \\
\downarrow \textit{trivial} & & & \nwarrow \S 4,\ \textit{Lemma 2} & \downarrow \textit{trivial} \\
(B_{2f}) & \xrightarrow{\ \textit{trivial}\ } & (B_2) & & (\tilde{B}'_2) \\
\uparrow \textit{trivial} & & \downarrow \S 4,\ \textit{Lemma 1} & & \uparrow \S 4,\ \textit{Lemma 3} \\
(B'_{2f}) & \xleftarrow{\ \S 4,\ \textit{Lemma 2}\ } & (B'_2) & \xrightarrow{\ \S 5,\ \textit{Lemma 4}\ } & (B'_{2g})
\end{array}$$

From here, we deduce immediately

Theorem. *Let S be a set and $\mathscr{M}$ a family of its subsets satisfying (B_1) and (B_3). Then the nine properties (B_2), (B'_2), (B_{2f}), (B'_{2f}), (B_{2g}), (B'_{2g}), $(\tilde{B}'_2)$, $(\tilde{B}'_{2f})$ and $(\tilde{B}'_{2g})$ are equivalent one to the other.*

6. DEFINITION OF A LA-DEPENDENCE STRUCTURE IN TERMS OF BASES

Theorem of § 5 can be re-stated in another form using the following

Definition. *An* A-*dependence structure $(S, \mathscr{B})$, where $\mathscr{B}$ is an* A-*independence covering of S, is said to be a* LA-*dependence structure if $(S, \mathscr{I}_{\mathscr{B}})$ is a* LA-*dependence structure (in the sense of* [2], *i.e. every element of $\mathscr{I}_{\mathscr{B}}$ is canonic).*

Thus, if $(S, \mathscr{B})$ is a LA-dependence structure, then $\mathscr{B}$ coincides, in view of $\mathscr{B}_{\mathscr{I}_{\mathscr{B}}} = \mathscr{B}$ (see § 2), with the family of the bases of S (in the sense of [2]).

The above mentioned main result can be then formulated as follows.

Theorem. *Let S be a set and $\mathscr{B} \neq \emptyset$ a family of its subsets satisfying the properties in one of the following groups:*

(i)	(B_1), (B_2), (B_3);	(iv)	(B'_2), (B_3);
(ii)	(B_1), (B_{2f}), (B_3);	(v)	(B'_{2f}), (B_3);
(iii)	(B_1), (B_{2g}), (B_3);	(vi)	(B'_{2g}), (B_3);
(vii)	$(\tilde{B}'_2)$, (B_3);		
(viii)	$(\tilde{B}'_{2f})$, (B_3);		
(ix)	$(\tilde{B}'_{2g})$, (B_3);		

Then $(S, \mathscr{B})$ is a LA-*dependence structure.*

7. FINAL REMARKS

Let us conclude the paper with a remark on the converse of Lemma 1 in § 5.

Lemma. *Let $(S, \mathscr{I})$ be an* A*-dependence structure. If every element of $\mathscr{M}_{\mathscr{I}}$ (i.e. every maximal independent set) is canonic, then $\mathscr{M}_{\mathscr{I}}$ possesses the property* (B_2').

Proof. Consider $B_1 \in \mathscr{M}_{\mathscr{I}}$, $B_2 \in \mathscr{M}_{\mathscr{I}}$ and $b_1 \in B_1 \setminus B_2$. In view of Lemma A, there is a subset B_2' of $B_2 \setminus B_1$ such that

$$[B_1 \setminus (b_1)] \cup B_2' \in \mathscr{I} \quad \text{and} \quad B_2 \subseteq C_{\mathscr{I}}([B_1 \setminus (b_1)] \cup B_2') .$$

Since B_2 is canonic, $S = C_{\mathscr{I}}(B_2) \subseteq C_{\mathscr{I}}([B_1 \setminus (b_1)] \cup B_2')$; we deduce that

$$[B_1 \setminus (b_1)] \cup B_2' \in \mathscr{M}_{\mathscr{I}} .$$

Moreover, evidently $B_2' \neq \emptyset$. Take an element $b_2 \in B_2'$ and consider

$$[B_1 \setminus (b_1)] \cup (b_2) \subseteq [B_1 \setminus (b_1)] \cup B_2''] .$$

Necessarily

$$b_1 \in C_{\mathscr{I}}([B_1 \setminus (b_1)] \cup (b_2)) ;$$

for, otherwise $B_1 \cup (b_2) \in \mathscr{I}$, in contradiction to the maximality of B_1, i.e. to $B_1 \in \mathscr{M}_{\mathscr{I}}$. But, then we have $B_1 \subseteq C_{\mathscr{I}}([B_1 \setminus (b_1)] \cup (b_2))$, and since B_1 is canonic,

$$S = C_{\mathscr{I}}(B_1) \subseteq ([B_1 \setminus (b_1)] \cup (b_2)) .$$

Hence, $[B_1 \setminus (b_1)] \cup (b_2) \in \mathscr{M}_{\mathscr{I}}$, i.e. $B_2' = (b_2)$, q.e.d.

Now, by virtue of Lemma and Theorem of § 6 we can derive the following

Theorem. *Let $(S, \mathscr{I})$ be an* A*-dependence structure such that every element of $\mathscr{M}_{\mathscr{I}}$ (every maximal independent set) is canonic. Then $(S, \mathscr{I})$ is a* LA*-dependence structure and thus every element of $\mathscr{I}$ (every independent set) is canonic.*

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Резюме

АКСИОМАТИЧЕСКОЕ ИССЛЕДОВАНИЕ БАЗИЦОВ В ПРОИЗВОЛЬНЫХ МНОЖЕСТВАХ

ВЛАСТИМИЛ ДЛАБ (Vlastimil Dlab), Прага

Г. Уитней определил в работе [5] матроид разными эквивалентными способами, в частности с помощью системы базицов. Матроид-конечное множество S с определённой системой подмножеств (независимых множеств), удовлетворяющих условиям (I_1) и (I_2), или – конечное множество S с системой подмножеств (базицов) имеющих следующие свойства:

(B_1) Никакое собственное подмножество базица не является базицом.
(B_2) Если B_1 и B_2 два базица, то для любого $b_1 \in B_1$ существует $b_2 \in B_2$ такое, что $[B_1 \setminus (b_1)] \cup b_2$ есть базис.

Добавлением условия (I_3), которое означает, что свойство множества быть независимым является свойством конечного характера, было определение матроида при помощи независимых множеств распространено на производные (бесконечные)множества. Это обобщенное понятие матроида совпадает с понятием LA-зависимостиной структуры, введенным автором в [2].

В данной работе понятие матроида обобщается на множества произвольной мощности в терминах базицов. Помимо свойства (B_2) автор определяет в теореме § 3, 11 других родственных свойств базицов:

$$(B'_2),\ (B_{2f}),\ (B'_{2f}),\ (B_{2g}),\ (B'_{2g}),\ (\tilde{B}_2),\ (\tilde{B}'_2),\ (\tilde{B}_{2f}),\ (\tilde{B}'_{2f}),\ (\tilde{B}_{2g}),\ (\tilde{B}'_{2g})^{*})$$

и доказывает, что свойства каждой из следующих 9 комбинаций

(i)	(B_1), (B_2), (B_3);	(iv)	(B'_2), (B_3);	(vii)	$(\tilde{B}'_2)$, (B_3);
(ii)	(B_1), (B_{2f}), (B_3);	(v)	(B'_{2f}), (B_3);	(viii)	$(\tilde{B}'_{2f})$, (B_3);
(iii)	(B_1), (B_{2g}), (B_3);	(vi)	(B'_{2g}), (B_3);	(ix)	$(\tilde{B}'_{2g})$, (B_3);

вполне характеризуют базисы (максимальные независимые множества) LA-зависимостиной структуры (обобщенного матроида) (теорема § 6), причём (B_3) означает следующее: Если каждое конечное подмножество множества I является подмножеством некоторого подходящим образом выбранного базица, то само I является подмножеством некоторого базица. В § 7 этот результат применяется к изучению общих A-зависимостиных структур.

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* Например свойство $(\tilde{B}'_{2g})$ означает следующее: Если B_1 и B_2 два базиса, то для любого подмножества $B'_1 \subseteq B_1 \setminus B_2$ существует подмножество $B'_2 \subseteq B_2 \setminus B_1$ такое, что B'_1 и B'_2 множества одинаковой мощности и $B'_1 \cup (B_2 \setminus B'_2)$ есть базис.